PIERMARCO CANNARSA

On a maximum principle for elliptic systems with constant coefficients

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On a Maximum Principle for Elliptic Systems with Constant Coefficients.

PIERMARCO CANNARSA (*)

1. Introduction.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( N \) be a positive integer. Let \( (\cdot, \cdot)_N, \| \cdot \|_N \) be the scalar product and the norm in \( \mathbb{R}^N \). We set 
\[
D_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n.
\]

Let \( H^1(\Omega, \mathbb{R}^N) \) be the usual Sobolev space of vectors \( u: \Omega \to \mathbb{R}^N \) with norm
\[
\| u \|_{H^1(\Omega, \mathbb{R}^N)} = \left( \int_{\Omega} \| u \|^2 \, dx + \int_{\Omega} \sum_{i=1}^n \| D_i u \|^2 \, dx \right)^{\frac{1}{2}}
\]
and let \( H^1_0(\Omega, \mathbb{R}^N) \) be the closure of \( C_0^\infty(\Omega, \mathbb{R}^N) \) with respect to norm (1.1).

Let \( L^{2,\lambda}(\Omega, \mathbb{R}^N), \, 0 < \lambda < n, \) be the Banach space defined as follows
\[
L^{2,\lambda}(\Omega, \mathbb{R}^N) = \left\{ u \in L^2(\Omega, \mathbb{R}^N) : \| u \|_{L^{2,\lambda}(\Omega, \mathbb{R}^N)} = \sup_{q > 0} q^{-\lambda} \int_{B(x,q) \cap \Omega} \| u \|^2 \, dy < +\infty \right\}
\]
(here \( B(x, q) = \{ y \in \mathbb{R}^n : \| x - y \| < q \} \)) and
\[
H^{1,\lambda}(\Omega, \mathbb{R}^N) = \{ u \in H^1(\Omega, \mathbb{R}^N) : D_i u \in L^{2,\lambda}(\Omega, \mathbb{R}^N), 1 \leq i \leq n \}.
\]


(1) We shall often omit the subscript \( N \) and write simply \( (\cdot), \| \cdot \| \).
$H^{1,\alpha}(\Omega, \mathbb{R}^n)$ is a Banach space with norm

$$
\|u\|_{H^{1,\alpha}(\Omega, \mathbb{R}^n)} = \|u\|_{L^2(\Omega, \mathbb{R}^n)} + \sum_{i=1}^{n} \|D_i u\|_{L^{\alpha}(\Omega, \mathbb{R}^n)}.
$$

Let $A_{ij}(x)$ $(i, j = 1, \ldots, n)$ be $N \times N$ matrices satisfying the ellipticity condition

\begin{equation}
\sum_{i,j=1}^{n} \xi_i^T A_{ij}(x) \xi_j \eta_x \geq \nu \|\xi\|_{\mathbb{R}^n}^2 \|\eta\|_{\mathbb{R}^n}^2, \quad \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n, \forall \eta \in \mathbb{R}^n.
\end{equation}

The following regularity theorem is proved in [1] (2)

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a $C^1$ (3) open set, $u \in H^{1,\alpha}(\Omega, \mathbb{R}^n)$, $f_i \in L^{2,\lambda}(\Omega, \mathbb{R}^n)$ $(0 < \lambda < n, i = 1, \ldots, n)$ and let $A_{ij}$ be continuous in $\bar{\Omega}$ and satisfy (1.2). Then, if $v$ is the solution of Dirichlet problem

\begin{equation}
\left\{
\begin{array}{l}
v - u \in H^1_0(\Omega, \mathbb{R}^n), \\
\sum_{i,j=1}^{n} (A_{ij}(x) D_i v) D_j \varphi \, dx = - \sum_{i=1}^{n} (f_i D_i \varphi) \, dx \quad \forall \varphi \in H^1_0(\Omega, \mathbb{R}^n),
\end{array}
\right.
\end{equation}

$v$ belongs to $H^{1,\alpha}(\Omega, \mathbb{R}^n)$ and

\begin{equation}
\|v\|_{H^{1,\alpha}(\Omega, \mathbb{R}^n)} \leq C_1 \left( \sum_{i=1}^{n} \|f_i\|_{L^{2,\lambda}(\Omega, \mathbb{R}^n)} + \|u\|_{H^{1,\alpha}(\Omega, \mathbb{R}^n)} \right).
\end{equation}

In this paper we prove a more specific regularity result which can be summarized as follows

**Theorem 1.11.** Let $\Omega \subset \mathbb{R}^n$ be a $C^1$ convex (4) open set, let $u$ belong to $H^{1,(n-2)} \cap L^{\infty}(\Omega, \mathbb{R}^n)$ and let $A^0_{ij}$ be $N \times N$ constant matrices satisfying

\begin{itemize}
  \item[(2)] In [1] the result is stated in the case of only one equation; it is known that it holds unchanged in the case of systems.
  \item[(3)] We say that a bounded open set $\Omega \subset \mathbb{R}^n$ is of class $C^1$ if for every point $x_0 \in \partial \Omega$ we can find an open neighbourhood $\Omega(x^0)$ and a $C^1$ homeomorphism $x \to \phi(x)$ which maps $\Omega(x^0)$ onto $B(0, 1)$, $\Omega(x^0) \cap \Omega$ onto the set \{\(x \in B(0, 1): x_n > 0\) and $\Omega(x^0) \cap \partial \Omega$ onto \{\(x \in B(0, 1): x_n = 0\).
  \item[(4)] The hypothesis that $\Omega$ is convex may be eliminated.
\end{itemize}
(1.2). Then, if $v$ is the solution of Dirichlet problem

$$v - u \in H^1_0(\Omega, \mathbb{R}^n),$$

(1.5) $$\int_{\Omega} \sum_{i,j=1}^{n} (A^0_{ij} D_i v | D_j v) \, dx = 0 \quad \forall \varphi \in H^1_0(\Omega, \mathbb{R}^n),$$

$v$ belongs to $H^{1,(n-2)} \cap L^\infty(\Omega, \mathbb{R}^n)$ and

$$\sup_{\Omega} \|v\| + \sum_{i=1}^{n} \|D_i v\|_{L^{n-1}(\Omega, \mathbb{R}^n)} \leq C_2 \left\{ \sup_{\Omega} \|u\| + \sum_{i=1}^{n} \|D_i u\|_{L^{n-1}(\Omega, \mathbb{R}^n)} \right\}.$$

A trivial consequence of theorem 1.II is the following maximum principle

**Theorem 1.11.** Let $\Omega \subset A \subset \mathbb{R}^n$, be two open sets and let $\Omega$ be convex (*4) and of class $C^1$. Let $u \in H^1 \cap L^\infty(A, \mathbb{R}^n)$ be such that

$$D_i u \in L^{2,n-2}(\Omega, \mathbb{R}^n), \quad 1 \leq i \leq n,$$

(1.7) $$\sum_{i=1}^{n} \|D_i u\|_{L^{n-1}(\Omega, \mathbb{R}^n)} \leq C_3 \sup_{A} \|u\|.$$

Then, if $v$ is the solution of Dirichlet problem (1.5), $v$ verifies the following inequality

$$\sup_{\Omega} \|v\| \leq C_4 \sup_{A} \|u\|.$$

Property (1.7) is quite usual in the study of nonlinear elliptic systems; consider, for example, the following problem.

Let $A_{ij}(x, \omega)$ $(i, j = 1, \ldots, n)$ be $N \times N$ bounded continuous matrices defined in $A \times \mathbb{R}^N$, satisfying the following ellipticity condition

(1.9) $$\sum_{i,j=1}^{n} (A_{ij}(x, \omega) \xi^i \xi^j) \geq \nu(K) \sum_{i=1}^{n} \|\xi^i\|^2, \quad v > 0,$$

$$\forall (x, \omega) \in A \times \{ \|w\| \leq \varepsilon \}, \forall \xi^1, \ldots, \xi^n \in \mathbb{R}^N.$$

Let $f: A \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}^3$ be measurable in $x \in A$ and continuous
in \((w, p)\); suppose also that \(f\) has quadratic growth

\[
\|f(x, w, p)\|_N \leq a(K) \|p\|_{n \mathcal{F}}^2 + b(K),
\]
\[
\forall (x, w, p) \in \mathcal{A} \times \{\|w\| < K\} \times R^{nN}.
\]

Finally, let \(Dw\) denote the vector \((D_1w, \ldots, D_nw)\) of \(R^{nN}\).

It is then known ([3]) that every solution \(u \in H^1 \cap L^\infty(A, R^N)\) of system

\[
\int \sum_{i,j=1}^n (A_{ij}(x, u) D_i u D_j \varphi)_N \, dx = \int (f(x, u, Du) \varphi)_N \, dx
\]
\[
\forall \varphi \in H^1_0 \cap L^\infty(A, R^N)
\]

which satisfies the following inequality (with \(M = \sup_{\mathcal{A}} \|u\|\))

\[
2Ma(M) < v(M)
\]

is Hölder continuous in \(A \setminus A_0\), where \(A_0\) is closed in \(A\) and such that \(H_{n,q}(A_0) = 0\) \(^{(5)}\) for a certain \(q \geq 2\).

The proof given in [3] needs a boundedness result of the following kind:

let \(u \in H^1 \cap L^\infty(A, R^N)\) be a solution of system (1.11) verifying (1.12);

let \(A_{ij}^0\) \((i, j = 1, \ldots, n)\) be constant \(N \times N\) matrices satisfying (1.2);

let \(v\) be the solution of Dirichlet problem

\[
\begin{cases}
\{ v - u \in H^1_0(B(x_0, r), R^N) \text{ with } B(x_0, 2r) \subset \subset A \text{ and } 0 < r < 1 \\
\int_{B(x_0, r)} \sum_{i,j=1}^n (A_{ij}^0 D_i u D_j \varphi) \, dx = 0 \quad \forall \varphi \in H^1_0(B(x_0, r), R^N).
\end{cases}
\]

Then

\[
v \in L^\infty(B(x_0, r), R^N) \quad \text{and} \quad \sup_{B(x_0, r)} \|v\| \leq C_5 \sup_{\mathcal{A}} \|u\|
\]

where \(C_5\) does not depend on \(x_0\) and \(r\).

\(^{(5)}\) \(H_\alpha, \alpha \geq 0,\) is the \(\alpha\)-dimensional Hausdorff measure.
In order to get (1.13), the proof of [3] recalls the maximum principle proved in [2], which depends on the possibility of representing $v$ by adequate potentials.

In section 3 we prove that (1.13) may be obtained in a simpler way, showing that $u$ verifies the hypotheses of Theorem 1.11.1.

This method can be extended to more general situations, such as elliptic systems of order $2m > 2$, even with continuous coefficients, and $C^1$ bounded open sets $\Omega$ not necessarily convex.

I would like to thank S. Campanato for the useful discussions we had on this subject.

2. Proof of Theorem 1.11.

Having fixed $y \in \Omega$, we define

$$d = \text{dist} (y, \partial \Omega) = \|y - z\| \quad \text{with} \quad z \in \partial \Omega .$$

As $v$ solves problem (1.5) and $A^0_{ij}$ are constant, the following inequality holds ([1], Lemma 7.1)

$$\int_{B(y,d)} \|v\|^2 \, dx \leq C(n) \left( \frac{\|v\|}{d} \right)^n \int_{B(y,d)} \|v\|^2 \, dx \quad \forall 0 < q < d . \quad (2.1)$$

On the other hand

$$\int_{B(y,d)} \|v\|^2 \, dx \leq \int_{B(y,2d) \cap \Omega} \|v\|^2 \, dx \leq \int_{B(y,2d) \cap \Omega} \|v\|^2 \, dx \leq C(n) \left[ d^n \sup_{\Omega} \|u\|^2 + \int_{B(y,2d) \cap \Omega} \|v - u\|^2 \, dx \right] . \quad (2.2)$$

As $v - u \in H^1_0(\Omega, \mathbb{R}^n)$, Poincaré inequality is valid ([4]):

$$\int_{B(y,2d) \cap \Omega} \|v - u\|^2 \, dx \leq C(n) \sum_{i=1}^n \|D_i (v-u)\|^2 \, dx \quad (\ast). \quad (2.3)$$

(\ast) As $\Omega$ is convex the constant $C(n)$ does not depend on $y$ (in general we shall write $C(n, v, \ldots)$ to mean a constant that depends on the algebraic data $n, v, \ldots$).
From (2.1), (2.2) and (2.3) we get

\[(2.4) \quad \frac{1}{\Omega} \int_{B(x,\varepsilon)} \|v\|^2 \, dx \leq \]

\[\leq C(n, \nu) \left[ \sup_{\Omega} \|u\|^2 + d^{2-n} \int_{B(x,\varepsilon) \cap \Omega} \sum_{i=1}^{n} \|D_i(v - u)\|^2 \, dx \right].\]

Theorem 1.1 implies that

\[v \in H^{1,(n-2)}(\Omega, \mathbb{R}^N)\]

and

\[(2.5) \quad \|v\|_{H^{1,(n-2)}(\Omega, \mathbb{R}^N)} \leq C_1 \|u\|_{H^{1,(n-2)}(\Omega, \mathbb{R}^N)}.\]

Combining (2.4) and (2.5) we prove (1.6) and the theorem.

3. Application to quasilinear systems.

Let \( \Lambda \subset \mathbb{R}^n \) be a bounded open set. Let \( A_{ij}(x, u) \) (1 \( \leq i, j \leq n \)) be \( N \times N \) bounded continuous matrices defined in \( \Lambda \times \mathbb{R}^N \), satisfying the ellipticity condition

\[(3.1) \quad \sum_{i,j=1}^{n} (A_{ij}(x, u) \xi^i \xi^j) > \nu(K) \sum_{i=1}^{n} \|\xi^i\|^2, \quad \nu > 0, \]

\[\forall (x, u) \in \Lambda \times \{\|u\| < K\}, \ \forall \xi^1, \ldots, \xi^n \in \mathbb{R}^N.\]

Let \( f: \Lambda \times \mathbb{R}^N \times \mathbb{R}^n \rightarrow \mathbb{R}^N \) be measurable in \( x \in \Lambda \), continuous in \((u, p)\) and with quadratic growth

\[(3.2) \quad \|f(x, u, p)\|_{\mathbb{R}^N} \leq a(K) \|p\|_{\mathbb{R}^n}^2 + b(K), \]

\[\forall (x, u, p) \in \Lambda \times \{\|u\| < K\} \times \mathbb{R}^n.\]

Let us consider the quasilinear system in divergence form

\[(3.3) \quad -\sum_{i,j=1}^{n} D_i(A_{ij}(x, u) D_j u) = f(x, u, Du), \quad \text{in} \ \Lambda.\]
The following lemma can be deduced from a «Caccioppoli inequality» proved in [3].

**Lemma 3.1.** Let \( u \in H^1 \cap L^\infty(A, \mathbb{R}^N) \) be a weak solution of system (3.3) satisfying the following inequality (with \( M = \sup_A \| u \| \))

\[
(3.4) \quad Ma(M) < \gamma(M).
\]

Then \( u \in H^1_{loc} (\mathbb{R}^N) \) and for every ball \( B(x^0, r) \subset B(x^0, 2r) \subset A \)

\[
(3.5) \quad \sum_{i=1}^n \| D_i u \|_{L^{n-1}(B(x^0, R^N))} < C' \sup_A \| u \|
\]

where \( C' \) depends on \( M \), but neither on \( r \) nor on \( x^0 \).

**Proof.** As \( u \in H^1 \cap L^\infty(A, \mathbb{R}^N) \) is a weak solution of (3.3)

\[
(3.6) \quad \int \sum_{i,j=1}^n (A_{ij}(x, u) D_j u | D_i \varphi) \, dx = \int (f(x, u, Du) | \varphi) \, dx
\]

\[\forall \varphi \in H^1_0 \cap L^\infty(A, \mathbb{R}^N).\]

Having fixed \( y \in B(x^0, r) \) and \( 0 < \sigma < r/2 \), we choose \( \theta \in C^\infty_0 (B(y, 2\sigma)) \) with \( 0 < \theta < 1, \theta = 1 \) in \( B(y, \sigma) \) and \( \| D\theta \| < 2/\sigma \).

If we substitute \( \varphi = \theta^2 u \) in (3.6), we get as in [3] the following «Caccioppoli inequality»:

\[
\int \sum_{i=1}^n \| D_i u \|^2 \, dx < C(\sigma) \left\{ \frac{1}{\sigma^2} \int_{B(y, 2\sigma)} \| u \|^2 \, dx + [b(M)]^2 \sigma^{n+2} \right\}
\]

Hence, if \( \sigma \) is such that

\[
[b(M)]^2 \sigma^4 < \sup_A \| u \|^2
\]

we get

\[
\int_{B(y, \sigma)} \sum_{i=1}^n \| D_i u \|^2 \, dx < C' \sigma^{n-2} \sup_A \| u \|^2.
\]

This proves (3.5) and the lemma.

**Remark 3.1.** Let \( u \in H^1 \cap L^\infty(A, \mathbb{R}^N) \) be as in Lemma 3.1 and consider a ball \( B(x^0, r) \subset B(x^0, 2r) \subset A, 0 < r < 1 \). Let \( A_{ij}^0 \) \( (i, j = 1, \ldots, n) \)
be $N \times N$ constant matrices satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \xi_i \xi_j (A_{ij} \eta \eta) \geq r \| \xi \|^2 \| \eta \|^2_N, \quad \forall \xi \in \mathbb{R}^n, \forall \eta \in \mathbb{R}^N.$$ 

Let $v$ be the solution of the following Dirichlet problem

$$\begin{cases}
\Delta v - u \in H^1_0(B(x_0, r), \mathbb{R}^N), \\
\int_{B(x_0, r)} \sum_{i,j} (A_{ij} D_i v | D_j v) \, dx = 0 \quad \forall \varphi \in H^1_0(B(x_0, r), \mathbb{R}^N).
\end{cases}$$

From Lemma 3.I and Theorem 1.III we draw the conclusion that

$$\sup_{B(x_0, r)} \| v \| \leq C^* \sup_{A} \| u \|.$$ 

Moreover, $C^*$ does not depend on $x_0$ and $r$.

The last statement can be shown by a homothetical argument.

REFERENCES


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