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On a Maximum Principle for Elliptic Systems with Constant Coefficients.

PIERMARCO CANNARSA (*)

1. Introduction.

Let $\Omega \subset R^n$ be a bounded open set and let N be a positive integer. Let $(|)_N$, $\| \ \|_N$ be the scalar product and the norm in R^N (1). We set $D_i = \partial/\partial x_i$, $i = 1, \ldots, n$.

Let $H^1(\Omega, R^N)$ be the usual Sobolev space of vectors $u\colon \Omega \to R^N$ with norm

$$\|u\|_{H^1(\Omega,R^N)} = \left\{ \int_\Omega \|u\|^2\, dx + \int_\Omega \sum_{i=1}^n \|D_i u\|^2\, dx \right\}^{\frac{1}{2}} \tag{1.1}$$

and let $H_0^1(\Omega, R^N)$ be the closure of $C_0^\infty(\Omega, R^N)$ with respect to norm (1.1).

Let $L^{2,\lambda}(\Omega, R^N)$, $0 \leqslant \lambda \leqslant n$, be the Banach space defined as follows

$$L^{2,\lambda}(\Omega, R^N) = \left\{ u \in L^2(\Omega, R^N)\colon \|u\|^2_{L^{2,\lambda}(\Omega,R^N)} = \sup_{\substack{\varrho>0 \\ x\in\bar\Omega}} \varrho^{-\lambda} \int_{B(x,\varrho)\cap\Omega} \|u\|^2\, dy < +\infty \right\}$$

(here $B(x, \varrho) = \{y \in R^n\colon \|x - y\| < \varrho\}$) and

$$H^{1,(\lambda)}(\Omega, R^N) = \{u \in H^1(\Omega, R^N)\colon D_i u \in L^{2,\lambda}(\Omega, R^N),\ 1 \leqslant i \leqslant n\}$$

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(1) We shall often omit the subscript N and write simply $(|)$, $\|\ \|$.

$H^{1,(\lambda)}(\Omega, R^N)$ is a Banach space with norm

$$\|u\|_{H^{1,(\lambda)}(\Omega,R^N)} = \|u\|_{L^2(\Omega,R^N)} + \sum_{i=1}^{n} \|D_i u\|_{L^{2,\lambda}(\Omega,R^N)} .$$

Let $A_{ij}(x)$ $(i, j = 1, \dots, n)$ be $N \times N$ matrices satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \xi_i \xi_j (A_{ij}(x)\eta|\eta)_N \geqslant \nu \|\xi\|_n^2 \|\eta\|_N^2 , \qquad \nu > 0 , \tag{1.2}$$
$$\forall x \in \bar{\Omega}, \ \forall \xi \in R^n, \ \forall \eta \in R^N .$$

The following regularity theorem is proved in [1] (2)

THEOREM 1.I. *Let $\Omega \subset\subset R^n$ be a C^1 (3) open set, $u \in H^{1,(\lambda)}(\Omega, R^N)$, $f_i \in L^{2,\lambda}(\Omega, R^N)$ $(0 \leqslant \lambda < n,\ i = 1, \dots, n)$ and let A_{ij} be continuous in $\bar{\Omega}$ and satisfy* (1.2). *Then, if v is the solution of Dirichlet problem*

$$\begin{cases} v - u \in H_0^1(\Omega, R^N) , \\ \displaystyle\int_\Omega \sum_{i,j=1}^{n} (A_{ij}(x) D_j v | D_i \varphi)\, dx = \int_\Omega \sum_{i=1}^{n} (f_i | D_i \varphi)\, dx \qquad \forall \varphi \in H_0^1(\Omega, R^N) , \end{cases} \tag{1.3}$$

v belongs to $H^{1,(\lambda)}(\Omega, R^N)$ and

$$\|v\|_{H^{1,(\lambda)}(\Omega,R^N)} \leqslant C_1 \left\{ \sum_{i=1}^{n} \|f_i\|_{L^{2,\lambda}(\Omega,R^N)} + \|u\|_{H^{1,(\lambda)}(\Omega,R^N)} \right\} . \tag{1.4}$$

In this paper we prove a more specific regularity result which can be summarized as follows

THEOREM 1.II. *Let $\Omega \subset\subset R^n$ be a C^1 convex* (4) *open set, let u belong to $H^{1,(n-2)} \cap L^\infty(\Omega, R^N)$ and let A_{ij}^0 be $N \times N$ constant matrices satisfying*

(2) In [1] the result is stated in the case of only one equation; it is known that it holds unchanged in the case of systems.

(3) We say that a bounded open set $\Omega \subset R^n$ is of class C^1 if for every point $x_0 \in \partial\Omega$ we can find an open neighbourhood $\Omega(x^0)$ and a C^1 homeomorfism $x \to \phi(x)$ which maps $\overline{\Omega(x^0)}$ onto $\overline{B(0, 1)}$, $\Omega(x^0) \cap \Omega$ onto the set $\{x \in B(0, 1) : x_n > 0\}$ and $\Omega(x^0) \cap \partial\Omega$ onto $\{x \in B(0, 1) : x_n = 0\}$.

(4) The hypothesis that Ω is convex may be eliminated.

(1.2). *Then, if v is the solution of Dirichlet problem*

$$
(1.5) \qquad \begin{cases} v - u \in H_0^1(\Omega, R^N), \\ \int_\Omega \sum_{i,j=1}^n (A_{ij}^0 D_j v | D_i \varphi)\, dx = 0 \qquad \forall \varphi \in H_0^1(\Omega, R^N), \end{cases}
$$

v belongs to $H^{1,(n-2)} \cap L^\infty(\Omega, R^N)$ and

$$
(1.6) \qquad \sup_\Omega \|v\| + \sum_{i=1}^n \|D_i v\|_{L^{2,n-2}(\Omega,R^N)} \leqslant C_2 \left\{ \sup_\Omega \|u\| + \sum_{i=1}^n \|D_i u\|_{L^{2,n-2}(\Omega,R^N)} \right\}.
$$

A trivial consequence of theorem 1.II is the following maximum principle

THEOREM 1.III. *Let $\Omega \subset \Lambda \subset\subset R^n$, be two open sets and let Ω be convex* [4] *and of class C^1. Let $u \in H^1 \cap L^\infty(\Lambda, R^N)$ be such that*

$$
(1.7) \qquad \begin{aligned} & D_i u \in L^{2,n-2}(\Omega, R^N), \qquad 1 \leqslant i \leqslant n, \\ & \sum_{i=1}^n \|D_i u\|_{L^{2,n-2}(\Omega,R^N)} \leqslant C_3 \sup_\Lambda \|u\|. \end{aligned}
$$

Then, if v is the solution of Dirichlet problem (1.5), *v verifies the following inequality*

$$
(1.8) \qquad \sup_\Omega \|v\| \leqslant C_4 \sup_\Lambda \|u\|.
$$

Property (1.7) is quite usual in the study of nonlinear elliptic systems; consider, for example, the following problem.

Let $A_{ij}(x, w)$ $(i, j = 1, \ldots, n)$ be $N \times N$ bounded continuous matrices defined in $\bar{\Lambda} \times R^N$, satisfying the following ellipticity condition

$$
(1.9) \qquad \begin{aligned} & \sum_{i,j=1}^n (A_{ij}(x, w)\xi^j | \xi^i) \geqslant \nu(K) \sum_{i=1}^n \|\xi^i\|^2, \qquad \nu > 0, \\ & \forall (x, w) \in \bar{\Lambda} \times \{\|w\| \leqslant K\}, \ \forall \xi^1, \ldots, \xi^n \in R^N. \end{aligned}
$$

Let $f\colon \Lambda \times R^N \times R^{nN} \to R^N$ be measurable in $x \in \Lambda$ and continuous in (w, p); suppose also that f has quadratic growth

$$\|f(x, w, p)\|_N \leqslant a(K)\|p\|^2_{nN} + b(K)\,, \qquad \forall (x, w, p) \in \Lambda \times \{\|w\| \leqslant K\} \times R^{nN}\,. \tag{1.10}$$

Finally, let Dw denote the vector $(D_1 w|\dots|D_n w)$ of R^{nN}.

It is then known ([3]) that every solution $u \in H^1 \cap L^\infty(\Lambda, R^N)$ of system

$$\int_\Lambda \sum_{i,j=1}^{n} (A_{ij}(x, u) D_j u | D_i \varphi)_N \, dx = \int_\Lambda (f(x, u, Du)|\varphi)_N \, dx \qquad \forall \varphi \in H^1_0 \cap L^\infty(\Lambda, R^N) \tag{1.11}$$

which satisfies the following inequality (with $M = \sup_\Lambda \|u\|$)

$$2Ma(M) < \nu(M) \tag{1.12}$$

is Hölder continuous in $\Lambda \setminus \Lambda_0$, where Λ_0 is closed in Λ and such that $H_{n-q}(\Lambda_0) = 0$ [5] for a certain $q > 2$.

The proof given in [3] needs a boundedness result of the following kind:

let $u \in H^1 \cap L^\infty(\Lambda, R^N)$ *be a solution of system* (1.11) *verifying* (1.12);

let A^0_{ij} $(i, j = 1, \dots, n)$ *be constant* $N \times N$ *matrices satisfying* (1.2);

let v *be the solution of Dirichlet problem*

$$\begin{cases} v - u \in H^1_0(B(x_0, r), R^N) \ \text{with}\ B(x_0, 2r) \subset\subset \Lambda \ \text{and}\ 0 < r \leqslant 1 \\ \displaystyle\int_{B(x_0,r)} \sum_{i,j=1}^{n} (A^0_{ij} D_j v | D_i \varphi)\, dx = 0 \qquad \forall \varphi \in H^1_0(B(x_0, r), R^N)\,. \end{cases}$$

Then

$$v \in L^\infty(B(x_0, r), R^N) \qquad \text{and} \qquad \sup_{B(x_0,r)} \|v\| \leqslant C_5 \sup_\Lambda \|u\| \tag{1.13}$$

where C_5 *does not depend on* x_0 *and* r.

[5] H_α, $\alpha \geqslant 0$, is the α-dimensional Hausdorff measure.

In order to get (1.13), the proof of [3] recalls the maximum principle proved in [2], which depends on the possibility of representing v by adequate potentials.

In section 3 we prove that (1.13) may be obtained in a simpler way, showing that u verifies the hypotheses of Theorem 1.III.

This method can be extended to more general situations, such as elliptic systems of order $2m\geqslant 2$, even with continuous coefficients, and C^1 bounded open sets Ω not necessarily convex.

I would like to thank S. Campanato for the useful discussions we had on this subject.

2. Proof of Theorem 1.II.

Having fixed $y \in \Omega$, we define

$$d = \operatorname{dist}(y, \partial\Omega) = \|y - z\| \qquad \text{with } z \in \partial\Omega\,.$$

As v solves problem (1.5) and A^0_{ij} are constant, the following inequality holds ([1], Lemma 7.I)

$$\int_{B(y,\varrho)} \|v\|^2\,dx \leqslant C(\nu)\left(\frac{\varrho}{d}\right)^n \int_{B(y,d)} \|v\|^2\,dx \qquad \forall 0<\varrho\leqslant d\,. \tag{2.1}$$

On the other hand

$$\int_{B(y,d)} \|v\|^2\,dx \leqslant \int_{B(z,2d)\cap\Omega} \|v\|^2\,dx \leqslant \leqslant C(n)\Big[d^n \sup_{\Omega}\|u\|^2 + \int_{B(z,2d)\cap\Omega} \|v-u\|^2\,dx\Big]\,. \tag{2.2}$$

As $v-u \in H^1_0(\Omega, R^N)$, Poincaré inequality is valid ([4]):

$$\int_{B(z,2d)\cap\Omega} \|v-u\|^2\,dx \leqslant C(n)\,d^2 \int_{B(z,2d)\cap\Omega} \sum_{i=1}^{n} \|D_i(v-u)\|^2\,dx \;(^6)\,. \tag{2.3}$$

(6) As Ω is convex the constant $C(n)$ does not depend on y (in general we shall write $C(n, \nu, \ldots)$ to mean a constant that depends on the algebraic data $n, \nu, \ldots$).

From (2.1), (2.2) and (2.3) we get

$$\frac{1}{\varrho^n}\int_{B(y,\varrho)} \|v\|^2\, dx \leqslant \tag{2.4}$$

$$\leqslant C(n,\nu)\left[\sup_{\Omega}\|u\|^2 + d^{2-n}\int_{B(z,2d)\cap\Omega}\sum_{i=1}^{n}\|D_i(v-u)\|^2\, dx\right].$$

Theorem 1.I implies that

$$v\in H^{1,(n-2)}(\Omega, R^N)$$

and

$$\|v\|_{H^{1,(n-2)}(\Omega,R^N)} \leqslant C_1\|u\|_{H^{1,(n-2)}(\Omega,R^N)}\,. \tag{2.5}$$

Combining (2.4) and (2.5) we prove (1.6) and the theorem.

3. Application to quasilinear systems.

Let $\Lambda\subset R^n$ be a bounded open set. Let $A_{ij}(x,u)$ $(1\leqslant i, j\leqslant n)$ be $N\times N$ bounded continuous matrices defined in $\bar{\Lambda}\times R^N$, satisfying the ellipticity condition

$$\sum_{i,j=1}^{n}(A_{ij}(x,u)\xi^j|\xi^i)\geqslant \nu(K)\sum_{i=1}^{n}\|\xi^i\|^2\,,\qquad \nu>0\,, \tag{3.1}$$

$$\forall (x,u)\in\bar{\Lambda}\times\{\|u\|\leqslant K\},\ \forall \xi^1,\ldots,\xi^n\in R^N\,.$$

Let $f\colon \Lambda\times R^N\times R^{nN}\to R^N$ be measurable in $x\in\Lambda$, continuous in (u,p) and with quadratic growth

$$\|f(x,u,p)\|_N\leqslant a(K)\|p\|_{nN}^2 + b(K)\,, \tag{3.2}$$

$$\forall (x,u,p)\in\Lambda\times\{\|u\|\leqslant K\}\times R^{nN}\,.$$

Let us consider the quasilinear system in divergence form

$$-\sum_{i,j=1}^{n}D_i\big(A_{ij}(x,u)\,D_j u\big) = f(x,u,Du)\,,\quad \text{in } \Lambda\,. \tag{3.3}$$

The following lemma can be deduced from a « Caccioppoli inequality » proved in [3].

LEMMA 3.I. *Let $u \in H^1 \cap L^\infty(\Lambda, R^N)$ be a weak solution of system (3.3) satisfying the following inequality* $\left(\textit{with } M = \sup\limits_{\Lambda} \|u\|\right)$

$$Ma(M) < \nu(M) . \tag{3.4}$$

Then $u \in H^{1,(n-2)}_{\text{loc}}(\Lambda, R^N)$ and for every ball $B(x^0, r) \subset B(x^0, 2r) \subset \Lambda$

$$\sum_{i=1}^{n} \|D_i u\|_{L^{2,n-2}(B(x^0 r), R^N)} \leqslant C' \sup_{\Lambda} \|u\| \tag{3.5}$$

where C' depends on M, but neither on r nor on x^0.

PROOF. As $u \in H^1 \cap L^\infty(\Lambda, R^N)$ is a weak solution of (3.3)

$$\int_{\Lambda} \sum_{i,j=1}^{n} \left(A_{ij}(x, u) D_j u | D_i \varphi\right) dx = \int_{\Lambda} \left(f(x, u, Du) | \varphi\right) dx \tag{3.6}$$

$$\forall \varphi \in H^1_0 \cap L^\infty(\Lambda, R^N) .$$

Having fixed $y \in \overline{B(x^0, r)}$ and $0 < \sigma < r/2$, we choose $\theta \in C_0^\infty(B(y, 2\sigma))$ with $0 \leqslant \theta \leqslant 1$, $\theta = 1$ in $B(y, \sigma)$ and $\|D\theta\| \leqslant 2/\sigma$.

If we substitute $\varphi = \theta^2 u$ in (3.6), we get as in [3] the following « Caccioppoli inequality »:

$$\int_{B(y,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 \, dx \leqslant C(\nu) \left\{ \frac{1}{\sigma^2} \int_{B(y,2\sigma)} \|u\|^2 \, dx + [b(M)]^2 \sigma^{n+2} \right\} .$$

Hence, if σ is such that

$$[b(M)]^2 \sigma^4 \leqslant \sup_{\Lambda} \|u\|^2$$

we get

$$\int_{B(y,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 \, dx \leqslant C' \sigma^{n-2} \sup_{\Lambda} \|u\|^2 .$$

This proves (3.5) and the lemma.

REMARK 3.I. Let $u \in H^1 \cap L^\infty(\Lambda, R^N)$ be as in Lemma 3.I and consider a ball $B(x^0, r) \subset B(x^0, 2r) \subset \Lambda$, $0 < r \leqslant 1$. Let A^0_{ij} $(i, j = 1, \dots, n)$

be $N \times N$ constant matrices satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \xi_i \xi_j (A_{ij}^0 \eta | \eta)_N \geqslant \nu \|\xi\|_n^2 \|\eta\|_N^2 , \qquad \forall \xi \in R^n, \ \forall \eta \in R^N .$$

Let v be the solution of the following Dirichlet problem

$$\begin{cases} v - u \in H_0^1(B(x_0, r), R^N) , \\ \displaystyle\int\limits_{B(x_0,r)} \sum_{i,j} (A_{ij}^0 D_j v | D_i \varphi)\, dx = 0 \qquad \forall \varphi \in H_0^1(B(x_0, r), R^N) . \end{cases}$$

From Lemma 3.I and Theorem 1.III we draw the conclusion that

$$\sup_{B(x_0,r)} \|v\| \leqslant C^* \sup_{A} \|u\| .$$

Moreover, C^* does not depend on x^0 and r.
The last statement can be shown by a homothetical argument.

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