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Partial Hölder Continuity of Solutions of Quasilinear Parabolic Systems of Second Order with Linear Growth.

SERGIO CAMPANATO (*)

1. Introduction.

Let Ω be a bounded open set of R^n , $n > 2$ ⁽¹⁾, with sufficiently smooth boundary $\partial\Omega$, for instance of class C^3 . Let N be an integer > 1 , $(|)$ and $\|\cdot\|$ the scalar product and the norm in R^N ⁽²⁾. If $u: \Omega \rightarrow R^N$, we set $Du = (D_1u | \dots | D_nu)$ where $D_i = \partial/\partial x_i$. In general $p = (p^1 | \dots | p^n)$, $p^i \in R^N$, denotes a vector of R^{nN} , x is a point of R^n , $t \in R$ and $X = (x, t)$.

$$B(x_0, \sigma) = \{x \in R^n: \|x - x_0\| < \sigma\}.$$

$$Q = \Omega \times (-T, 0) \text{ with } T > 0.$$

If $X_0 = (x_0, t)$, we set

$$Q(X_0, \sigma) = B(x_0, \sigma) \times (t_0 - \sigma^2, t_0).$$

We say that $Q(X_0, \sigma) \subset\subset Q$ if

$$B(x_0, \sigma) \subset\subset \Omega \quad \text{and} \quad \sigma^2 < t_0 + T \leq T$$

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⁽¹⁾ This is just to fix ideas; the case $n = 2$ can be dealt with by trivial modifications.

⁽²⁾ In general $(|)_k$, $\|\cdot\|_k$ are the scalar product and the norm in R^k . We shall omit the index k if there is not ambiguity of writing.

$H^{k,p}$ and $H_0^{k,p}$ are the usual Sobolev spaces. If $p = 2$ we write simply H^k and H_0^k .

Let us consider the quasilinear parabolic system of second order

$$(1.1) \quad -\sum_{ij=1}^n D_i(A_{ij}(X, u) D_j u) + \frac{\partial u}{\partial t} = -\sum_{i=1}^n D_i f^i(X, u) + f^0(X, u, Du)$$

where A_{ij} are $N \times N$ matrices which are uniformly continuous and bounded in $\bar{Q} \times R^N$ and satisfy the strong ellipticity condition

$$(1.2) \quad \begin{aligned} \sum_{ij} (A_{ij}(X, u) p^j | p^i) &\geq \nu \sum_i \|p^i\|^2 \quad (\nu > 0), \\ \forall (X, u, p) &\in \bar{Q} \times R^N \times R^{nN}, \end{aligned}$$

$f^i(X, u)$, $i = 1, \dots, n$, and $f^0(X, u, p)$ are vectors of R^N , measurable in $X \in Q$ and continuous in u and (u, p) respectively. Suppose that f^i, f^0 have linear growth

$$(1.3) \quad \|f^i(X, u)\| \leq g_i(X) + c\|u\|, \quad i = 1, \dots, n$$

$$(1.4) \quad \|f^0(X, u, p)\| \leq g_0(X) + c\{\|u\| + \sum_i \|p^i\|\}$$

with

$$(1.5) \quad g = \left(\sum_i D_i g_i + g_0 \right) \in L^2(-T, 0, H^{-1}(\Omega)).$$

We set, for the sake of brevity,

$$(1.6) \quad Eu = -\sum_{ij} D_i(A_{ij}(X, u) D_j u) + \frac{\partial u}{\partial t},$$

$$(1.7) \quad F = -\sum_i D_i f^i(X, u) + f^0(X, u, Du),$$

$$(1.8) \quad a(u, \varphi) = \int_Q \sum_{ij} (A_{ij} D_j u | D_i \varphi) - \left(u \left| \frac{\partial \varphi}{\partial t} \right. \right) dX,$$

$$(1.9) \quad \langle F, \varphi \rangle = \int_Q \sum_i (f^i | D_i \varphi) + (f^0 | \varphi) dX,$$

$$(1.10) \quad W(Q) = L^2(-T, 0, H_0^1(\Omega, R^N)) \cap H^1(-T, 0, L^2(\Omega, R^N)).$$

A solution of system (1.1) is a vector $u \in L^2(-T, 0, H^1(\Omega, R^N))$ such that

$$(1.11) \quad \begin{aligned} a(u, \varphi) &= \langle F, \varphi \rangle, \\ \forall \varphi \in W(Q): \quad \varphi(x, -T) &= \varphi(x, 0) = 0 \quad \text{in } \Omega. \end{aligned}$$

A solution of Cauchy-Dirichlet problem

$$(1.12) \quad \begin{aligned} Eu &= F && \text{in } Q \\ u &= 0 && \text{on } \partial\Omega \times (-T, 0) \\ u(x, -T) &= 0 && \text{in } \Omega \end{aligned}$$

is a vector $u \in L^2(-T, 0, H_0^1(\Omega, R^N))$ such that

$$(1.13) \quad \begin{aligned} a(u, \varphi) &= \langle F, \varphi \rangle, \\ \forall \varphi \in W(Q): \quad \varphi(x, 0) &= 0 \quad \text{in } \Omega. \end{aligned}$$

It is known that, even if g is smooth, there are solutions of system (1.1) which fail to be Hölder continuous in Q ⁽³⁾. We shall prove, in section 3, the following partial Hölder continuity result:

THEOREM 1.I. *If $u \in L^2(-T, 0, H^1(\Omega, R^N))$ is a solution of system (1.1) and*

$$(1.14) \quad \begin{aligned} g_i &\in L^p(Q), \quad i = 1, \dots, n, \\ g_0 &\in L^p(-T, 0, L^{pn/(n+2)}(\Omega)) \quad \text{with } p > n + 2, \end{aligned}$$

then there is a set $Q_0 \subset Q$, closed in Q , such that

$$(1.15) \quad \mathcal{M}_n(Q_0) = 0$$

$$(1.16) \quad u \in C^{0,\alpha}(Q \setminus Q_0, R^N), \quad \forall \alpha < 1 - \frac{n+2}{p}$$

and for every open subset $A \subset \subset Q \setminus Q_0$

$$(1.17) \quad [u]_{\alpha, \bar{A}} \leq c \left\{ 1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 dX \right\}^{(4)}.$$

⁽³⁾ i.e. on every compact set $K \subset Q$.

⁽⁴⁾ C depends on the $L^2(Q)$ -norm of g_i, g_0 and on the distance of $\bar{A}$ from the parabolic boundary of Q .

Here $\mathcal{M}_n$ is the n -dimensional Hausdorff measure with respect to the metric

$$(1.18) \quad \delta(X, Y) = \max \{ \|x - y\|, |t - \tau| \}, \quad X = (x, t), \quad Y = (y, \tau)$$

and also Hölder continuity in (1.16) is related to this metric.

The previous result is proved also in [9] with a different technique, for the special case $f^i = f^0 = 0$ ⁽⁵⁾.

The method of this paper can be extended to the case of systems of order $2m \geq 2$ and to more general growth conditions on the vectors f^i, f^0 .

In order to prove theorem 1.I we need a local L^p regularity result of this kind: We set

$$(1.19) \quad \xi = (n + 2) \left(1 - \frac{2}{p} \right),$$

$$(1.20) \quad \Phi(X_0, \sigma) = \sigma^\xi + \int_{Q(X_0, \sigma)} \|u\|^2 + \sum_i \|D_i u\|^2 + \sigma^{-2} \|u - u_\sigma\|^2 dX$$

where u_σ is the average of u on $Q(X_0, \sigma)$

$$u_\sigma = \int_{Q(X_0, \sigma)} u(X) dX.$$

THEOREM 1.II. *If $u \in L^2(-T, 0, H^1(\Omega, R^N))$ is a solution of system (1.1) and (1.14) holds, then we can find $q > 2$ such that $\forall Q(X_0, 2\sigma) \subset\subset Q$ with $\sigma \leq 1$*

$$(1.21) \quad \left(\int_{Q(X_0, \sigma)} \sum_i \|D_i u\|^q dX \right)^{2/q} \leq c \sigma^{-(n+2)} \Phi(X_0, 2\sigma).$$

Theorem 1.II easily follows, via Hölder inequality, from the following local L^q result proved in [5]:

THEOREM 1.III. *If u is a solution of system (1.1), we can find*

⁽⁵⁾ In this case α can be every real number less than 1, because we are in the situation $g_i, g_0 \in L^\infty(Q)$.

$p_0, 2 < p_0 \leq 2^*$ ⁽⁶⁾, such that if

$$(1.22) \quad \begin{aligned} g_i &\in L^q(Q), \quad i = 1, \dots, n, \\ g_0 &\in L^q(-T, 0, L^2(\Omega)), \quad 2 \leq q < p_0, \end{aligned}$$

then $\forall Q(X_0, 2\sigma) \subset\subset Q$ with $\sigma \leq 1$

$$(1.23) \quad \begin{aligned} &\left(\int_{Q(X_0, \sigma)} \sum_i \|D_i u\|^q dX \right)^{1/q} \leq c \left(\int_{Q(X_0, 2\sigma)} \sum_i |g_i|^q dX \right)^{1/q} + \\ &+ c\sigma^{1+n(1/q-1/2)} \left\{ \int_{t_0-4\sigma^2}^{t_0} \|g_0\|_{L^2(B(x_0, 2\sigma))}^q dt \right\}^{1/q} + c\sigma^{(n+2)(1/q-1/2)} \{\Phi(X_0, 2\sigma)\}^{1/2}. \end{aligned}$$

In fact, by Hölder inequality

$$\left(\int_{Q(X_0, 2\sigma)} \sum_i |g_i|^q dX \right)^{1/q} \leq c\sigma^{(n+2)(1/q-1/p)} \left(\int_Q \sum_i |g_i|^p dX \right)^{1/p}.$$

In the same way, as $pn/(n+2) > 2$ and we can suppose $q \leq p$,

$$\left(\int_{t_0-4\sigma^2}^{t_0} \|g_0\|_{L^2(B(x_0, 2\sigma))}^q dt \right)^{1/q} \leq c\sigma^{2(1/q-1/p) + (n/2)(1-2(n+2)/pn)} \left(\int_{-T}^0 \|g_0\|_{L^{pn/(n+2)}(\Omega)}^p dt \right)^{1/p}.$$

From (1.23), as $\sigma \leq 1$, we then get

$$\left(\int_{Q(X_0, \sigma)} \sum_i \|D_i u\|^q dX \right)^{1/q} \leq c\sigma^{(n+2)(1/q-1/p)} + c\sigma^{(n+2)(1/q-1/2)} [\Phi(X_0, 2\sigma)]^{1/2}$$

and therefore (1.21).

2. Some lemmas.

We list in this section a few lemmas that will be used in the rest of the work.

We set $Q(\sigma) = Q(0, \sigma)$ and $B(\sigma) = B(0, \sigma)$.

⁽⁶⁾ $2^* = 2n/(n-2)$ is the Sobolev exponent.

LEMMA 2.I. For every $u \in L^2(-\sigma^2, 0, H^1(B(\sigma), R^N)) \cap H^{\frac{1}{2}}(-\sigma^2, 0, L^2(B(\sigma), R^N))$ the following inequality holds

$$(2.1) \quad \int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^2 dX \leq c \sigma^2 \left\{ \int_{Q(\sigma)} \sum_i \|D_i u\|^2 dX + \int_{-\sigma^2}^0 dt \int_{-\sigma^2}^0 d\xi \int_{B(\sigma)} \frac{\|u(x, t) - u(x, \xi)\|^2}{|t - \xi|^2} dx \right\}.$$

Inequality (2.1) is well known; we give the proof for the reader's convenience:

$$\begin{aligned} \int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^2 dX &\leq \int_{Q(\sigma)} dx dt \int_{Q(\sigma)} \|u(x, t) - u(y, \xi)\|^2 dy d\xi \leq \\ &\leq c \left\{ \int_{-\sigma^2}^0 dt \int_{B(\sigma)} dx \int_{B(\sigma)} \|u(x, t) - u(y, t)\|^2 dy + \right. \\ &\quad \left. + \sigma^{-2} \int_{-\sigma^2}^0 dt \int_{-\sigma^2}^0 d\xi \int_{B(\sigma)} \|u(y, t) - u(y, \xi)\|^2 dy \right\} \leq \\ &\leq c \sigma^2 \left\{ \int_{Q(\sigma)} \sum_i \|D_i u\|^2 dX + \int_{-\sigma^2}^0 dt \int_{-\sigma^2}^0 d\xi \int_{B(\sigma)} \frac{\|u(y, t) - u(y, \xi)\|^2}{|t - \xi|^2} dy \right\}. \end{aligned}$$

Let B_{ij} be $N \times N$ constant matrices which satisfy the strong ellipticity condition

$$(2.2) \quad \sum_{ij} (B_{ij} p^j | p^i) \geq v \sum_i \|p^i\|^2, \quad v > 0, \\ \forall p \in R^{nN}.$$

Let f^i belong to $L^2(Q(\sigma), R^N)$ and let $v \in L^2(-\sigma^2, 0, H^1(B(\sigma), R^N))$ be a solution of the following parabolic system

$$(2.3) \quad \int_{Q(\sigma)} \sum_{ij} (B_{ij} D_j v | D_i \varphi) - \left(v \left| \frac{\partial \varphi}{\partial t} \right| \right) dX = \int_{Q(\sigma)} \sum_i (f^i | D_i \varphi) dX, \\ \forall \varphi \in C_0^\infty(Q(\sigma), R^N).$$

LEMMA 2.II. *If hypotheses (2.2), (2.3) hold, then for every $\tau \in (0, 1)$*

$$(2.4) \quad \int_{Q(\tau\sigma)} \sum_i \|D_i v\|^2 dX \leq c \left\{ \tau^{n+2} \int_{Q(\sigma)} \sum_i \|D_i v\|^2 dX + \int_{Q(\sigma)} \sum_i \|f^i\|^2 dX \right\},$$

$$(2.5) \quad \int_{Q(\tau\sigma)} \|v - v_{\tau\sigma}\|^2 dX \leq c \left\{ \tau^{n+4} \int_{Q(\sigma)} \|v - v_\sigma\|^2 dX + \tau^2 \sigma^2 \int_{Q(\sigma)} \sum_i \|f^i\|^2 dX \right\}.$$

PROOF. In $Q(\sigma)$, $v = V + W$ where W is the solution of Cauchy-Dirichlet problem

$$(2.6) \quad \begin{aligned} & W \in L^2(-\sigma^2, 0, H_0^1(B(\sigma), R^N)), \\ & \int_{Q(\sigma)} \sum_{ij} (B_{ij} D_j W | D_i \varphi) - \left(W \left| \frac{\partial \varphi}{\partial t} \right. \right) dX = \int_{Q(\sigma)} \sum_i (f^i | D_i \varphi) dX, \\ & \forall \varphi \in W(Q(\sigma)): \varphi(x, 0) = 0 \text{ in } B(\sigma), \end{aligned}$$

whereas

$$(2.7) \quad \begin{aligned} & V \in L^2(-\sigma^2, 0, H^1(B(\sigma), R^N)), \\ & \int_{Q(\sigma)} \sum_{ij} (B_{ij} D_j V | D_i \varphi) - \left(V \left| \frac{\partial \varphi}{\partial t} \right. \right) dX = 0, \quad \forall \varphi \in C_0^\infty(Q(\sigma), R^N). \end{aligned}$$

It is known [11] that $W \in H^{\frac{1}{2}}(-\sigma^2, 0, L^2(B(\sigma), R^N))$ and

$$(2.8) \quad \int_{Q(\sigma)} \sum_i \|D_i W\|^2 dX + \int_{-\sigma^2}^0 dt \int_{B(\sigma)} d\xi \int \frac{\|W(x, t) - W(x, \xi)\|^2}{|t - \xi|^2} dx \leq c \int_{Q(\sigma)} \sum_i \|f^i\|^2 dX$$

where c is invariant with respect to the homothetical transformation

$$(2.9) \quad x = \sigma y, \quad t = \sigma^2 \xi$$

V verifies the following inequalities, as shown in [1]: $\forall \tau \in (0, 1)$

$$(2.10) \quad \int_{Q(\tau\sigma)} \sum_i \|D_i V\|^2 dX \leq c \tau^{n+2} \int_{Q(\sigma)} \sum_i \|D_i V\|^2 dX,$$

$$(2.11) \quad \int_{Q(\tau\sigma)} \|V - V_{\tau\sigma}\|^2 dX \leq c \tau^{n+4} \int_{Q(\sigma)} \|V - V_\sigma\|^2 dX,$$

here c is invariant under transformation (2.9).

(2.4) follows from (2.10) and (2.8) in a standard way. On the other hand, from (2.11) we get

$$(2.12) \quad \int_{Q(\tau\sigma)} \|v - v_{\tau\sigma}\|^2 dX \leq \\ \leq c \left\{ \tau^{n+4} \int_{Q(\sigma)} \|v - v_\sigma\|^2 dX + \int_{Q(\tau\sigma)} \|W - W_{\tau\sigma}\|^2 dX + \tau^2 \int_{Q(\sigma)} \|W - W_\sigma\|^2 dX \right\}.$$

But lemma 2.I and (2.8) imply that

$$(2.13) \quad \int_{Q(\tau\sigma)} \|W - W_{\tau\sigma}\|^2 dX + \tau^2 \int_{Q(\sigma)} \|W - W_\sigma\|^2 dX \leq c\tau^2\sigma^2 \sum_{Q(\sigma)}^i \|f^i\|^2 dX.$$

Then (2.5) follows from (2.12), (2.13).

LEMMA 2.III. *If $v \in L^2(Q(\sigma), R^N)$ then for every $\tau \in (0, 1)$*

$$(2.14) \quad \int_{Q(\tau\sigma)} \|v\|^2 dX \leq c \left\{ \tau^{n+2} \int_{Q(\sigma)} \|v\|^2 dX + \int_{Q(\sigma)} \|v - v_\sigma\|^2 dX \right\}$$

(2.14) is a trivial consequence of the following estimate

$$\int_{Q(\tau\sigma)} \|v\|^2 dX \leq c \left\{ \int_{Q(\sigma)} \|v - v_\sigma\|^2 dX + \text{meas } Q(\tau\sigma) \|v_\sigma\|^2 \right\}.$$

LEMMA 2.IV. *Let φ, ψ be non negative functions defined in $(0, \sigma]$, let α be positive, $A > 1$, B and $M \geq 0$ and suppose that $\forall \tau \in (0, 1)$ and $\forall \varrho \leq \sigma$*

$$(2.15) \quad \begin{aligned} \varphi(\tau\varrho) &\leq A\tau^\alpha\varphi(\varrho) + \psi(\varrho), \\ \psi(\tau\varrho) &\leq B\tau^\alpha\psi(\varrho) + M, \end{aligned}$$

then $\forall \tau \in (0, 1)$ and $\forall \varepsilon \in (0, \alpha)$

$$(2.16) \quad \varphi(\tau\sigma) \leq A\tau^{\alpha-\varepsilon} \{\varphi(\sigma) + KB\psi(\sigma)\} + CM$$

where K, C depend on A, α, ε .

The proof is the same as in lemma 2.IV of [4].

LEMMA 2.V. *Let φ, ω_1 defined in $(0, d]$, and ω_2 , defined in $(0, +\infty)$ be non negative and nondecreasing functions. Let A, α be positive constants and $0 \leq \beta < \alpha$. Suppose that $\forall \tau \in (0, 1)$ and $\forall \sigma \in (0, d]$*

$$(2.17) \quad \varphi(\tau\sigma) \leq \{A\tau^\alpha + \omega_1(\sigma) + \omega_2(\sigma^{-\beta}\varphi(\sigma))\} \cdot \varphi(\sigma).$$

If for a fixed $\varepsilon \in (0, \alpha - \beta)$ we can find $\sigma_\varepsilon \in (0, d]$ such that

$$(2.18) \quad \omega_1(\sigma_\varepsilon) + \omega_2(\sigma_\varepsilon^{-\beta}\varphi(\sigma_\varepsilon)) < (1 + A)^{-\alpha/\varepsilon}$$

then $\forall \tau \in (0, 1)$

$$(2.19) \quad \varphi(\tau\sigma_\varepsilon) \leq B\tau^{\alpha-\varepsilon}\varphi(\sigma_\varepsilon), \quad B = (1 + A)^{(\alpha-\varepsilon)/\varepsilon}.$$

See for example [6], lemma 1.IV.

3. The partial Hölder continuity theorem.

In this section we prove theorem 1.I.

Suppose hypotheses (1.2), (1.3), (1.4), (1.14) hold and let $u \in L^2(-T, 0, H^1(\Omega, R^N))$ be a solution in Q of system (1.1).

As

$$p > n + 2 \Rightarrow g_i \in L^2(Q), \quad i = 0, \dots, n$$

we get from (1.3), (1.4)

$$f^0(X, u, Du) \in L^2(Q, R^N) \quad \text{and} \quad f^i(X, u) \in L^2(Q, R^N), \quad i = 1, \dots, n.$$

Then [11] and a standard localizing argument (see [5], n. 4) imply that for every cylinder $Q^* = \Omega^* \times (-\lambda T, 0)$ with $\Omega^* \subset\subset \Omega$ and $\lambda \in (0, 1)$

$$u \in H^{\frac{1}{2}}(-\lambda T, 0, L^2(\Omega^*, R^N))$$

and

$$(3.1) \quad \int_{Q^*} \sum_i \|D_i u\|^2 dX + \int_{-\lambda T}^0 dt \int_{-\lambda T}^0 d\eta \int_{\Omega^*} \frac{\|u(x, t) - u(x, \eta)\|^2}{|t - \eta|^2} dx \leq \\ \leq c \left\{ 1 + \int \|u\|^2 + \sum_i \|D_i u\|^2 dX \right\}.$$

Here c depends on the $L^2(Q)$ -norms of g_i, g_0 and on the distance of Q^* from the parabolic boundary of Q ⁽⁷⁾.

Hence, by lemma 2.I, $\forall Q(X_0, \sigma) \subset \subset Q$

$$(3.2) \quad \sigma^{-2} \int_{Q(X_0, \sigma)} \|u - u_\sigma\|^2 dX \leq c \left\{ 1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 dX \right\}$$

where c depends on the $L^2(Q)$ -norms of g_i, g_0 and on the distance of $Q(X_0, \sigma)$ from the parabolic boundary of Q .

For the hypotheses we formulated on matrices $A_{ij}(X, u)$ we can find a bounded continuous function $\omega(\eta)$, defined for $\eta \geq 0$, which is increasing, concave, such that $\omega(0) = 0$ and $\forall X, Y \in \bar{Q}$ and $\forall u, v \in R^N$

$$(3.3) \quad \left\{ \sum_{ij} \|A_{ij}(X, u) - A_{ij}(Y, v)\|^2 \right\}^{\frac{1}{2}} \leq \omega(\delta^2(X, Y) + \|u - v\|^2)$$

where $\delta(X, Y)$ is the parabolic distance (1.18).

We can now prove the following lemma:

LEMMA 3.I. *If u is a solution of system (1.1) under the hypotheses (1.14), then $\forall Q(X_0, \sigma) \subset \subset Q$ with $\sigma \leq 2$, $\forall \tau \in (0, 1)$ and $\forall \lambda \in (n, \xi)$*

$$(3.4) \quad \Phi(X_0, \tau\sigma) \leq K \Phi(X_0, \sigma) \{ \tau^\lambda + \sigma^\varepsilon + [\omega(c\sigma^{-n} \Phi(X_0, \sigma))]^{1-2/r} \}$$

where

$$(3.5) \quad \varepsilon = 2 \left(1 - \frac{2}{p} \right)$$

and ω is defined as in (3.3).

PROOF. By theorem 1.II, we can find $r > 2$ such that $\forall Q(X_0, 2\sigma) \subset \subset Q$ with $\sigma \leq 1$

$$(3.6) \quad \left[\int_{Q(X_0, \sigma)} \left(\sum_i \|D_i u\|^2 \right)^{r/2} dX \right]^{2/r} \leq c \sigma^{(n+2)(2/r-1)} \Phi(X_0, 2\sigma).$$

⁽⁷⁾ $\Omega \times \{-T\} \cup \partial\Omega \times (-T, 0)$.

Consider $Q(X_0, 2\sigma) \subset\subset Q$ with $\sigma \leq 1$. For the sake of brevity we set

$$A_{ij}^0 = A_{ij}(X_0, u_{Q(X_0, \sigma)}),$$

$$a_0(u, \varphi) = \int_{Q(X_0, \sigma)} \sum_{ij} (A_{ij}^0 D_j u | D_i \varphi) - \left(u \left| \frac{\partial \varphi}{\partial t} \right. \right) dX.$$

In $Q(X_0, \sigma)$ we write $u = v + w$ where

$$w \in L^2(t_0 - \sigma^2, t_0, H_0^1(B(x_0, \sigma), R^N)),$$

$$(3.7) \quad a_0(w, \varphi) = \int_{Q(X_0, \sigma)} \sum_{ij} ([A_{ij}^0 - A_{ij}(X, u)] D_j u | D_i \varphi) dX + \int_{Q(X_0, \sigma)} (f^0 | \varphi) dX,$$

$$\forall \varphi \in W(Q(X_0, \sigma)): \varphi(x, t_0) = 0 \quad \text{in } B(x_0, \sigma),$$

whereas

$$v \in L^2(t_0 - \sigma^2, t_0, H^1(B(x_0, \sigma), R^N)),$$

$$(3.8) \quad a_0(v, \varphi) = \int_{Q(X_0, \sigma)} \sum_i (f^i | D_i \varphi) dX, \quad \forall \varphi \in C_0^\infty(Q(X_0, \sigma), R^N).$$

As

$$f^0(X, u, Du) \in L^2(Q)$$

there is one and only one solution w and

$$(3.9) \quad \int_{Q(X_0, \sigma)} \sum_i \|D_i w\|^2 dX + \int_{t_0 - \sigma^2}^{t_0} dt \int_{t_0 - \sigma^2}^{t_0} d\eta \int_{B(x_0, \sigma)} \frac{\|w(x, t) - w(x, \eta)\|^2}{|t - \eta|^2} dx \leq$$

$$\leq c \int_{Q(X_0, \sigma)} \sum_{ij} \|A_{ij}^0 - A_{ij}(X, u)\|^2 \cdot \sum_i \|D_i u\|^2 dX + c\sigma^2 \int_{Q(X_0, \sigma)} \|f^0(X, u, Du)\|^2 dX.$$

By Hölder inequality

$$\int_{Q(X_0, \sigma)} |g_0|^2 dX \leq c \left[\int_{-T}^0 dt \left(\int_{\Omega} |g_0|^{pn/(n+2)} dx \right)^{(n+2)/n} \right]^{2/p} \sigma^{2(1-2/p) + n - (2/p)(n+2)}.$$

Then, if we take into account (1.14) and if we set

$$(3.10) \quad \varepsilon = 2 \left(1 - \frac{2}{p}\right)$$

we have

$$(3.11) \quad \sigma^2 \int_{Q(X_0, \sigma)} \|f^0(X, u, Du)\|^2 dX \leq c \sigma^\varepsilon \Phi(X_0, \sigma).$$

On the other hand, from (3.3), (3.6) and the fact that ω is concave ⁽⁸⁾, we get

$$(3.12) \quad \begin{aligned} & \int_{Q(X_0, \sigma)} \sum_{ij} \|A_{ij}^0 - A_{ij}(X, u)\|^2 \cdot \sum_i \|D_i u\|^2 dX \leq \\ & \leq \left[\int_{Q(X_0, \sigma)} \left(\sum_i \|D_i u\|^2 \right)^{r/2} dX \right]^{2/r} \cdot \left[\int_{Q(X_0, \sigma)} \omega(\sigma^2 + \|u - u_\sigma\|^2) dX \right]^{1-2/r} \leq \\ & \leq c \Phi(X_0, 2\sigma) \left[\omega \left(\sigma^2 + \int_{Q(X_0, \sigma)} \|u - u_\sigma\|^2 dX \right) \right]^{1-2/r} \leq^{(9)} \\ & \leq c \Phi(X_0, 2\sigma) [\omega(c \sigma^{-n} \Phi(X_0, \sigma))]^{1-2/r}. \end{aligned}$$

From (3.9), (3.11), (3.12) and lemma 2.I we draw the conclusion that $\forall \tau \in (0, 1]$

$$(3.13) \quad \begin{aligned} & \int_{Q(X_0, \sigma)} \sum_i \|D_i w\|^2 dX + (\tau \sigma)^{-2} \int_{Q(X_0, \tau \sigma)} \|w - w_{\tau \sigma}\|^2 dX \leq \\ & \leq c \Phi(X_0, 2\sigma) \{ \sigma^\varepsilon + [\omega(c \sigma^{-n} \Phi(X_0, \sigma))]^{1-2/r} \}. \end{aligned}$$

If we use lemma 2.II, then we get the following estimate on v : $\forall \tau \in (0, 1)$ and $\varrho \leq \sigma$

$$(3.14) \quad \begin{aligned} & \int_{Q(X_0, \tau \varrho)} \sum_i \|D_i v\|^2 dX + (\tau \varrho)^{-2} \int_{Q(X_0, \tau \varrho)} \|v - v_{\tau \varrho}\|^2 dX \leq \\ & \leq c \tau^{n+2} \int_{Q(X_0, \varrho)} \sum_i \|D_i v\|^2 + \varrho^{-2} \|v - v_\varrho\|^2 dX + c \int_{Q(X_0, \varrho)} \sum_i \|f^i(X, u)\|^2 dX. \end{aligned}$$

$$^{(8)} \quad \int_{Q(X_0, \sigma)} \omega(\varphi) dX \leq \omega \left(\int_{Q(X_0, \sigma)} \varphi dX \right).$$

⁽⁹⁾ By (1.19) and the fact that $\sigma \leq 1$.

On the other and, by (1.3) and Hölder inequality

$$\int_{Q(X_0, \varrho)} \sum_i \|f^i(X, u)\|^2 dX \leq c \left\{ \varrho^\varepsilon + \int_{Q(X_0, \varrho)} \|u\|^2 dX \right\} = c\psi(X_0, \varrho).$$

Lemma 2.III implies that $\forall \tau \in (0, 1)$ and $\varrho \leq \sigma$

$$(3.15) \quad \psi(X_0, \tau\varrho) \leq c\tau^\varepsilon \psi(X_0, \varrho) + c\sigma^\varepsilon \Phi(X_0, \sigma).$$

From (3.14), (3.15) and lemma 2.IV we conclude that $\forall \lambda \in (n, \xi)$ and $\forall \tau \in (0, 1)$

$$(3.16) \quad \begin{aligned} & \int_{Q(X_0, \tau\sigma)} \sum_i \|D_i v\|^2 + (\tau\sigma)^{-2} \|v - v_{\tau\sigma}\|^2 dX \leq \\ & \leq c\tau^\lambda \int_{Q(X_0, \sigma)} \sum_i \|D_i v\|^2 + \sigma^{-2} \|v - v_\sigma\|^2 dX + c\Phi(X_0, \sigma)\{\tau^\lambda + \sigma^\varepsilon\}. \end{aligned}$$

As $u = v + w$, from (3.13), (3.16) we get by a standard argument that $\forall \tau \in (0, 1)$

$$(3.17) \quad \begin{aligned} \Phi(X_0, \tau\sigma) - \psi(X_0, \tau\sigma) & \leq \\ & \leq c\Phi(X_0, 2\sigma)\{\tau^\lambda + \sigma^\varepsilon + [\omega(c\sigma^{-n}\Phi(X_0, 2\sigma))]^{1-2/r}\}. \end{aligned}$$

The previous inequality is trivial for $1 \leq \tau < 2$ and we can add $\psi(X_0, \tau\sigma)$ to the left hand side because by (3.15)

$$\psi(X_0, \tau\sigma) \leq c\Phi(X_0, \sigma)\{\tau^\lambda + \sigma^\varepsilon\}.$$

Therefore we have proved (3.4).

We define

$$(3.18) \quad Q_0 = \left\{ X \in Q : \minlim_{\sigma \rightarrow 0} \sigma^{-n} [\Phi(X, \sigma) - \sigma^\varepsilon] > 0 \right\}.$$

For a well known theorem (Lebesgue)

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \int_{Q(X, \sigma)} \|u(Y) - u_\sigma\|^2 dY &= 0 \quad \text{a.e. in } Q, \\ \lim_{\sigma \rightarrow 0} \sigma^{-n} \int_{Q(X, \sigma)} \|u(Y)\|^2 + \sum_i \|D_i u(Y)\|^2 dY &= 0 \quad \text{a.e. in } Q \end{aligned}$$

and then

$$\text{meas. } Q_0 = 0 .$$

We can even say something more. We define the Hausdorff measure $\mathcal{M}_\alpha$ with respect to the metric δ , as usual,

$$(3.19) \quad \mathcal{M}_\alpha(E) = \liminf_{\sigma \rightarrow 0} \left\{ \sum_i \delta^\alpha(E_i) : \bigcup_i E_i \supset E \text{ and } \delta(E_i) < \sigma \right\}$$

where $\delta(E_i)$ is the diameter of E_i with respect to δ . Then, if we argue as in [9] ⁽¹⁰⁾, we can show that

$$(3.20) \quad \mathcal{M}_n(Q_0) = 0 .$$

LEMMA 3.II. — *If u is a solution of system (1.1) and hypotheses (1.2), (1.3), (1.4), (1.14) are satisfied, then $\forall X_0 \in Q \setminus Q_0$ and $\forall \eta \in (0, \xi - n)$ we can find $\sigma_\eta < 1$ and $r > 0$, with $Q(X_0, r + \sigma_\eta) \subset Q$, such that $\forall Y \in Q(X_0, r)$ and $\forall \tau \in (0, 1)$*

$$(3.21) \quad \Phi(Y, \tau \sigma_\eta) \leq c \tau^{\xi - \eta} \Phi(Y, \sigma_\eta) .$$

In particular, Q_0 is closed in Q .

PROOF. Having chosen $X_0 \in Q \setminus Q_0$, we define ⁽¹¹⁾

$$(3.22) \quad \begin{aligned} \omega_1(t) &= K t^\xi , \\ \omega_2(t) &= K [\omega(ct)]^{1-2/r} , \\ G(X_0, \sigma) &= \omega_1(\sigma) + \omega_2(\sigma^{-n} \Phi(X_0, \sigma)) . \end{aligned}$$

As $X_0 \in Q \setminus Q_0$

$$(3.23) \quad \minlim_{\sigma \rightarrow 0} G(X_0, \sigma) = 0 .$$

Having fixed $\eta \in (0, \xi - n)$, we choose $\lambda = \xi - \eta/2$ and $\eta_0 = \eta/2$.

⁽¹⁰⁾ Proof of Theorem 2.

⁽¹¹⁾ K is the constant which appears (3.4).

Therefore

$$\lambda \in (n, \xi), \quad \eta_0 \in (0, \lambda - n), \quad \lambda - \lambda_0 = \xi - \eta.$$

By (3.23) we can find $\sigma_\eta < 1$ such that $Q(X_0, \sigma_\eta) \subset\subset Q$ and

$$(3.24) \quad G(X_0, \sigma_\eta) < (1 + K)^{-\lambda/\eta_0}.$$

As $Y \rightarrow G(Y, \sigma_\eta)$ is continuous in Q , we can find r such that $Q(X_0, r + \sigma_\eta) \subset Q$ and

$$(3.25) \quad G(Y, \sigma_\eta) < (1 + K)^{-\lambda/\eta_0}, \quad \forall Y \in Q(X_0, r).$$

Then, $\forall Y \in Q(X_0, r)$ and $\forall \sigma \leq 1$ such that $Q(Y, \sigma) \subset\subset Q$, inequality (3.1) and condition (3.25) hold; therefore hypotheses of lemma 2.IV are satisfied with

$$\begin{aligned} \varphi(\sigma) &= \Phi(Y, \sigma), \\ \alpha &= \lambda, \quad \beta = n, \quad \varepsilon = \eta_0, \\ \omega_1, \omega_2 &\text{ are defined as in (3.22).} \end{aligned}$$

Hence $\forall \tau \in (0, 1)$ and $\forall Y \in Q(X_0, r)$

$$(3.26) \quad \Phi(Y, \tau\sigma^\eta) \leq c\tau^{\lambda-\eta_0}\Phi(Y, \sigma_\eta) = c\tau^{\xi-\eta}\Phi(Y, \sigma_\eta).$$

In particular

$$\lim_{\sigma \rightarrow 0} \sigma^{-n} \Phi(Y, \sigma) = 0.$$

Therefore

$$X_0 \in Q \setminus Q_0 \Rightarrow Q(X_0, r) \subset Q \setminus Q_0$$

which means that $Q \setminus Q_0$ is open and so Q_0 is closed in Q .

Now the partial Hölder continuity theorem easily follows from lemma 3.II. In fact, recalling the definition of Φ , from (3.21) we deduce that, if $X_0 \in Q \setminus Q_0$, $Y \in Q(X_0, r)$ and $\tau \in (0, 1)$, then

$$\int_{Q(Y, \tau\sigma_\lambda)} \|u - u_{\tau\sigma_\lambda}\|^2 dX \leq c\tau^{\lambda+2} \Phi(Y, \sigma_\lambda), \quad \forall n < \lambda < \xi.$$

By (3.2)

$$\Phi(Y, \sigma_\lambda) \leq c \left\{ 1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 dX \right\}.$$

Therefore $\forall Y \in Q(X_0, r)$ and $\tau \in (0, 1)$

$$\int_{Q(Y, \tau\sigma_\lambda)} \|u - u_{\tau\sigma_\lambda}\|^2 dX \leq c\tau^{\lambda+2} \left\{ 1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 dX \right\}.$$

By [8], the previous inequality implies that $u \in C^{0,\alpha}(\overline{Q(X_0, r)})$ ⁽¹²⁾
 $\forall \alpha < 1 - (n+2)/p$ and

$$[u]_{\alpha, \overline{Q(X_0, r)}}^2 \leq c \left\{ 1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 dX \right\}.$$

This proves the theorem.

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⁽¹²⁾ Hölder continuity with respect to the metric δ defined in (1.18).

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