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Endomorphism Rings of Valued Vector Spaces.

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The study of valued vector spaces as a tool to obtain information about the socles of abelian p-groups has proved extremely useful. We continue this approach with the study of the endomorphism rings of valued vector spaces. As we shall see, they resemble—as expected—endomorphism rings of abelian p-groups.

We discuss two aspects of the endomorphism rings.

Firstly, we investigate the relationship between two valued vector spaces whose endomorphism rings are isomorphic. Our theorem is analogous to the Baer-Kaplansky theorem on abelian p-groups, [3], but in the present case isomorphism is replaced by a weaker notion: a vector space isomorphism which does not necessarily preserve valuation, only inequalities between values.

Secondly, we give a ring-theoretical characterization of endomorphism rings of valued vector spaces in general. This result is motivated by Liebert's theorem [4] on the endomorphism rings of separable abelian p-groups, and its analogue for homogeneous separable torsion-free groups by Metelli and Salce [5]. In our proof, we use ideas from both of these papers.

We consider vector spaces V over a fixed field Φ which are equipped with valuations into a totally ordered set (or class) Γ. We assume that Γ has a maximum element ∞ and that every non-empty subset of Γ has a supremum. A valuation of V is a function $v\colon V \to \Gamma$ such

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that

(i) $v(a) = \infty$ exactly if $a = 0$;

(ii) $v(\alpha a) = v(a)$ for all $\alpha \neq 0$ in Φ and all $a \in V$;

(iii) $v(a + b) \geqq \min\big(v(a), v(b)\big)$ for all $a, b \in V$.

A morphism between two valued vector spaces is a Φ-linear map which does not decrease values. We write morphisms on the left. The set of all morphisms from V to V forms a Φ-algebra, the endomorphism ring of V, which we denote End V. We shall regard End V as an abstract Φ-algebra, without valuation.

For unexplained terminology and elementary properties of valued vector spaces we refer to [1] and [2].

1. Valued vector spaces with isomorphic endomorphism rings.

Let V and W be valued vector spaces. A map $\varphi\colon V \to W$ is called a *pseudo-isomorphism* if

(a) it is a vector space isomorphism;

(b) $v(a) \leqq v(b)$ for $a, b \in V$ if and only if $v(\varphi a) \leqq v(\varphi b)$.

Notice that if $\xi \in \mathrm{End}\, V$, then $\varphi\xi\varphi^{-1} \in \mathrm{End}\, W$, and the correspondence

$$\xi \to \varphi\xi\varphi^{-1}$$

gives rise to a ring isomorphism between End V and End W. We wish to show that, conversely, every isomorphism between the endomorphism rings is induced by a pseudo-isomorphism of the valued vector spaces.

THEOREM 1. *Let V and W be valued vector spaces and ψ*: End $V \to$ End W *an algebra isomorphism. Then there exists a pseudo-isomorphism $\varphi\colon V \to W$ such that*

$$\psi\xi = \varphi\xi\varphi^{-1} \quad \textit{for all } \xi \in \mathrm{End}\, V\,.$$

For brevity, write $E = \mathrm{End}\, V$ and $E^* = \mathrm{End}\, W$, and set $\xi^* = \psi\xi$ for $\xi \in \mathrm{End}\, V$.

If ε is a primitive idempotent in E, then ε^* is one in E^*. As εV is 1-dimensional, its non-zero elements have the same value, so we can preorder the set of primitive idempotents of E by setting

$$\varepsilon \leqq \varrho \quad \text{to mean} \quad v(\varepsilon V) \leqq v(\varrho V)\,.$$

Notice that $\varepsilon \leqq \varrho$ if and only if $\varrho E \varepsilon \neq 0$. Consequently, ψ preserves the preorder of primitive idempotents, i.e. $\varepsilon \leqq \varrho$ exactly if $\varepsilon^* \leqq \varrho^*$.

Let us select in the support of V, (supp $V = \{v(a) | a \neq 0 \text{ in } V\}$), a strictly descending chain $\{v(a_\sigma)\}_{\sigma<\lambda}$ ($a_\sigma \in V$) which is inversely well-ordered in the ordering of Γ, and which is cofinal in supp V in the sense that for every $\gamma \in$ supp V there is a $\sigma < \lambda$ such that $v(a_\sigma) \leqq \gamma$. As Φa_σ is a summand of V, some primitive idempotent $\varepsilon_\sigma \in E$ satisfies $\varepsilon_\sigma V = \Phi a_\sigma$; moreover, we can choose these ε_σ ($\sigma < \lambda$) to be pairwise orthogonal. It is evident that there are endomorphisms $\xi_{\sigma\varrho}$ of V (for all $\sigma < \varrho < \lambda$) such that

$$\xi_{\sigma\varrho} a_\varrho = a_\sigma \quad \text{and} \quad \xi_{\sigma\varrho} a_\tau = 0 \quad \text{for } \tau \neq \varrho\,.$$

The endomorphisms ε_σ and $\xi_{\sigma\varrho}$ satisfy:

(i) ε_σ are pairwise orthogonal primitive idempotents;

(ii) $\xi_{\sigma\varrho}\varepsilon_\varrho = \xi_{\sigma\varrho} = \varepsilon_\sigma \xi_{\sigma\varrho}$ for all $\sigma < \varrho$;

(iii) $\xi_{\sigma\varrho}\xi_{\varrho\pi} = \xi_{\sigma\pi}$ for all $\sigma < \varrho < \pi$.

It is clear that the endomorphisms ε_ϱ^* and $\xi_{\sigma\varrho}^*$ in E^* satisfy the same conditions. By primitivity, $\varepsilon_\varrho^* W$ is 1-dimensional, and (ii) implies that $\xi_{\sigma\varrho}^*$ will map $\varepsilon_\varrho^* W$ onto $\varepsilon_\sigma^* W$. We wish to show that, for every $\sigma < \lambda$, we can select a $c_\sigma \in \varepsilon_\sigma^* W$ such that $\xi_{\sigma\varrho}^* c_\varrho = c_\sigma$ for all $\sigma < \varrho$. Suppose that these c_σ have been so chosen for every $\sigma < \tau$. If $\tau - 1$ exists, choose c_τ so as to satisfy $\xi_{\tau-1,\tau}^* c_\tau = c_{\tau-1}$. If τ is a limit ordinal $< \lambda$, choose c_τ to satisfy $\xi_{\sigma\tau}^* c_\tau = c_\sigma$ for some $\sigma < \tau$. Then, for $\sigma < \varrho < \tau$, $\xi_{\varrho\tau}^* c_\tau = \alpha c_\varrho$ for some $\alpha \in \Phi$, and (iii) guarantees that $\alpha = 1$.

We are now ready to define a map $\varphi \colon V \to W$. Given $0 \neq a \in V$, pick an a_σ and an $\eta \in E$ such that $a = \eta a_\sigma$ (our choice of the a_σ ensures that this is possible), and let

$$\varphi \colon a \to \eta^* c_\sigma\,.$$

This φ is well-defined, for if $a = \xi a_\varrho$ for some $\xi \in E$ and $\varrho > \sigma$, then $\xi_{\sigma\varrho} a_\varrho = a_\sigma$ implies $(\xi - \eta\xi_{\sigma\varrho})\xi_\varrho = 0$ whence $(\xi^* - \eta^* \xi^*_{\sigma\varrho})\xi^*_\varrho = 0$ and $\xi^* c_\varrho = \eta^* c_\sigma$. It is straightforward to check that φ is a vector space isomorphism between V and W. To show that it is a pseudo-isomorphism, note that if $v(a) \leqq v(b)$ for $a, b \in V$, then $\chi a = b$ for some $\chi \in E$. Write $a = \xi \alpha_\sigma$, $b = \eta a_\sigma$; then $(\chi\xi - \eta)\varepsilon_\sigma = 0$ which implies $(\chi^* \xi^* - \eta^*)\varepsilon^*_\sigma = 0$. Hence $\chi^*(\varphi a) = \varphi b$, and thus $v(\varphi a) \leqq v(\varphi b)$, indeed. Finally, every $c \in W$ is of the form $c = \eta^* c_\sigma$ for some $\eta^* \in E^*$ and σ. Consequently,

$$\xi^* c = \xi^* \eta^* c_\sigma = \varphi(\xi\eta a_\sigma) = \varphi\xi\varphi^{-1}\varphi(\eta a_\sigma) = \varphi\xi\varphi^{-1} c\,,$$

completing the proof of the theorem.

It is straightforward to check that if φ is a pseudo-automorphism of V, then

$$f\colon v(a) \to v(\varphi a)$$

is an order-automorphism of supp V. In certain cases, e.g. if supp V is well-ordered, the identity map is the only order-automorphism. In these cases, a pseudo-automorphism of V is necessarily an automorphism; therefore Theorem 1 implies that then all automorphisms of End V are inner. (This is well known for ordinary vector spaces V where supp V may be viewed as a singleton.)

2. A characterization of endomorphism rings.

Our next purpose is to characterize intrinsically those rings which are endomorphism rings of valued vector spaces.

Let $E = \text{End}\, V$ be the endomorphism ring of a valued vector space V. By I we shall denote the set of primitive idempotents in E, and by E_0 the left ideal of E generated by I. First we show that the following hold:

(i) For all $\eta \in E$ and $\varrho \in I$, there is a $\sigma \in I$ such that $\sigma\eta\varrho = \eta\varrho$.

(ii) For all $\varrho, \sigma \in I$, either $\varrho E \sigma \neq 0$ or $\sigma E \varrho \neq 0$; whichever is not 0 is a 1-dimensional subspace of E.

(iii) If, for some $\xi \in E$, $\varrho, \sigma \in I$ both $\xi E \varrho \neq 0$ and $\varrho E \sigma \neq 0$, then their product $\xi E \varrho E \sigma \neq 0$ either.

(iv) E is complete and Hausdorff in the topology where a subbase of neighbourhoods of 0 consists of left annihilators of elements of I, and E_0 is dense in E in this topology.

In fact, (i) follows from the fact that $\eta\varrho V$ is an at most 1-dimensional subspace of V, so it is an indecomposable summand of V.

To prove (ii), observe that $\varrho E\sigma V = 0$ if and only if σV cannot be mapped onto ϱV. This is the case exactly if $v(\varrho) < v(\sigma)$. If $v(\varrho) \geqq v(\sigma)$, then $\varrho E\sigma V = \varrho V$.

As $\varrho E\sigma V = \varrho V$ and $\xi E\varrho V \neq 0$ implies $\xi E\varrho E\sigma \neq 0$, (iii) is clear.

Finally, to prove (iv), note that all endomorphism rings are complete Hausdorff in the finite topology. In the present case, $\{\eta \in E | \eta a = 0\} = E(1-\sigma)$ for any $a \in V$ where σ is the projection to the first summand in $V = \Phi a \oplus V'$ (for a suitable subspace V' of V). In other words, the finite topology of E has a subbase of neighbourhoods of 0 consisting of left annihilators $\operatorname{Ann}\varrho$ of $\varrho \in I$ in E. Given $\eta \in E$ and $a_1, \ldots, a_n \in V$, the subspace W spanned by the $a_1, \ldots, a_n$ is a summand of V. If π is a projection $V \to W$ and $U = \{\eta \in E | \eta a_1 = \ldots = \eta a_n = 0\}$ is the corresponding neighbourhood of 0, then $1 - \pi \in U$. As U is a left ideal of E, $\eta - \eta\pi \in U$; here $\eta\pi \in E_0$ since π is the sum of n primitive idempotents. The density of E_0 in E is now clear.

What we have said so far establishes the necessity part of the following theorem.

THEOREM 2. *Let E be a Φ-algebra with* 1. *E is the endomorphism ring of a valued vector space V if and only if E satisfies conditions* (i)-(iv).

In order to prove sufficiency, let E satisfy (i)-(iv). The set I of primitive idempotents, which is not empty by (iv), can be preordered by setting, for $\varrho, \sigma \in I$,

$$\varrho \leqq \sigma \quad \text{if and only if} \quad \sigma E\varrho \neq 0 .$$

This relation $\leqq$ is trivially reflexive (as $\sigma \in \sigma E\sigma$). Its transitivity follows from (iii), while (ii) implies that it is a total preorder. The classes $[\varrho]$ under the equivalence $\varrho \sim \sigma$ exactly if $\varrho \leqq \sigma$ and $\sigma \leqq \varrho$ form a totally ordered set Γ, and we define a function $f: I \to \Gamma$ by $f(\varrho) = [\varrho]$. We can now adjoin to Γ new elements, so that Γ will contain ∞ and suprema for all of its non-empty subsets.

We give a detailed proof in case Γ has a smallest element, and then indicate how this has to be modified in the general case.

Let $[\varepsilon]$ be the smallest element in Γ. Define

$$V = E\varepsilon .$$

For $\varrho \in I$, ϱV is never 0, because (ii) and the minimality of $[\varepsilon]$ imply that $\varrho V = \varrho E\varepsilon$ is 1-dimensional. We assign the value $f(\varrho)$ to the elements of ϱV:

$$v(\varrho a) = f(\varrho) \qquad (0 \neq a \in V) .$$

This definition is unambiguous, for if $\varrho\xi\varepsilon = a = \sigma\eta\varepsilon$ for some $\varrho, \sigma \in I$ and $\xi, \eta \in E$, then both $\varrho a = a$ and $\sigma a = a$, thus $0 \neq \varrho\sigma \in \varrho E\sigma$ and $0 \neq \sigma\varrho \in \sigma E\varrho$ imply $[\varrho] = [\sigma]$. Furthermore, (i) guarantees that we can assign a value to every $a \in V$.

In order to ascertain that $v\colon V \to \Gamma$ yields a valuation of the vector space V, we have to verify that $v(a + b) \geqq \min\big(v(a), v(b)\big)$ for all $a, b \in V$. Set $a = \xi\varepsilon$, $b = \eta\varepsilon$ $(\xi, \eta \in E)$; by (i) $\varrho a = a$ and $\sigma b = b$ for some $\varrho, \sigma \in I$. Again by (i), $\tau(a + b) = a + b$ for some $\tau \in I$; therefore either $\tau a \neq 0$ or $\tau b \neq 0$ (or else $a + b = 0$ in which case there is nothing to prove). Hence either $\tau\varrho \neq 0$ or $\tau\sigma \neq 0$, that is either $f(\varrho) \leqq f(\tau)$ or $f(\sigma) \leqq f(\tau)$, as desired.

Every $\xi \in E$ induces an endomorphism $\bar{\xi}$ of V via

$$\bar{\xi}a = \bar{\xi}(\eta\varepsilon) = (\xi\eta)\varepsilon$$

where $a = \eta\varepsilon$ $(\eta \in E)$. To see that $\bar{\xi}$ does not decrease values, suppose that $\varrho, \sigma \in I$ satisfy $\varrho(\eta\varepsilon) = \eta\varepsilon$ and $\sigma(\xi\eta\varepsilon) = \xi\eta\varepsilon$ (cf. (i)). It is enough to consider the case $\xi\eta\varepsilon \neq 0$; then $\xi\eta\varepsilon = \sigma\xi\eta\varepsilon = \sigma\xi\varrho\eta\varepsilon$ implies $\sigma\xi\varrho \neq 0$ whence $f(\varrho) \leqq f(\sigma)$, i.e. $v(\eta\varepsilon) \leqq v(\xi\eta\varepsilon)$. If $\xi \in E$ induces the zero endomorphism of V, i.e. if $\bar{\xi}V = \xi E\varepsilon = 0$, then in view of $\varrho E\varepsilon \neq 0$ for all $\varrho \in I$ we conclude from (iii) that $\xi E\varrho = 0$ for all $\varrho \in I$. We infer that $\xi \in \operatorname{Ann}\varrho$ for all $\varrho \in I$, so the Hausdorff property in (iv) guarantees that $\xi = 0$. Consequently, E can be viewed as a subring of $\operatorname{End} V = F$.

For every $\varrho \in I$, $\varrho E\varepsilon = \varrho V$ is 1-dimensional in view of (ii). Thus we can write

$$\varrho V = \Phi x \qquad \text{for some } x = \xi\varepsilon \in V$$

$(\xi \in E)$. Given any $\mu \in F$, we then have $\mu x = \chi\varepsilon$ for some $\chi \in E$. By (i), some $\sigma \in I$ satisfies $\sigma\chi\varepsilon = \chi\varepsilon$, where $f(\sigma) \geqq f(\varrho)$ since μ does not decrease values. Hence $\sigma E\varrho \neq 0$ which, along with $\varrho E\varepsilon \neq 0$ implies $\sigma E\varrho \cdot \varrho E\varepsilon \neq 0$, as stipulated by (iii). Hence $\sigma E\varrho \cdot \varrho E\varepsilon = \Phi(\chi\varepsilon)$, and there is a $\xi \in E$ such that $\xi x = \mu x$. For any $y = \eta\varepsilon \in V$ we now have

$$(\mu\bar{\varrho})y = (\mu\bar{\varrho})\eta\varepsilon = \mu(\varrho\eta\varepsilon) = \mu(\alpha x) =$$

$$= \xi(\alpha x) = \xi(\varrho\eta\varepsilon) = (\xi\varrho)(\eta\varepsilon) = (\overline{\xi\varrho})y$$

(where we set $\varrho\eta\varepsilon = \alpha x$ for some $\alpha \in \Phi$), showing that $\mu\bar{\varrho} = \overline{\xi\varrho}$. Hence it is evident that

$$F\varrho = E\varrho \quad \text{for every } \varrho \in I .$$

This implies at once

$$FE_0 = E_0 .$$

Next, let π be a primitive idempotent in F. Then $\pi V = \Phi z$ for a suitable $z \in V$. By (i), $\sigma z = z$ for some $\sigma \in I$ whence we obtain

$$\mathrm{Ann}_F \pi = \mathrm{Ann}_F \sigma .$$

We conclude that the finite topology of F has a subbase consisting of left annihilators (in F) of the elements of I. It also follows that the finite topology of F induces the given topology of E.

We proceed to verify that E_0 is dense in F. Let $\mu \in F$ and $U = \cap \mathrm{Ann}_F \varrho_j$ ($\varrho_j \in I$; $j = 1, \ldots, n$) a neighbourhood of 0 in F. Since by (iv) E_0 is dense in E, there is an $\eta \in E_0$ such that $1 - \eta \in U \cap E$. Manifestly, U is a left ideal of F, so $\mu(1 - \eta) \in U$, i.e. $\mu - \mu\eta \in U$. Here $\mu\eta \in E_0$ as is evident from $FE_0 = E_0$, so E_0 is dense in F, indeed.

To conclude this proof, note that $E_0 \leqq E \leqq F$ and E_0 is dense in F. By (iv), E is complete, so necessarily $E = F$, as we wished to prove.

If Γ fails to have a smallest element, then we proceed as follows:

Just as in the proof of Theorem 1, we choose a decreasing chain of primitive idempotents which is cofinal in I. As in Liebert's theorem on the endomorphism rings of separable abelian p-groups [4], we use the fact that $\varrho E\sigma$ is 1-dimensional if $\sigma < \varrho$ in our chain to define a direct system of left ideals of the form $E\sigma$, and we take V to be the direct limit of this system.

Now the Hausdorff property of the topology on E ensures that E is canonically embedded as a subring of End $V = F$. In the proof above that $F\varrho = E\varrho$ for every $\varrho \in I$, we replace the minimal idempotent ε by some σ in our chain and use the canonical embedding of E_σ in F to conclude that $FE_0 = E_0$.

Finally, the proof that the finite topology on F induces the given topology on E, and that E_0 is dense in F follows exactly as in the proof above.

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