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An Analysis of Karp's Interpolation Theorem and the Notion of k-Consistency Property.

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SUMMARY - In this paper, some counterexamples are exhibited to show that an improvement of Karp's interpolation theorem cannot be achieved with the use of Karp's notion of k-consistency property.

SOMMARIO - Nel presente articolo vengono presentati alcuni controesempi per mostrare che non si può rafforzare il teorema di interpolazione di Karp usando la nozione di Karp di proprietà di k-consistenza.

Karp's interpolation theorem [4] is an extension to the infinitary language $L_{k,k}$, k a strong limit cardinal of denumerable cofinality, of Craig's interpolation theorem, in sprite of Malitz's limiting results [5] on interpolation theorems for infinitary languages.

Indeed Karp's theorem holds with respect to the notion of ω-satisfiability in ω-chains of models, which is weaker than that of satisfiability [4].

Karp's theorem can be stated as follows.

If $F_1 \to F_2$ is an ω-valid sentence of $L_{k,k}$ then there is a sentence F, called the interpolant for $F_1 \to F_2$, such that each extralogical symbol occurring in F occurs both in F_1 and in F_2, and the sentences $F_1 \to F$ and $F \to F_2$ are both valid.

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Due to the before mentioned result of Malitz, the hypothesis of the theorem cannot be weakened to the validity of $F_1 \to F_2$.

On the other hand an improvement of Karp's result strengthening the conclusion to the ω-validity of $F_1 \to F$ and of $F \to F_2$ was obtained by E. Cunningham [1] using a new notion of consistency property, the one she called chain consistency property.

In [1] it was already stated that Karp's notion of k-consistency property was not adequate to prove the improvement of the interpolation theorem.

In this paper we will further analyze this point providing new counterexamples to point out the inadequacy of the notion of k-consistency property for an improvement of Karp's interpolation theorem.

Notice that to give a counterexample to the improvement of Karp's interpolation theorem it is not that straightforward, since we should end up with either $F_1 \to F$ or $F \to F_2$ valid but not ω-valid sentences and the usual examples against interpolation theorems become of little use: ω-chains of models and sentences in $L_{k,k} - L_{k,\omega}$ are to play a key role in the counterexamples we are looking for, since for the sentences in $L_{k,\omega}$ ω-validity is the same as validity.

To understand better the key points of Karp's theorem, let us follow her proof of the theorem.

The notion of consistency property and the related model existence theorem are central features of the proof.

For reference, let us recall the notion of consistency property that can be stated as follows.

Let $k = \bigcup \{k_n : n \in \omega\}$, with $2^{k_n} \leqslant k_{n+1}$.

S is a consistency property for $L_{k,k}$ with respect to the sets of new individual constants C_n with $n \in \omega$, $|C_n| = k_n$, iff $\forall s \in S$, $|s| \leqslant k$ and there is $n \in \omega$ such that s is a set of sentences in the language L_n obtained from $L_{k,k}$ by adding the constants in $\bigcup \{C_i : i \in n\}$ and all of the following conditions hold:

C0) If Z is an atomic sentence, either $Z \notin s$ or $-Z \notin s$, and if Z is of the form $t \neq t$, t a constant, then $Z \notin s$;

C1) If $\{--F_i : i \in I\} \subset s \in S$ and $|I| < k$, then $s \cup \{F_i : i \in I\} \in S$;

C2) If $\{\& \overline{F}_i : i \in I\} \subset s \in S$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ we have that $0 < |\overline{F}_i| \leqslant k_m$, then $s \cup \left(\bigcup \{\overline{F}_i : i \in I\}\right) \in S$;

C3) If $\{-\&\overline{F}_i\colon i\in I\}\subset s\in S$ and $|I|<k$ and there is $m\in\omega$ such that for all $i\in I$ we have that $0<|\overline{F}_i|\leqslant k_m$, then there is a function $f\in\times\{\overline{F}_i\colon i\in I\}$ such that $s\cup\{-f(i)\colon i\in I\}\in S$;

C4) If $\{\forall\overline{v}_i\overline{F}_i\colon i\in I\}\subset s\in S$ and $|I|<k$ and there is $m\in\omega$ such that for all $i\in I$ we have that $0<|\overline{v}_i|\leqslant k_m$, then for all $n\in\omega$ we have that $s\cup\{F_i(\overline{v}_i/f)\colon f$ is a function from $\bigcup\{\overline{v}_i\colon i\in I\}$ into $\bigcup\{C_j\colon j\leqslant n\},\ i\in I\}\in S$;

C5) If $\{-\forall\overline{v}_iF_i\colon i\in I\}\subset s\in S$ and $|I|<k$ and there is $m\in\omega$ such that for all $i\in I$ we have that $0<|\overline{v}_i|\leqslant k_m$ and s is a set of sentences in the language L_{n-1} for some $n\in\omega$ with $m\leqslant n$, then for all $1-1$ functions f from $\bigcup\{\overline{v}_i\colon i\in I\}$ into C_n we have that $s\bigcup\{-F_i(\overline{v}_i/f)\colon i\in I\}\in S$;

C6) *a*) If $\{c_i=d_i\colon i\in I\}\subset s\in S$ and $|I|<k$ and c_i and d_i are constants in some $\bigcup\{C_j\colon j\leqslant n\}$, then $s\cup\{d_i=c_i\colon i\in I\}\in S$;

b) If $\{Z_i(c_i),\ c_i=d_i\colon i\in I\}\subset s\in S$ where $Z_i(c_i)$ is an atomic or negated atomic sentence and c_i, d_i are constants in some $\cup\{C_j\colon j\leqslant n\}$ and $|I|<k$, then $s\cup\{Z_i(d_i)\colon i\in I\}\in S$.

We are assuming, without loss of generality, that our sentences and sets of sentences are such that no variable occur in more than one set of variables immediatly after a quantifier. For the notation and for the notion of ω-satisfiability for sentences we refer also to [2].

The Model Existence Theorem for a consistency property states that any set of sentences in a consistency property is ω-satisfiable.

To state the interpolation theorem in terms of ω-satisfiability instead of ω-validity, we have to consider the negation of the sentence mentioned in the statement of the theorem that can now be rewritten as follows: If there is no interpolant for $F_1\to F_2$ then the sentence $-(F_1\to F_2)$ is ω-satisfiable.

To deal with consistency properties, we have to consider sets of sentences and not just a single sentence as $F_1\to F_2$. We should then extend the notion of interpolant to sets of sentences. The adequate definition turns out to be the following one:

Let $\overline{F}$ be a set of sentences and $(\overline{F}_1,\overline{F}_2)$ a partition of $\overline{F}$, then we say that the sentence F is an interpolant for $\overline{F}$ with respect to the partition $(\overline{F}_1,\overline{F}_2)$ if the extralogical symbols in F occur both in $\overline{F}_1$ and in $\overline{F}_2$ and $\overline{F}_1\cup\{-F\}$ and $\overline{F}_2\cup\{F\}$ are not satisfiable.

To obtain Karp's interpolation theorem we let $\overline{F}$ be $\{F_1, -F_2\}$ which is ω-satisfiable iff so is $-(F_1 \to F_2)$, and we let $\overline{F}_1$ be $\{F_1\}$, so that $\{F_1, -F\}$ is not satisfiable iff $F_1 \to F$ is valid, and we let $\overline{F}_2$ be $\{-F_2\}$ so that $\{-F_2, F\}$ is not satisfiable iff $F \to F_2$ is valid.

To prove Karp's interpolation theorem it now amounts to prove that:

The sets $\overline{F}$ of sentences such that there is a partition $(\overline{F}_1, \overline{F}_2)$ of $\overline{F}$ without interpolant for $\overline{F}$ with respect to $(\overline{F}_1, \overline{F}_2)$, are a consistency property.

Thus $\overline{F}$ becomes ω-satisfiable, due to the model existence theorem, and Karp's interpolation theorem is proved.

The improvement of Karp's interpolation theorem corresponds to consider an «ω-interpolant» for $\overline{F}$ with respect to $(\overline{F}_1, \overline{F}_2)$, that is a sentence F for which, besides the usual restriction on the extra-logical symbols, the following holds: $\overline{F}_1 \cup \{-F\}$ and $\overline{F}_2 \cup \{F\}$ are not ω-satisfiable.

The new result would be:

The set Γ of sets $\overline{F}$ of sentences such that there is a partition $(\overline{F}_1, \overline{F}_2)$ of $\overline{F}$ without ω-interpolant for $\overline{F}$ with respect to $(\overline{F}_1, \overline{F}_2)$, is a consistency property.

This result fails, even if it almost goes through.

By this we mean that if we try to prove each one of the clauses for Γ to be a consistency property, we easily succed to prove almost all of them (C0)- C1)- C3)- C3)- C4)- C6), i.e. the clauses for atomic sentences, negation, conjunction, disjunction, universal quantification, identity).

The only clause that presents a problem (that turns out to be unsolvable within the present notion of consistency property) is C5), the existential quantification clause.

Indeed to prove C5) for Γ to be a consistency property we should argue as follows.

Let $\{-\forall \overline{v}_i F_i \colon i \in I\} \subset \overline{F} \in \Gamma$ and let $|I| < k$ and suppose that there is $m \in \omega$ such that for all $i \in I$ we have that $0 < |\overline{v}_i| < k_m$. Let $(\overline{F}_1, \overline{F}_2)$ be a partition of $\overline{F}$ and let $I_1 = \{i \colon i \in I$ and $-\forall \overline{v}_i F_i \in \overline{F}_1\}$, $I_2 = I - I_1$. Suppose that there is a $1-1$ function f from $\bigcup \{\overline{v}_i \in I\}$ in C_n such that $\overline{F}' = \overline{F} \cup \{-F_i(\overline{v}_i/f) \colon i \in I\} \notin \Gamma$, i.e. for all partitions of $\overline{F}'$ there is an ω-interpolant. Let $\overline{F}'_1 = \overline{F}_1 \cup \{-F_i(\overline{v}_i/f) \colon i \in I_1\}$ and $\overline{F}'_2 = \overline{F}_2 \cup \{-F_i(\overline{v}_i/f) \colon i \in I_2\}$. Our goal would be to show that then there is also an ω-interpolant for $\overline{F}$ with respect to $(\overline{F}_1, \overline{F}_2)$, which would contradict the assumptions and prove clause C5).

Let F be an ω-interpolant for $\bar{F}'$ with respect to $(\bar{F}_1, \bar{F}_2)$.

Since f is $1-1$, the extralogical symbols that occur both in $\bar{F}'_1$ and in $\bar{F}'_2$ are the same as those occurring both in $\bar{F}_1$ and in $\bar{F}_2$. Therefore, as far as the condition on the symbols is concerned, F could be an ω-interpolant for $\bar{F}$ with respect to $(\bar{F}_1, \bar{F}_2)$.

To complete the proof we should show that from the facts that $\bar{F}'_1 \cup \{-F\}$ and $\bar{F}'_2 \cup \{F\}$ are not ω-satisfiable, it follows that $\bar{F}_1 \cup \cup \{-F\}$ and $\bar{F}_2 \cup \{F\}$ are not ω-satisfiable.

This could be done if we had the following lemma:

If $\{-\forall \bar{v}_j F_j\colon j \in J\} \subset s$ and s is ω-satisfiable then also $s \cup \cup \{-F_j(\bar{v}_j/f)\colon j \in J\}$ is ω-satisfiable, where f is a $1-1$ function from $\bigcup \{\bar{v}_j\colon j \in J\}$ to a set of individual constants not occuring in s.

Since s is ω-satisfiable, there is an ω-chain of models $\bar{M}$ such that $\bar{M} \models^{\omega} s$, and in particular for each $j \in J$ $\bar{M} \models^{\omega} -\forall \bar{v}_j F_j$. This implies that there is a bounded assignment $\bar{b}_j$ to the variables in $\bar{v}_j$ such that $\bar{M}, \bar{b}_j \models^{\omega} -F_j$ and also $\bar{M}, \bar{b}_j(f^{-1}) \models^{\omega} s \cup \{-F_j(\bar{v}_j/f)\}$.

If J is finite we are done for $\bar{b} = \cup \{\bar{b}_j\colon j \in J\}$ is still a bounded assignment and we would have: $\bar{M}, \bar{b}(f^{-1}) \models^{\omega} s \cup \{-F_j(\bar{v}_j/f)\colon j \in J\}$. But if J is not finite, $\bar{b}$ needs not to be a bounded assignment, and the lemma fails as we shall see with a counterexample.

This is indeed the key point, the failure of which will cause the impossibility to improve Karp's theorem using the present notion of consistency property as we shall see with a further counterexample.

Now let us return our attention to the first counterexample.

We need an infinite set s of existential sentences $\exists \bar{v}_j F_j$ which are ω-satisfiable but not satisfiable (for otherwise we know that also $s \cup \{F_j(\bar{v}_j/f)\colon j \in J\}$ is again satisfiable and hence ω-satisfiable) such that $s \cup \{F_j(\bar{v}_j/f)\colon j \in J\}$ is not ω-satisfiable.

From now on, let us work in a language $L_{k,k}$ whose only extralogical symbol is P, a binary predicate, and $k = \beth_\infty$ the first strong limit cardinal of cofinality ω after ω.

For each $j \in \omega$ we let $\exists \bar{v}_j F_j$ be:

$$\begin{aligned}
&\exists \{v_i\colon i \leqslant j\}\Big[(\forall v_{j+1} - P(v_{j+1}, v_{j+1}))\ \& \\
&\quad \&(\forall v_{j+2} \forall v_{j+3}(P(v_{j+2}, v_{j+3}) \vee P(v_{j+3}, v_{j+2}) \vee v_{j+2} = v_{j+3}))\ \& \\
&\quad \&(\forall v_{j+4} \forall v_{j+5} \forall v_{j+6}\big((P(v_{j+4}, v_{j+5})\ \&\ P(v_{j+5}, v_{j+6})) \to P(v_{j+4}, v_{j+6})\big))\ \& \\
&\quad \&(\forall v_{j+7} \exists v_{j+8} P(v_{j+7}, v_{j+8}))\ \&\ (\forall \{v_{j+9+n}\colon n \in \omega\} \\
&\qquad\qquad - \&\ \{P(v_{j+9+n}, v_{j+10+n})\colon n \in \omega\})\ \&\ (\&\ \{P(v_i, v_{ji+1})\colon i < j\})\Big].
\end{aligned}$$

These sentences are ω-satisfied in the ω-chain of models $\bar{M} = \langle M_i : i \in \omega \rangle$ where M_i is the structure with the natural numbers less or equal to i as universe and P is interpreted in the strict total order relation R on the universes.

Indeed $\forall v_{j+1} — P(v_{j+1}, v_{j+1})$ requires R to be antireflexive,

$$\forall v_{j+2} \forall v_{j+3} (P(v_{j+2}, v_{j+3}) \vee P(v_{j+3}, v_{j+2}) \vee v_{j+2} = v_{j+3})$$

requires R to be tricotomic,

$$\forall v_{j+4} \forall v_{j+5} \forall v_{j+6} ((P(v_{j+4}, v_{j+5}) \,\&\, P(v_{j+5}, v_{j+6})) \to P(v_{j+4}, v_{j+6}))$$

requires R to be transitive,

$$\forall v_{j+7} \exists v_{j+8} P(v_{j+7}, v_{j+8})$$

requires R to admit an infinite ascending chain,

$$\exists \{v_i : i \leqslant j\} \,\&\, \{P(v_i, v_{i+1}) : i < j\}$$

requires the domain of R to contain at least j elements, and

$$\forall \{v_{j+9+n} : n \in \omega\} — \& \{P(v_{j+9+n}, v_{j+10+n}) : n \in \omega\}$$

makes the formula ω-satisfiable and not satisfiable for it does not contradict the fact that R admits an infinite ascending chain only if we consider bounded assignments.

So $\bar{M} \models^{\omega} \{\exists \bar{v}_j F_j : j \in \omega\}$ and also $\bar{M}, \bar{b}_j \models^{\omega} F_j(\bar{v}_j)$ where $\bar{b}_j$ maps $v_i \in \bar{v}_j$ into i.

But there is no bounded assignment $\bar{b}$ such that $\bar{M}, \bar{b} \models^{\omega} \{F_j(\bar{v}_j) : j \in \omega\}$ for in this case both the set $\{P(v_i, v_{i+1}) : i \in \omega\}$ and the formula $— \& \{P(v_{j+9+n}, v_{j+10+n}) : n \in \omega\}$ should be ω-satisfiable, which is impossible in any ω-chain of models under any bounded assignment.

This counterexample shows that the proposed lemma fails.

Now we turn to exibit a counterexample to show that the set Γ of sets $\bar{F}$ of sentences such that there is a partition $(\bar{F}_1, \bar{F}_2)$ of $\bar{F}$ without ω-interpolant for $\bar{F}$ with respect to $(\bar{F}_1, \bar{F}_2)$ is not a consistency property.

Let $\bar{F} = \{\exists \bar{v}_j F_j : j \in \omega\}$ where $\exists \bar{v}_j F_j$ is as in the previous counterexample. Let $\bar{F}_1 = \emptyset$ and $\bar{F}_2 = \bar{F}$.

CLAIM. There is no ω-interpolant for $\overline{F}$ with respect to $(\overline{F}_1, \overline{F}_2)$.

Indeed if F were an ω-interpolant for $\overline{F}$ with respect to $(\overline{F}_1, \overline{F}_2)$ then no extralogical symbols should occur in F since no one occurs both in $\overline{F}_1$ and in $\overline{F}_2$. Furthermore F should be ω-valid since $\overline{F}_1 \cup \cup \{-F\} = -F$ should be not ω-satisfiable. Therefore $\overline{F}_2 \cup \{F\}$ is not ω-satisfiable iff $\overline{F}_2$ is not ω-satisfiable. But $\overline{F}_2 = \overline{F}$ which is ω-satisfiable, and this contradiction proves the claim.

Hence $\overline{F} \in \Gamma$.

For Γ to be a consistency property with respect to the sets o constants $\{C_i : i \in \omega\}$, we should have that also $\overline{F}' = \overline{F} \cup \{F_j(\overline{v}_j/f) : j \in \omega\} \in \Gamma$, where f is a $1-1$ function from $\cup\{\overline{v}_j : j \in \omega\}$ to C_1.

Let $(\overline{F}'_1, \overline{F}'_2)$ be any partition of $\overline{F}'$.

We have to consider three cases.

a) Either there is a maximum index j_1 of the sentences $F_j(\overline{v}_j/f)$ in $\overline{F}'_1$, or there is no such sentence in $\overline{F}'_1$.

b) Either there is a maximum index j_2 of the sentences $F_j(\overline{v}_j/f)$ in $\overline{F}'_2$, or there is no such sentence in $\overline{F}'_2$.

c) None of the previous cases.

In case *a*) we know that $\overline{F}'_2$ is not ω-satisfiable, so if we take F to be $\forall x(x = x)$, F would be an ω-interpolant for it does not contain any extralogical symbol and both $\overline{F}'_1 \cup \{-F\}$ and $\overline{F}'_2 \cup \{F\}$ are not ω-satisfiable.

Case *b*) is analogous: this time $\overline{F}'_1$ is not satisfiable so that $-\forall x (x = x)$ would be an ω-interpolant.

Case *c*) is even easier for in this case both $\overline{F}'_1$ and $\overline{F}'_2$ are not ω-satisfiable and any sentence without extralogical symbols is an ω-interpolant.

Thus in any case there is an ω-interpolant for $\overline{F}'$ with respect to any partition $(\overline{F}'_1, \overline{F}'_2)$ and Γ is not a consistency property.

To conclude let us remark that the shortcoming of the current notion of consistency property is not just in relation to the proof of an improvement of Karp's interpolation theorem. The counterexamples that we have exhibited are based on a set s of ω-satisfiable existential sentences that becomes not ω-satisfiable once the instances of the existential sentences are added to it, i.e. such that $s \cup \{F(\overline{v}_F/f) : \exists \overline{v}_F F \in s\}$, where f is a $1-1$ function from the variables in $\cup \{\overline{v}_F : \exists \overline{v}_F F \in s\}$ into some C_n, is not ω-satisfiable.

Thus what it turns out to be of relevance is the fact that the set of the sets of sentences in some L_n which are ω-satisfiable is not a consistency property, even though in the opposite direction it holds that any set of sentences in a consistency property is ω-satisfiable.

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