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# A Completeness Theorem for the General Interpreted Modal Calculus $MC^\nu$ of A. Bressan.

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SUMMARY - A new semantics is introduced for Bressan's modal calculus $MC^\nu$ based on types of all finite levels. By this semantics we extend a completeness theorem of Zane Parks concerning the first order segment of $MC^\nu$ deprived of the description operator, to a completeness theorem for (possibly contingent) modal theories based on the full calculus $MC^\nu$.

## 1. Introduction.

In [11] Zane Parks gives a completeness theorem for the first order part of Bressan's calculus $MC^\nu$ deprived of descriptions. In this paper we extend Z. Parks' theorem to $MC^\nu$ itself (and every theory based on it, i.e. every $MC^\nu$-theory) treating types and descriptions too. Unlike [11] this paper deals also with contingent (i.e. not modally closed) theories based on $MC^\nu$. This is achieved by identifying particular semantical structures which are sound for the definition of $\mathscr{T}$-validity, where $\mathscr{T}$ is an arbitrary $MC^\nu$-theory.

The calculus $MC^\nu$ is based on the modal language $ML^\nu$. In [1] a semantics for $ML^\nu$ is introduced: starting from $\nu$ sets $D_1$ to $D_\nu$ of (typed) individuals and a set $\Gamma$ of elementary possible cases (elsewere called worlds, or points), for every type $t$ it is defined the set $QI_t$ of quasi intensions—briefly $QIs$—of type $t$ on which variables and constants of this type can be valued.

More precisely, $QI_i$, the set (of individual concepts) on which individual variables (of type $i$) run, is $(\Gamma \to D_i)$ $(i = 1, \ldots, \nu)$; $QI_{(t_1,\ldots,t_n)}$,

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on which relational variables (of type $(t_1, \ldots, t_n)$) run, is $(\Gamma \to \mathscr{P}(QI_{t_1} \times \times \ldots \times QI_{t_n}))$; and $QI_{(t_1,\ldots,t_n:t_0)}$, on which functional variables (of type $(t_1, \ldots, t_n : t_0)$) run, is $(QI_{t_1} \times \ldots \times QI_{t_n} \to QI_{t_0})$ (1).

Those interpretations for $ML^\nu$ which are based on a structure of quasi intensios as the above one will be called *standard interpretations*; in these interpretations every $QI_t$ is uniquely determined by the choice of the sets $D_1$ to $D_\nu$ and $\Gamma$.

Following [7], our definition of $\mathscr{T}$-validity shall refer to a wider class of interpretations, the so-called general interpretations. In general interpretations the set of objects with a certain type is not uniquely determined by the sets of objects of lower type-level; for instance, if $O_t$ and $O_{t'}$ are the sets of the objects of type $t$ and $t'$ respectively, then $O_{(t:t')}$ will be an arbitrary subset of $(O_t \to O_{t'})$ (closed with respect to definable functions). In particular, the set of individual concepts of type $r$ will be an arbitrary subset of $(\Gamma \to D_r)$, like in [11].

The proof of the completeness theorem for $MC^\nu$-theories is an Henkin type proof; i.e., starting from a consistent set $K$ of formulas, a general interpretation defined by means of linguistic entities is constructed in which $K$ is satisfiable. Such a general interpretation is denumerable and hence a form of the Löwenheim-Skolem theorem holds.

Let us remark that the semantics introduced in this way is essentially non-extensional, i.e. the extension of a $QI$ $\xi$ in a possible case $\gamma$, in general, does not depend only on the extension (in $\gamma$) of the parts of $\xi$, but on the whole intension of them (for more details, see N. Belnap's foreword to [1], or [3]).

At the end of N. 9 some hints are briefly given for the proof of a completeness theorem for contingent theories based on $MC^\nu$, i.e. theories whose proper axioms are arbitrary (possibly not modally closed). Of course, in the interpretations of such theories a particular elementary case is privileged, the so-called real case. Contingent theories are very important in view of axiomatizations of physical theories. Indeed, only some of these admit modally closed axiomatization in $MC^\nu$; in astronomy, for instance, contingent axioms are needed—see [2].

Let us remark that in [1] NN. 52, 53 the (modally closed) calculus $MC^\nu_\varrho$ is introduced by which contingent theories can be investigated. In view of this possibility, the completeness theorem for modally

(1) If $A_0$ to $A_n$ are sets, we denote their cartesian product by $A_0 \times A_1 \times \times \ldots \times A_n$, the class of subset of $A_0$ by $\mathscr{P}A_0$, and the set of mappings of $A_1 \times \ldots \times A_n$ into $A_0$ by $A_1 \times \ldots \times A_n \to A_0$. Furthermore, we denote the empty set by $\emptyset$.

closed theories covers all possibilities. However we shall consider contingent theories indipendently because they seem interesting at least from a formal point of view.

## 2. The modal language $ML^\nu$.

The modal language $ML^\nu$ is based on a type system $\tau^\nu$ that contains $\nu$ *individual types*: $1, \ldots, \nu$. We denote $n$-tuples by $\langle \ldots \rangle$ and define $\tau^\nu$ recursively by

(*a*) $\{1, \ldots, \nu\} \subset \tau^\nu$ and

(*b*) if $t_1, \ldots, t_n \in \tau^\nu$ and $t_0 \in \tau^\nu \cup \{0\}$, then $\langle t_1, \ldots, t_n, t_0 \rangle \in \tau^\nu$.

We call *types* the elements of $\tau^\nu \cup \{0\}$ $(= \bar{\tau}^\nu)$. A type $t$ of the form $\langle t_1, \ldots, t_n, t_0 \rangle$ with $t_0 = 0$ $[t_0 \neq 0]$ is called (and used as) a *relation* [*function*] *type* and, following Carnap, is denoted by $(t_1, \ldots, t_n)$ $[(t_1, \ldots, t_n : t_0)]$.

The symbols of the modal language $ML^\nu$ are the variables $v_{tn}$ and the constants $c_{tn}$ $\big($where $n \in Z^+$ $(= \{1, 2, \ldots\})$ and $t \in \tau^\nu\big)$, « $\sim$ » (not), « $\wedge$ » (and), « $\Box$ » (necessarily), « $=$ » (identity), reversed iota « $\imath$ » for descriptions, the comma, and the parentheses.

The class $\mathscr{E}_t$ of the designators or wfes (*well formed expressions*) of type $t$ $(\in \bar{\tau}^\nu)$ for $ML^\nu$ is defined recursively by the following (formation) rules $(f_1)$ to $(f_8)$ where $n$ $[t_0]$ runs over $Z^+$ $[\bar{\tau}^\nu]$ and $t, t_1, \ldots, t_n$ run over $\tau^\nu$.

$(f_1)$ $v_{tn}, c_{tn} \in \mathscr{E}_t$;

$(f_2)$ if $\Delta, \Delta' \in \mathscr{E}_t$, then $\Delta = \Delta' \in \mathscr{E}_0$;

$(f_r)$ if $\Delta_1 \in \mathscr{E}_{t_1}, \ldots, \Delta_n \in \mathscr{E}_{t_n}$ and $\Delta \in \mathscr{E}_{\langle t_1, \ldots, t_n, t_0 \rangle}$, then $(\Delta(\Delta_1, \ldots, \Delta_n)) \in \mathscr{E}_{t_0}$;

$(f_4)$ if $\Delta \in \mathscr{E}_0$, then $((\imath v_{tn})\Delta) \in \mathscr{E}_t$;

$(f_{5\text{-}8})$ if $\Delta, \Delta' \in \mathscr{E}_0$, then $(\sim\Delta)$, $(\Delta \wedge \Delta')$, $((v_{tn})\Delta)$, $(\Box\Delta) \in \mathscr{E}_0$.

The connectives $\vee$, $\supset$, and $\equiv$, $\exists$, and $\Diamond$ (it is possible) are understood to be introduced in the usual way; furthermore we use $(\forall x_1, \ldots, x_n)p$ and $(\exists x_1, \ldots, x_n)p$ as metalinguistic abbreviations of $(x_1) \ldots (x_n)p$ and $\sim (\forall x_1, \ldots, x_n) \sim p$ respectively. In order to drop parentheses we consider $(\imath x)$, $(x)$, $\sim$, $\wedge$, $\vee$, $\supset$, and $\equiv$ as having decreasing cohesive powers and we use also dots to devide expressions.

A wff (*well formed formula*) $p$ is said to be *modally closed* if it is contructed starting out from some wffs $\Box p_1, \ldots, \Box p_n$ by means of $\sim$, $\wedge$, $(v_{tn})$, and $\Box$. The *modal closure* of $p$ is $p$ or $\Box p$ according to whether $p$ is modally closed or not. The modal closure of the (*extensional*) *closure* $(\forall x_1, \ldots, x_n)p$ of $p$ is called the *total closure* of $p$.

If $a$ and $x$ are respectively a term and a variable of the same type, then we denote by $\Delta[x/a]$ the result of substituting occurrences of $a$ for free occurrences of $x$ in the wfe $\Delta$ after having performed changes of bound variables in it in order to obtain $a$ free for $x$ in (an « equivalent » of) $\Delta$.

Using the above convention we put

$$(2.1) \qquad (\exists_1 x)p \equiv_D (\exists x)(p \wedge (y)(p[x/y] \supset x = y)) .$$

Furthermore we assume that every expression used in what follows has a type, i.e. it is well formed. This will make several explanations unnecessary. For instance, if we speak of the term $\Delta(\Delta')$, where $\Delta \in \mathscr{E}_{(t':t)}$, this implies $\Delta' \in \mathscr{E}_{t'}$.

## 3. $ML^\nu$-interpretations.

In [1] N. 6 a semantical system is introduced for $ML^\nu$: $\nu + 1$ nonempty sets $D_1, \ldots, D_\nu$, and $D_{\nu+1} = \Gamma$ are fixed (for $i = 1, \ldots, \nu$, $D_i[\Gamma]$ is called the $i$-th individual domain [the set of elementary possible cases or $\Gamma$-cases]) and the class $QI_t$ of the quasi-intensions—briefly $QIs$—of type $t$ $(\in \bar{\tau}^\nu)$ based on $D_1$ to $D_{\nu+1}$ is defined recursively by the conditions (3.1-3) below $(n \in Z^+;\ t_0, t_1, \ldots, t_n \in \tau^\nu)$;

$$(3.1) \qquad QI_0 = \mathscr{P}(\Gamma) , \qquad QI_r = (\Gamma \to D_r) \qquad (r = 1, \ldots, \nu);$$

$$(3.2) \qquad QI_{(t_1,\ldots,t_n)} = \mathscr{P}(QI_{t_1} \times \ldots \times QI_{t_n} \times \Gamma) \ (= (\Gamma \to \mathscr{P}(QI_{t_1} \times \ldots \times QI_{t_n})));$$

$$(3.3) \qquad QI_{(t_1,\ldots,t_n:t_0)} = (QI_{t_1} \times \ldots \times QI_{t_n} \to QI_{t_0}) .$$

Furthermore a function $a^\nu$, of domain $\tau^\nu$, is considered such that $a_t^\nu =_D a^\nu(t) \in QI_t$; $a_t^\nu$ is called the *non-existing object* of type $t$ $(\in \tau^\nu)$ because it serves to give a designatum to descriptions in the $\Gamma$-cases in which they do not fulfil their conditions of exact uniqueness. In [1] (p. 19) $a^\nu_{(t_1,\ldots,t_n)}$ is assumed to be the empty set, and in addition the counterdomain of $a^\nu_{(t_1,\ldots,t_n:t_0)}$ is assumed to be $\{a^\nu_{t_0}\}$. These (natural) conditions are conventional and we can omit them.

DEF. 3.1. *In connection with the above sets $D_1$ to $D_{\nu+1}$ and function $a^\nu$ we assume $\mathcal{Q}\mathcal{I}_t$ $(t \in \bar{\tau}^\nu)$ to fulfil* (3.1′-3′) *where* (3.i′) *is what* (3.i) *becomes if we substitute $QI_t$ with $\mathcal{Q}\mathcal{I}_t$ and the equality sign* « = » *with the inclusion sign* « ⊆ ». *If $a^\nu_t \in \mathcal{Q}\mathcal{I}_t$ for every $t \in \tau^\nu$, then we put*

$$D = \{\mathcal{Q}\mathcal{I}_t \colon t \in \bar{\tau}^\nu\}\,, \qquad \mathcal{D} = \langle D, a^\nu\rangle\,, \tag{3.4}$$

*and we say that $D$ is a QI-structure and $\mathcal{D}$ a QI-system.*

An *interpretation for $ML^\nu$* (briefly, an *$ML^\nu$-interpretation*) is then an ordered pair $\mathcal{M} = \langle \mathcal{D}, M\rangle$ where $\mathcal{D}$ is a $QI$-system and $M$ is a valuation of the constants of $ML^\nu$, that is a function which, for all $t \in \tau^\nu$, assigns an element of $\mathcal{Q}\mathcal{I}_t$ to every constant of type $t$. If $V$ is any valuation of the variables of $ML^\nu$ on the $ML^\nu$-interpretation $\mathcal{M}$ (briefly, $V$ is an *$\mathcal{M}$-valuation*), then the ordered pair $\mathcal{S} = \langle \mathcal{M}, V\rangle$ will be said an *$ML^\nu$-system*. As usual, if $V$ and $V'$ are two valuation such that $V(x) = V'(x)$ for $x \notin \{x_1, \dots, x_n\}$ and $V'(x_i) = \xi_i$, then we denote $V'$ by $V\begin{pmatrix} x_1, \dots, x_n \\ \xi_1, \dots, \xi_n \end{pmatrix}$.

Every $QI$-system, or $ML^\nu$-interpretation, or $ML^\nu$-system, based on a $QI$-structure in which $\mathcal{Q}\mathcal{I}_t = QI_t$ for all $t \in \bar{\tau}^\nu$, will be called *standard* (2).

Furthermore, in order to introduce a semantics for theories having some contingent axioms, we consider *contingent $ML^\nu$*-interpretations and systems; these are ordered pairs $\langle \mathcal{M}, \gamma_R\rangle$ and $\langle \mathcal{S}, \gamma_R\rangle$ respectively, where $\gamma_R$ is an arbitrarily fixed possible case, the so-called *real case*.

DEF. 3.2. *For $\gamma \in \Gamma$ and $\xi, \eta \in \mathcal{Q}\mathcal{I}_t$ with $t \in \bar{\tau}^\nu$ we say that $\xi$ and $\eta$ are equivalent QIs of type $t$ in the case $\gamma$ (with respect to the QI-structure $D$), and we write*

$$\xi =^t_{D,\gamma} \eta \qquad (\text{or } \xi =^t_\gamma \eta)\,, \tag{3.5}$$

*if one of the following conditions holds:*

(*a*) *$t \in \{1, \dots, \nu\}$ or $t = (t_1, \dots, t_n)$, and $\xi(\gamma) = \eta(\gamma)$;*

(*b*) *$t = (t_1, \dots, t_n \colon t_0)$ and $\xi(\xi_1, \dots, \xi_n) =^{t_0}_\gamma \eta(\xi_1, \dots, \xi_n)$ for all n-tuples $\langle \xi_1, \dots, \xi_n\rangle \in (Q\mathcal{I}_{t_1} \times \dots \times Q\mathcal{I}_{t_n})$;*

(*c*) *$t = 0$ and $\xi \cap \{\gamma\} = \eta \cap \{\gamma\}$.*

(2) If $D(= \{QI_t \colon t \in \bar{\tau}^\nu\})$ and $D'(= \{\mathcal{Q}\mathcal{I}_t \colon t \in \bar{\tau}^\nu\})$ are two $QI$-structures based on the same sets $D_1, \dots, D_\nu$, and $\Gamma$, and $D$ is standard, then for $t \in \{0, 1, \dots, \nu\}$, $\mathcal{Q}\mathcal{I}_t$ is a subset of $QI_t$ and, in general, $\mathcal{Q}\mathcal{I}_t$ can be embedded in $QI_t$. However, for the sake of simplicity, we shall write $\mathcal{Q}\mathcal{I}_t \subseteq QI_t$ for all $t$.

The proof of Theor. 10.2 in [1], concerning standard $QI$-structures, can be trivially adapted to demonstrate the following theorem in which $D$ is an arbitrary $QI$-structure.

THEOR. 3.1. *Let* $t \in \bar{\tau}^\nu$ *and* $\xi, \eta \in Q\mathscr{I}_t$ $(\in D)$; *then* $\xi =_{D,\gamma} \eta$ *for every* $\gamma$ *iff* $\xi = \eta$.

## 4. Designation rules for $ML^\nu$ in connection with $ML^\nu$-systems.

For every $ML^\nu$-system $\mathscr{S}$ $(= \langle \mathscr{M}, V \rangle)$ or contingent $ML^\nu$-system $\langle \mathscr{S}, \gamma_R \rangle$ we associate every $\Delta \in \mathscr{E}_t$ with an *intensional designatum* $\tilde{\Delta}$ in $QI_t$. This designatum is unique, as will appear from the nature of the rules. Hence we denote it by $\mathrm{des}_{\mathscr{S}}(\Delta)$ or $\mathrm{des}_{\mathscr{M}V}(\Delta)$. We define it recursively by the rules $(d_1)$ to $(d_9)$ below which are extensions to $ML^\nu$-systems of the rules $(\delta_1)$ to $(\delta_9)$ in [1], NN. 8, 11. For the sake of simplicity the equalities $\tilde{\Delta}_i = \mathrm{des}_{\mathscr{S}}(\Delta_i)$ $(i = 0, 1, \dots, n)$, and $\tilde{R} = \mathrm{des}_{\mathscr{S}}(R)$, are assumed in the following table; furthermore, the afore-mentioned recursion consists of an induction on the number $\nu_\Delta$ of occurrences of $\imath$ in $\Delta$ and, in connection with a given value of $\nu_\Delta$, of an induction on the length $l_\Delta$ of $\Delta$.

| Rule | If $\Delta$ is | Then $\tilde{\Delta}(= \mathrm{des}_{\mathscr{S}}(\Delta))$ is |
|---|---|---|
| $(d_1)$ | $v_{tn}$ or $c_{tn}$ | $V(v_{tn})$ or $M(c_{tn})$, respectively; |
| $(d_2)$ | $\Delta_1 = \Delta_2(\Delta_1, \Delta_2 \in \mathcal{E}_t)$ | $\{\gamma \in \Gamma: \tilde{\Delta}_1 =^t_\gamma \tilde{\Delta}_2\}$; |
| $(d_3)$ | $R(\Delta_1, \dots, \Delta_2)(\in \mathcal{E}_0)$ | $\{\gamma \in \Gamma: \langle \tilde{\Delta}_1, \dots, \tilde{\Delta}_n, \gamma \rangle \in \tilde{R}\}$; |
| $(d_4)$ | the term $\Delta_0(\Delta_1, \dots, \Delta_n)$ | $\tilde{\Delta}_0(\tilde{\Delta}_1, \dots, \tilde{\Delta}_n)$; |
| $(d_{5\text{-}6})$ | $\sim \Delta_1$ or $\Delta_1 \wedge \Delta_2$ | $\Gamma - \tilde{\Delta}_1$ or $\tilde{\Delta}_1 \cap \Lambda_2$, respectively; |
| $(d_7)$ | $(x)\Delta_1$ where $x$ is $v_{tn}$ | $\bigcap_{\xi \in \mathscr{Q}\mathscr{I}_t} \mathrm{des}_{\mathscr{M}V'}(\Delta_1)$, where $V' = V\begin{pmatrix} x \\ \xi \end{pmatrix}$; |
| $(d_8)$ | $\Box \Delta_1$ | $\Gamma$ if $\tilde{\Delta}_1 = \Gamma$, $\emptyset$ otherwise; |
| $(d_9)$ | $(\imath x)\Delta_1$ where $x$ is $v_{tn}$ | the element $\eta$ of $QI_t$ (possibly not in $\mathscr{Q}\mathscr{I}_t$) that fulfils conditions $(\alpha)$ and $(\beta)$ below. |

$(\alpha)$ If $\gamma \in \mathrm{des}_{\mathscr{S}}(\sim (\exists_1 x)\Delta_1)$, then $\eta = {}^t_{D,\gamma} a^\nu_t$.

$(\beta)$ If $\gamma \in \mathrm{des}_{\mathscr{S}}((\exists_1 x)\Delta_1)$, $\xi \in \mathscr{Q}\mathscr{I}_t$, and $\gamma \in \mathrm{des}_{\mathscr{M}V'}(\Delta_1)$ for $V' = V\begin{pmatrix} x \\ \xi \end{pmatrix}$, then $\eta = {}^t_{D,\gamma} \xi$.

Remark first that in $(d_7)$ the intersection is considered for $\xi \in \mathscr{Q}\mathscr{I}_t$ and not for $\xi \in QI_t$. Second, remark that $\mathrm{des}_{\mathscr{S}}(\Delta)$ may fail to be in $\mathscr{Q}\mathscr{I}_t$, which is unsatisfactory; however we are almost exclusively interested in general interpretations—see N. 6—in which, as we shall show, $\mathrm{des}_{\mathscr{S}}(\Delta)$ is always in $\mathscr{Q}\mathscr{I}_t$ and hence the fact that $\mathrm{des}_{\mathscr{S}}(\Delta)$ does not necessarily belong to $\mathscr{Q}\mathscr{I}_t$ constitues no trouble for us.

For $\mathscr{S}$ standard, $\mathrm{des}_{\mathscr{S}}(\Delta)$ is the quasi intensional designatum of $\Delta$ according to NN. 8, 11 in [1]; in particular Theor. 11.1 in [1] holds. It is straightforward to check that this theorem can be extended to every $ML^{\nu}$-system:

THEOR. 4.1. *For every choice of $\mathscr{M}$ and $V$, conditions* ($\alpha$) *and* ($\beta$) *are fulfilled by exactly one $\eta \in QI_t$.*

DEF. 4.1. *Let $\langle \mathscr{M}, \gamma_R \rangle$ be a contingent $ML^{\nu}$-interpretation, $p$ a wff, and $K$ a class of wffs. We say that*

($a$) *$p$ $[K]$ is $\gamma$-satisfiable in $\langle \mathscr{M}, \gamma_R \rangle$ (or $\mathscr{M}$) if, for some $\mathscr{M}$-valuation $V$, $\gamma \in \mathrm{des}_{\mathscr{M}V}(p)$ $\left[\gamma \in \bigcap_{q \in K} \mathrm{des}_{\mathscr{M}V}(q)\right]$;*

($b$) *$p$ is $\gamma$-true in $\langle \mathscr{M}, \gamma_R \rangle$* (or *$\mathscr{M}$*) *if, for every $\mathscr{M}$-valuation $V$, $\gamma \in \mathrm{des}_{\mathscr{M}V}(p)$; ($b'$) $\langle \mathscr{M}, \gamma_R \rangle$* (or *$\mathscr{M}$*) *is a $\gamma$-model for $K$ if every formula in $K$ is $\gamma$-true in $\langle \mathscr{M}, \gamma_R \rangle$;*

($c$) *$p$ is true in $\langle \mathscr{M}, \gamma_R \rangle$ $[\mathscr{M}]$ if it is $\gamma_R$-true [$\gamma$-true for every $\gamma \in \Gamma$]; ($c'$) $\langle \mathscr{M}, \gamma_R \rangle$ $[\mathscr{M}]$ is a model for $K$ if every $p$ in $K$ is true in $\langle \mathscr{M}, \gamma_R \rangle$ $[\mathscr{M}]$.*

The following theorem is obviously true.

THEOR. 4.2. *If* (1) *$x$ is $v_{tn}$, $\Delta$ is a wfe, and $a$ is a term of type $t$,* (2) *$V$ is an $\mathscr{M}$-valutation, and* (3) *$\xi = \mathrm{des}_{\mathscr{M}V}(a)$ and $V' = V\binom{x}{\xi}$, then* $\mathrm{des}_{\mathscr{M}V}(\Delta[x/a]) = \mathrm{des}_{\mathscr{M}V'}(\Delta)$.

## 5. An axiom system for the modal calculus $MC^{\nu}$ based on $ML^{\nu}$.

The axiom schemes A5.1-17 below for $MC^{\nu}$ are written following more [4] than [1]. For them we assume that (1) $p$ and $q$ are wffs, (2) $\Delta$ is a term, and (3) $x, y, z, x_1$ to $x_n$, $F, G, f$, and $g$ are distinct variables.

A5.1-3 AS12.1-3 in [$B$] that are equivalent to tautologies.

A5.4 $(x)(p \supset q) \supset .(x)p \supset (x)q$ .

A5.5 $\Box (p \supset q) \supset .\Box p \supset \Box q$ .

A5.6 $p \supset (x)p$, where $x$ is not free in $p$.

A5.7 $p \supset \Box p$, where $p$ is modally closed.

A5.8 $(x)p \supset p[x/\Delta]$ .

A5.9 $\Box p \supset p$ (3) .

A5.10, 11 $x = x$; $x = y \wedge y = z \supset x = z$ .

A5.12 $\Box x = y \supset \Delta[z/x] = \Delta[z/y]$ .

A5.13 $F = G \equiv (\forall x_1, \ldots, x_n).F(x_1, \ldots, x_n) \equiv G(x_1, \ldots, x_n)$ .

A5.14 $f = g \equiv (\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ .

A5.15 $(\exists F)(\forall x_1, \ldots, x_n).p \equiv F(x_1, \ldots, x_n)$ .

A5.16 $(\exists f)\ (\forall x_1, \ldots, x_n) \Delta = f(x_1, \ldots, x_n)$ .

A5.17 (a) $(\exists_1 x)p \wedge p[x/y] \supset y = (\imath x)p$;
(b) $\sim (\exists_1 x)p \supset (\imath x)p = a_t^*$, where $x$ is $v_{tn}$ and
$a_t^* =_D (\imath v_{tn}) v_{tn} \neq v_{tn}$ .

In addition to A5.1-17, we assume:

(*) *if $p$ is an axiom of $MC^\nu$, then $(x)p$ and $\Box p$ are axioms of $MC^\nu$, for every variable $x$ of $ML^\nu$.*

E. Omodeo has proved in [10] that descriptions can be eliminated from $MC^\nu$ according to the Russel method, only provided we replace $MC^\nu$ with the calculus $MC_\imath^\nu$ which has those among the axioms above that do not contain $\imath$, and has an additional axiom (no such axioms

(3) A5.5, A5.7, and A5.9 tell us that the modal calculus $MC^\nu$ is based on the Lewis system S5—see [9]. The semantical counterpart of this features is in the designation rule ($d_8$) in N4.

are wanted in extensional logic). The last axiom can be

$$A5.17' \quad (z)(\exists y)\,\square\,[(\exists x)p \wedge (x)(p \supset x = y) . \vee . \sim(\exists_1 x)p \wedge y = z] \,.$$

In [1] further axioms are considered for $MC^\nu$; there are, for instance, the axiom of choice, an axiom asserting the existence of a contingent attribute, and two conventional axioms which are the syntactical counterpart of the conventional conditions on the non-existing object. However we prefer to consider the « minimal » version of $MC^\nu$ (i.e. that based on A5.1-17) and hence to enclose other versions, like that considered in [1], in the wider concept of $MC^\nu$-theory.

We say that the theory $\mathscr{T}$ is an *$MC^\nu$-theory* if $\mathscr{T}$ has the symbols of $ML^\nu$ except some (perhaps all) constants, and the axioms of $\mathscr{T}$ are those of $MC^\nu$—to be called *logical axioms*—and other wffs to be called *proper axioms*. An $MC^\nu$-theory $\mathscr{T}$ is said to be *modally closed* if such are its proper axioms; otherwise $\mathscr{T}$ is said to be *contingent*.

The only deduction rule in $MC^\nu$ (and $MC^\nu$-theories) is the Modus Ponens. The definitions of wffs *deducible from $K$ in $\mathscr{T}$* ($K \vdash_{\mathscr{T}}$), and *theorems of $\mathscr{T}$* ($\vdash_{\mathscr{T}}$) are as usual; furthermore we will omit the subscript $\mathscr{T}$ in $\vdash_{\mathscr{T}}$ when no confusion can arise.

It is very easy to realize that, if $s_1, \ldots, s_n$ is any string of modal quantifiers (that is, $s_i$ is $\square$ or $\diamond$) and $\mathscr{T}$ [$\mathscr{T}'$] is an arbitrary [a modally closed] $MC^\nu$-theory, then

$$(5.1) \qquad \vdash_{\mathscr{T}} s_1, \ldots, s_n p \equiv s_n p \qquad \text{and} \qquad K \vdash_{\mathscr{T}} p \Rightarrow \{(x)q \colon q \in K\} \vdash_{\mathscr{T}} (x)p\,;$$

$$(5.2) \qquad K \vdash_{\mathscr{T}'} p \Rightarrow \{\square q \colon q \in K\} \vdash_{\mathscr{T}'} \square p \qquad \text{and} \qquad \vdash_{\mathscr{T}'} (x)\,\square p \equiv \square(x)p\,.$$

Some contingent assertions concerning the real world—such as « at the instant $t$ the earth has angular velocity $w$ »—constitute some postulate of e.g. astronomy. The easiest way of treating such postulates is to give them modally closed forms by use of the calculus $MC^\nu_\varrho$—see NN. 52, 53 in [1]—which is also an $MC^\nu$-theory. On the basis of this remark first a completeness theorem will be proved in connection with modally closed theories; then it will be extended to contingent theories, for greater (admittely formal) generality.

A (contingent or not) $ML^\nu$-interpretation in which the axioms of the $MC^\nu$-theory $\mathscr{T}$ are true is said to be a *model* of $\mathscr{T}$ (briefly, a *$\mathscr{T}$-model*). It is straightforward to check that the theorems of an $MC^\nu$-theory $\mathscr{T}$ are true in every $\mathscr{T}$-model.

## 6. General $ML^\nu$-interpretations.

DEF. 6.1. *Assume that* (1) $\mathscr{S} = \langle \mathscr{M}, V\rangle$ *is an* $ML^\nu$*-system*, (2) $n \geqslant 0$, $x_1$ *to* $x_n$ *are distinct variables of the respective types* $t_1$ *to* $t_n$, *and* $X = \{x_1, \dots, x_n\}$, (3) $\Delta \in \mathscr{E}_{t_0}$ *and either*

(4) $t_0 = 0$ *and* $\xi$ *is the set of the* $(n+1)$*-tuples* $\langle \xi_1, \dots, \xi_n, \gamma\rangle$ *such that* $\xi_i \in \mathscr{Q}\mathscr{I}_{t_i}$ $(i = 1, \dots, n)$, $\gamma \in \Gamma$, *and* $\gamma \in \mathrm{des}_{\mathscr{M}V'}(\Delta)$ *vith* $V' = V\begin{pmatrix} x_1, \dots, x_n \\ \xi_1, \dots, \xi_n\end{pmatrix}$, *so that* $\xi \in QI_{(t_1, \dots, t_n)}$; *or*

(4′) $t_0 \in \tau^\nu$ *and* $\xi$ *is the element of* $QI_{(t_1, \dots, t_n : t_0)}$ *such that, for every choice of* $\xi_i \in \mathscr{Q}\mathscr{I}_{t_i}$ $(i = 1, \dots, n)$, $\xi(\xi_1, \dots, \xi_n) = \mathrm{des}_{\mathscr{M}V'}(\Delta)$ *where* $V' = V\begin{pmatrix} x_1, \dots, x_n \\ \xi_1, \dots, \xi_n\end{pmatrix}$.

*Then* $\xi$ *is said to be the QI denoted by* $\Delta$ *with respect to* $X$ *and* $\mathscr{S}$, *and this fact is expressed by* $\xi = d(\Delta, X, \mathscr{S}) = d(\Delta, X, \mathscr{M}, V)$.

DEF. 6.2. *The QI* $\xi$ *is said to be definable with respect to the* $ML^\nu$*-interpretation* $\mathscr{M}$ *if there is a wfe* $\Delta$ *of* $ML^\nu$, *a finite set* $X$ *of variables, and an* $\mathscr{M}$*-valuation* $V$ *for which* $\xi = d(\Delta, X, \mathscr{M}, V)$.

DEF. 6.3. (*a*) *The* $ML^\nu$*-interpretation* $\mathscr{M}$ *is said to be general if, for* $t \in \bar{\tau}^\nu$, *every QI of type* $t$, *definable with respect to* $\mathscr{M}$, *is in* $\mathscr{Q}\mathscr{I}_t$;

(*b*) *the* $ML^\nu$*-interpretation* $\mathscr{M}$ *is said to be weakly general if it becomes general by adding* $\mathscr{Q}\mathscr{I}_0$ *with the set* $D_0 = \{d(p, \emptyset, \mathscr{M}, V) : p \in \mathscr{E}_0$ and $V$ is an $\mathscr{M}$-valuation$\}$ (4).

In the sequel we shall say that the $ML^\nu$-system $\mathscr{S}$ $(= \langle \mathscr{M}, V\rangle)$ is general or weakly general if $\mathscr{M}$ is general or weakly general respectively.

Recalling rules $(d_1)$ to $(d_9)$ in N. 4, we easily see that $\mathrm{des}_{\mathscr{M}V}(\Delta) = d(\Delta, \emptyset, \mathscr{M}, V)$, and hence, if $\mathscr{M}$ is a general interpretation, then

(4) In [7] an interpretation is said to be general if its domain contains the designatum of every expression. This simple definition is equivalent (over the system in [7]) to ours; indeed, in the calculus investigated in [7], the $d(\Delta, \{x_1, \dots, x_n\}, \mathscr{M}, V)$ is nothing else than $\mathrm{des}_{\mathscr{M}V}(\lambda x_1, \dots, x_n)\Delta$. In [1] the operator $\lambda$ is defined by means of the operator $\imath$, thus the non-existing object may appear and hence the more elaborated definition 6.3 is needed.

the designatum of every expression is in the domain of $\mathscr{M}$; so that the function $\mathrm{des}_{\mathscr{M}V}$ is satisfactory.

THEOR. 6.1. (*a*) AA5.1-14 *are true in every* $ML^\nu$*-interpretation*; (*b*) $\mathscr{M}$ *is a weakly general* $ML^\nu$*-interpretation iff* $\mathscr{M}$ *is an* $MC^\nu$*-model.*

PROOF. To prove part (*a*) is a matter of routine. Furthermore, we see that, if $\mathscr{M}$ is weakly general, then it is an $MC^\nu$-model; indeed the truth of AA5.15-17 follows directly from Defs. 6.1-3 and rule $(d_9)$ in N. 4. Now, in order to prove the other half of (*b*), first remark that, as is substantially shown in [1] (Theor. 40.1),

$$(6.1)\qquad \begin{array}{l} \vdash_{\overline{MC^\nu}} (\exists F)(\forall x_1, \ldots, x_n)\,\Box.F(x_1, \ldots, x_n) \equiv p\ , \quad \text{and} \\ \vdash_{\overline{MC^\nu}} (\exists f)\ (\forall x_1, \ldots, x_n)\,\Box\ f\ (x_1, \ldots, x_n) = \Delta\ (^5) \end{array}$$

for every choice of the wff $p$, term $\Delta \in \mathscr{E}_{t_0}$, and variables $x_1, \ldots, x_n$ (of the respectives types $t_1, \ldots, t_n$), $F$, and $f$, with $F$ [$f$] not free in $p$ [$\Delta$]. In addition $\mathscr{M}$ is an $MC^\nu$-model by an hypothesis; hence for every $p, x_1, \ldots, x_n$, and $F$ as above, and $\mathscr{M}$-valuation $V$, $\mathrm{des}_{\mathscr{M}V}(q) = \Gamma$, where $q$ is $(\exists F)(\forall x_1, \ldots, x_n)\,\Box.F(x_1, \ldots, x_n) \equiv p$; but this is equivalent to the existence of a $\xi \in \mathscr{Q}\mathscr{I}_{(t_1,\ldots,t_n)}$ such that, for all $n$-tuples $\langle \xi_1, \ldots, \xi_n \rangle$ with $\xi_i \in \mathscr{Q}\mathscr{I}^*_{t_i}$, $\langle \xi_1, \ldots, \xi_n, \gamma \rangle \in \xi$ iff $\gamma \in \mathrm{des}_{\mathscr{M}V'}(p)$ with $V' = V\begin{pmatrix} x_1, \ldots, x_n \\ \xi_1, \ldots, \xi_n \end{pmatrix}$, hence $d(p, \{x_1, \ldots, x_n\}, \mathscr{M}, V) = \xi \in \mathscr{Q}\mathscr{I}_{(t_1,\ldots,t_n)}$. By similar reasonings (using $(6.1)_2$) one can easily see that $d(\Delta, \{x_1, \ldots, x_n\}, \mathscr{M}, V) \in \mathscr{Q}\mathscr{I}_{(t_1,\ldots,t_n:t_0)}$ for every term $\Delta \in \mathscr{E}_{t_0}$, variables $x_1, \ldots, x_n$, and $\mathscr{M}$-valuation $V$. Q.E.D.

The following theorem refers, throught its assumption (2), to countable $QI$-structures.

THEOR. 6.2. *Assume that* (1) $\mathscr{M}$ *is an* $ML^\nu$*-interpretation,* (2) $V_0$ *is an* $\mathscr{M}$*-valuation such that, for every* $\xi \in \mathscr{Q}\mathscr{I}_t$ $(t \in \bar{\tau}^\nu)$ $\xi = \mathrm{des}_{\mathscr{M}V_0}(\Delta')$ *for some wfe* $\Delta'$, *and* (3) $\eta = d(\Delta, X, \mathscr{M}, V)$; *then there exists a wfe* $\Delta_0$ *such that* $\eta = d(\Delta_0, X, \mathscr{M}, V_0)$.

PROOF. Let us first remark that $d(\Delta, X, \mathscr{M}, V) = d(\Delta, X, \mathscr{M}, V')$ if $V'(x) = V(x)$ for every variable $x$ free in $\Delta$ and not belonging to $X$,

(5) Remark that in the proof of (6.1) in [1], A5.17 is effectively needed and hence it is not possible to strengten this half of part (*b*) by requiring $\mathscr{M}$ to be only a model of the part of $MC^\nu$ based on A5.1-16.

and hence we can suppose $V_0(x)$ arbitrary, for $x$ as above, and equal to $V(x)$ otherwise. Let $y_1, \ldots, y_n$ be the variables not in $X$ and free in $\Delta$, and let $\eta_i = V(y_i)$ $(i = 1, \ldots, n)$. By hypothesis (2) there exist $n$ wfes $a_1, \ldots, a_n$ such that $\eta_i = \mathrm{des}_{\mathscr{M}V_0}(a_i)$. Let $\Delta_0 = \Delta[y_1/a_1, \ldots, y_n/a_n]$. Then the equality $d(\Delta, X, \mathscr{M}, V) = d(\Delta_0, X, \mathscr{M}, V_0)$ is a straightforward consequence of Theor. 4.2. Q.E.D.

## 7. Statement of the completeness theorem. Saturated sets.

If $p$ is a wff of the $MC^\nu$-theory $\mathscr{T}$, then $p$ is said to be *$\mathscr{T}$-valid* —briefly $\models_{\mathscr{T}} p$—iff $p$ is true in every general $\mathscr{T}$-model.

Now we can write our completeness theorem for modally closed $MC^\nu$-theories, which will be proved in the next sections.

THEOR. 7.1. *For every wff $p$ of the modally closed $MC^\nu$-theory $\mathscr{T}$, $\models_{\mathscr{T}} p$ iff $\vdash_{\mathscr{T}} p$.*

Let us first recall some standard definitions. A set $K$ of (closed or open) formulas is said to be *$\mathscr{T}$-consistent* if there is a formula which is not deducible from $K$ in $\mathscr{T}$. Of course, a maximal $\mathscr{T}$-consistent set $K$ contains every formula deducible from it in $\mathscr{T}$ and, in particular, every theorem of $\mathscr{T}$.

A language $\mathscr{L}'$ is called an *extension* of the language $\mathscr{L}$ if it is obtained from $\mathscr{L}$ by adding a (possibly empty) set of new constants for each type (if $\mathscr{L}$ and $\mathscr{L}'$ are based on a type system). An *$\omega$-extension* is an extension in which for each type the set of added constants is denumerable.

DEF. 7.1. *Let the theory $\mathscr{T}$ be based on the language $\mathscr{L}$ and let $\mathscr{L}'$ be an extension of $\mathscr{L}$. Then the set $H$ of wffs of $\mathscr{L}'$ is said to be $\mathscr{T}$-$\mathscr{L}'$-saturated provided conditions* (i) *and* (ii) *below hold:*

(i) *$H$ is maximal $\mathscr{T}'$-consistent, where $\mathscr{T}'$ is the extension of $\mathscr{T}$ obtained by adding $\mathscr{T}$ with the logical axioms involving all constants of $\mathscr{L}'$,*

(ii) *if $(\exists x)p \in H$, then for some constant $a$ of $\mathscr{L}'$ $p[x/a] \in H$.*

Remark that, by the maximality of $H$, (ii) is equivalent to

(ii′) *if $p[x/a] \in H$ for all constants $a$ of $\mathscr{L}'$, then $(x)p \in H$.*

LEMMA 7.1. *Assume that* (1) $H$ *is a* $\mathscr{T}$-$ML^\nu$-*saturated set of wffs*, (2) $\Gamma_H$ *is defined by means of*

$$(7.1) \qquad \Gamma_H = \{\gamma : \gamma \text{ is a } \mathscr{T}\text{-}ML^\nu\text{-saturated set of wffs and } \{p : \Box p \in H\} \subseteq \gamma\},$$

(3) $\gamma_1 \in \Gamma_H$, *and* (4) $\mathscr{T}$ *is modally closed. Then* $\Box p \in \gamma_1$ *iff, for all* $\gamma_2 \in \Gamma_H$, $p \in \gamma_2$.

PROOF. First we assume ($\alpha$) $\Box p \in \gamma_1$, $p \notin \gamma_2$ ($\in \Gamma_H$). By assumption (1), (7.1), and Def. 7.1, $H$, $\gamma_1$, and $\gamma_2$ are maximal $\mathscr{T}$-consistent. Hence $\sim p \in \gamma_2$ so that $\Box p \notin H$. Then $\sim \Box p \in H$, hence $\Box \sim \Box p \in H$ and $\sim \Box p \in \gamma_1$ which contrasts to $\Box p \in \gamma_1$. We conclude that, if $\Box p \in \gamma_1$, then ($\beta$) $p \in \gamma_2$ for all $\gamma_2 \in \Gamma_H$. We now conversely assume ($\beta$) and $\Box p \notin \gamma_1$. Let $K_1 = \{q : \Box q \in H\}$. $K_1$ satisfies condition (ii$'$) of Def. 7.1; indeed, if $r[x/a] \in K_1$ for ever constant $a$ (of the same type of $x$), then $\Box\, r[x/a] \in H$ for every constant $a$ and, by the $\mathscr{T}$-$ML^\nu$-saturation of $H$, $(x) \Box\, r \in H$; but this is equivalent to $\Box\, (x) r \in H$, and hence $(x) r \in K_1$. The closure $K_2$ of $K_1$ through $\vdash_{\mathscr{T}}$ (i.e. $\{q : K_1 \vdash_{\mathscr{T}} q\}$) satisfies condition (ii$'$) of Def. 7.1 too; indeed, if $K_1 \vdash_{\mathscr{T}} r[x/a]$ for every constant $a$, then $H \vdash_{\mathscr{T}} \Box\, r[x/a]$ for every $a$ and $\Box\, r[x/a] \in H$ for every $a$, so that $(x) r \in K_1$ and $(x) r \in K_2$. Now, the closure $K_3$ of $K_1 \cup \{\sim p\}$ through $\vdash_{\mathscr{T}}$ can be shown to satisfy condition (ii$'$) of Def. 7.1. Indeed, $K_1 \cup \{\sim p\} \vdash_{\mathscr{T}} r[x/a]$ for every $a$, implies $K_1 \vdash_{\mathscr{T}} \sim p \supset r[x/a]$ for every $a$, and $K_1 \vdash_{\mathscr{T}} (y)(\sim p \supset r[x/y])$ for a suitable variable $y$ not free in $p$; hence, $K_1 \vdash_{\mathscr{T}} \sim p \supset (y) r[x/y]$ and $K_1 \cup \{\sim p\} \vdash_{\mathscr{T}} (y) r[x/y]$, which is equivalent to $K_1 \cup \{\sim p\} \vdash_{\mathscr{T}} (x) r$. $K_3$ is also $\mathscr{T}$-consistent; otherwise $K_1 \vdash_{\mathscr{T}} p$, $H \vdash_{\mathscr{T}} \Box p$, $\Box\Box p \in H$, and $\Box p \in \gamma_1$, which contrasts to an hypothesis. Using the proof of Theor. 3 in [8], a $\mathscr{T}$-$ML^\nu$-saturated extension of $K_1 \cup \{\sim p\}$ can be constructed. Of course this contradictes the hypothesis ($\beta$). Q.E.D.

## 8. The Henkin construction on which the proof of the completeness theorem is based.

Let $\mathscr{T}$ be a modally closed $MC^\nu$-theory, and let $K$ be a $\mathscr{T}$-consistent set of formulas given arbitrarily. In this section we construct an $ML^\nu$-interpretation $\mathscr{M}_0$ in which the set $K$ will be proved to be satisfiable—see N. 9; in N. 9 we also prove that $\mathscr{M}_0$ is a general $\mathscr{T}$-model, so that the completeness theorem will follow in a standard way.

The first step is to prove that the set $K$ has a $\mathscr{T}$-$\mathscr{L}$-saturated extension $H$, for some extension $\mathscr{L}$ of $ML^\nu$. This can be easily achieved following, for instance, [9] (pp. 160, 161); furthermore, by replacing every constant $c_{tn}$ in $\mathscr{T}$ by $c_{t,2n}$, we can identify $\mathscr{L}$ with $ML^\nu$ (6).

Now, in order to construct an interpretation $\mathscr{M}_0$, we first identify $D_r$ with $\mathscr{E}_r$—see N. 3—($r = 1, \dots, \nu$) and $\Gamma$ with the set $\Gamma_H$ defined in (7.1), i.e. we identify individuals with individual expressions and possible cases with $\mathscr{T}$-$ML^\nu$-saturated sets of formulas.

For $\gamma \in \Gamma$ we consider the equivalence relation $\approx_\gamma$ in $D_1 \cup \dots \cup D_\nu$ such that $\Delta_1 \approx_\gamma \Delta_2$ iff $\Delta_1 = \Delta_2 \in \gamma$. For $\Delta \in \mathscr{E}_r$ and $\gamma \in \Gamma$ let $Q_\Delta(\gamma)$ be a particular term $\Delta'$ in $\mathscr{E}_r$ such that $\Delta' \approx_\gamma \Delta$. Thus we have associated every $\Delta \in \mathscr{E}_r$ with a function $Q_\Delta$ to be dealt with as the $QI$ of $\Delta$. We now give $Q_\Delta$ a meaning also for an arbitrary non-individual wfe $\Delta$ in $ML^\nu$, recursively, by means of the conditions (8.1-3) below.

(8.1) If $\Delta \in \mathscr{E}_{(t_1,\dots,t_n)}$, $Q_\Delta = \{\langle Q_{\Delta_1}, \dots, Q_{\Delta_n}, \gamma\rangle : \Delta(\Delta_1, \dots, \Delta_n) \in \gamma\}$.

(8.2) If $\Delta \in \mathscr{E}_{(t_1,\dots,t_n:t_0)}$, $Q_\Delta$ is the mapping of
$\{Q_{\Delta_1}\colon \Delta_1 \in \mathscr{E}_{t_1}\} \times \dots \times \{Q_{\Delta_n}\colon \Delta \in \mathscr{E}_{t_n}\}$ into $\{Q_{\Delta_0}\colon \Delta_0 \in \mathscr{E}_{t_0}\}$
for which $Q_\Delta(Q_{\Delta_1}, \dots, Q_{\Delta_n}) = Q_{\Delta(\Delta_1,\dots,\Delta_n)}$.

(8.3) If $\Delta \in \mathscr{E}_0$, then $Q_\Delta = \{\gamma\colon \Delta \in \gamma\}$.

Now the $ML^\nu$-interpretation $\mathscr{M}_0$ and the $ML^\nu$-system $\mathscr{S}_0$ $(= \langle \mathscr{M}_0, V_0\rangle)$ can be defined by means of

$$(8.4) \qquad \mathscr{Q}\mathscr{I}_t = \{Q_\Delta\colon \Delta \in \mathscr{E}_t\} \quad (t \in \bar{\tau}^\nu)\,,$$
$$a_t^\nu = Q_{a_t^*} \quad (t \in \tau^\nu)\,, \qquad M_0(c_{tn}) = Q_{c_{tn}}, \qquad V_0(v_{tn}) = Q_{v_{tn}}\,.$$

(6) In [7] the (correspondent of the) set $H$ is required only to be maximal $\mathscr{T}$-consistent and, in general, saturated sets are not considered. Our departure from [7] is necessary because of the different uses of the operator $\imath$; in [7] (which follows [5]), $\imath$ is a choice operator and, in particular, the formula $(\exists x)A(x) \supset A((\imath x)A(x))$ is a valid formula. From the designation rule $(d_9)$ we see that the above formula may fail to be true if we refer to an $ML^\nu$-interpretation.

THEOR. 8.1. *For* $t \in \tau^\nu$ *and* $\Delta, \Delta' \in \mathscr{E}_t$ *we have* $\Delta = \Delta' \in \gamma$ *iff* $Q_\Delta =_\gamma^t Q_\Delta$ *—see* (3.5).

PROOF. In the case $t \in \{1, \ldots, \nu\}$ the thesis is trivial.

Case $t = (t_1, \ldots, t_n)$. Since $\gamma$ is maximal consistent, by A5.13 $\Delta = \Delta' \in \gamma$ iff $[\Delta(\Delta_1, \ldots, \Delta_n) \equiv \Delta'(\Delta_1, \ldots, \Delta_n)] \in \gamma$ for all $\langle \Delta_1, \ldots, \Delta_n \rangle \in \mathscr{E}_{t_1} \times \ldots \times \mathscr{E}_{t_n}$. This holds iff for everyone of these $n$-tuples the conditions $\Delta(\Delta_1, \ldots, \Delta_n) \in \gamma$ and $\Delta'(\Delta_1, \ldots, \Delta_n) \in \gamma$ are equivalent, and this holds iff $Q_\Delta =_\gamma^t Q_{\Delta'}$—cf. (8.1).

Case $t = (t_1, \ldots, t_n : t_0)$. Let the thesis hold for $t = t_0$ as an inductive hypothesis. By A5.14 $\Delta = \Delta' \in \gamma$ iff $\Delta(\Delta_1, \ldots, \Delta_n) = \Delta'(\Delta_1, \ldots, \Delta_n) \in \gamma$ for every $n$-tuple $\langle \Delta_1, \ldots, \Delta_n \rangle \in \mathscr{E}_{t_1} \times \ldots \times \mathscr{E}_{t_n}$. By the inductive hypothesis this holds iff $Q_{\Delta(\Delta_1, \ldots, \Delta_n)} =_\gamma^{t_0} Q_{\Delta'(\Delta_1, \ldots, \Delta_n)}$ for all $n$-tuples above. By (8.2) this holds in turn iff $Q_\Delta =_\gamma^t Q_{\Delta'}$. Q.E.D.

Theors. 8.1 and 3.1 obviously imply the following

THEOR. 8.2. *For* $t \in \tau^\nu$ *and* $\Delta, \Delta' \in \mathscr{E}_t$, *we have* $\Delta = \Delta' \in \gamma$ *for every* $\gamma \in \Gamma$ *iff* $Q_\Delta = Q_{\Delta'}$.

COROLLARY 8.1. *Every* $\xi \in \mathscr{Q}\mathscr{I}_t$ $(t \in \tau^\nu)$ *has the form* $Q_c$ *for some constant* $c$ *of type* $t$.

PROOF. By (8.4), $\xi$ is $Q_\Delta$ for some term $\Delta$; furthermore, $(\exists x) \Box (\Delta = x)$ is provable in $MC^\nu$; hence by the $\mathscr{T}$-$ML^\nu$-saturation of $H$, there is a constant $c$ such that $\Box (\Delta = c) \in H$, that is, by (7.1), $\Delta = c \in \gamma$ for all $\gamma \in \Gamma$. The thesis follows now from Theor. 8.2. Q.E.D.

THEOR. 8.3. *For every wfe* $\Delta$ *of* $ML^\nu$, $\mathrm{des}_{\mathscr{S}_0}(\Delta) = Q_\Delta$; *and hence, by* (8.3),

(8.5) $\quad p \in \gamma$ *iff* $\gamma \in \mathrm{des}_{\mathscr{S}_0}(p)$, *for every wff* $p$.

PROOF. We use an induction on the number $\nu_\Delta$ of occurrences of $\imath$ in $\Delta$, and for every $n$ $(\geqslant 0)$ we treat the wfes $\Delta$ with $\nu_\Delta = n$ by induction on their lengths $l_\Delta$. For $l_\Delta = 1$, $\Delta$ is $c_{tn}$ or $v_{tn}$, hence the thesis follows by $(8.4)_4$.

Case 1: $\Delta$ is the term $\Delta'(\Delta_1, \ldots, \Delta_n)$. By the inductive hypothesis and (8.2), $\mathrm{des}_{\mathscr{S}_0}(\Delta) = \mathrm{des}_{\mathscr{S}_0}(\Delta')(\mathrm{des}_{\mathscr{S}_0}(\Delta_1), \ldots, \mathrm{des}_{\mathscr{S}_0}(\Delta_n)) = Q_{\Delta'}(Q_{\Delta_1}, \ldots, Q_{\Delta_n}) = Q_\Delta$.

Case 2$a$: $\Delta$ is the wff $R(\Delta_1, \ldots, \Delta_n)$. Then by the designation rule $(d_3)$, the inductive hypothesis, and (8.1)–(8.3), $\gamma \in \mathrm{des}_{\mathscr{S}_0}(\Delta) \Leftrightarrow$

$\Leftrightarrow \langle \mathrm{des}_{\mathscr{S}_0}(\Delta_1), \dots, \mathrm{des}_{\mathscr{S}_0}(\Delta_n), \gamma\rangle \in \mathrm{des}_{\mathscr{S}_0}(R) \Leftrightarrow \langle Q_{\Delta_1}, \dots, Q_{\Delta_n}, \gamma\rangle \in Q_R \Leftrightarrow$
$\Leftrightarrow R(\Delta_1, \dots, \Delta_n) \in \gamma \Leftrightarrow \gamma \in Q_\Delta$.

Case $2b$: $\Delta$ is $\Delta_1 = \Delta_2$ with $\Delta_1, \Delta_2 \in \mathscr{E}_t$. Then by rule $(d_2)$, the inductive hypothesis, Theor. 8.1, and (8.3), $\gamma \in \mathrm{des}_{\mathscr{S}_0}(\Delta) \Leftrightarrow \mathrm{des}_{\mathscr{S}_0}(\Delta_1) =^t_\gamma$ $=^t_\gamma \mathrm{des}_{\mathscr{S}_0}(\Delta_2) \Leftrightarrow Q_{\Delta_1} =^t_\gamma Q_{\Delta_2} \Leftrightarrow \Delta_1 = \Delta_2 \in \gamma \Leftrightarrow \gamma \in Q_\Delta$.

Case $2c$: $\Delta$ is $\sim p$. Then $\mathrm{des}_{\mathscr{S}_0}(\Delta) = \Gamma - \mathrm{des}_{\mathscr{S}_0}(p)$ by rule $(d_5)$. Furthermore, since every $\gamma \in \Gamma$ is maximal consistent, by (8.3), $Q_\Delta =$ $= \Gamma - Q_p$. The thesis now follows by the inductive hypothesis.

Case $2d$: $\Delta$ is $p \wedge q$. The proof is similar to the above one.

Case $2e$: $\Delta$ is $\Box p$. Then by rule $(d_8)$, the inductive hypothesis, Lemma 7.1, and (8.3), $\gamma \in \mathrm{des}_{\mathscr{S}_0}(\Delta) \Leftrightarrow \gamma_1 \in \mathrm{des}_{\mathscr{S}_0}(p)$ for all $\gamma_1$ $(\in \Gamma) \Leftrightarrow$ $\Leftrightarrow \gamma_1 \in Q_p$ for all $\gamma_1 \Leftrightarrow p \in \gamma_1$ for all $\gamma_1 \Leftrightarrow \Delta \in \gamma \Leftrightarrow \gamma \in Q_\Delta$.

Case $2f$: $\Delta$ is $(x)p$ where $x$ is $v_{tn}$. Then, by rule $(d_7)$, $\gamma \in \mathrm{des}_{\mathscr{S}_0}(\Delta)$ iff for all $\xi \in \mathscr{Q}\mathscr{I}_t$, $\gamma \in \mathrm{des}_{\mathscr{M}_0 V'}(p)$ with $V' = V\begin{pmatrix} x \\ \xi \end{pmatrix}$. Since, by Corollary 8.1, every $\xi \in \mathscr{Q}\mathscr{I}_t$ is a $Q_b$ (that is $\mathrm{des}_{\mathscr{S}_0}(b)$) for some constant $b$, and Theor. 4.2 holds, the last condition holds iff for every constant $b \in \mathscr{E}_t$, $\gamma \in \mathrm{des}_{\mathscr{S}_0}(p[x/b])$, which by the inductive hypothesis is equivalent to $\gamma \in Q_{p[x/b]}$ for all constants $b \in \mathscr{E}_t$ and hence, by (8.3), to $p[x/b] \in \gamma$ for the same constants. Since $\gamma$ is $\mathscr{T}$-$ML^\nu$-saturated, this holds iff $(x)p \in \gamma$ and hence, by (8.3), iff $\gamma \in Q_\Delta$.

We conclude that the thesis holds for $\nu_\Delta = 0$. Now fix $n > 0$ and let the thesis hold for $\nu_\Delta < n$; and assume $\nu_\Delta = n$.

Case 3: $\Delta$ has the form $(\imath x)p$ where $x$ is $v_{tn}$. By the inductive hypothesis the thesis holds for $p$ and every wff $q$ that contains $p$ and has no occurrences of $\imath$ outside $p$. Let $q$ be $(\exists_1 x)p$. Remark that by Theor. 4.1, the transitivity of the relation $=^t_\gamma$, and rule $(d_9)$, it suffices to prove that $Q_\Delta$ is equivalent to $a^\nu_t$ in the cases $\gamma_1 \notin \mathrm{des}_{\mathscr{S}_0}(q)$, and to some $QI$ $\xi$ such that $\gamma_2 \in \mathrm{des}_{\mathscr{M}_0 V'}(p)$, for $V' = V\begin{pmatrix} x \\ \xi \end{pmatrix}$, in the cases $\gamma_2 \in \mathrm{des}_{\mathscr{S}_0}(q)$.

Let $\gamma_1 \in \mathrm{des}_{\mathscr{S}_0}(\sim q)$. By the inductive hypothesis, $\gamma_1 \in Q_{\sim q}$, i.e. $\sim q \in$ $\in \gamma_1$; hence, by A5.17 $(b)$, $(\imath x)p = a^*_t \in \gamma_1$. Then, by Theor. 8.1,

$$(8.6) \qquad Q_\Delta =^t_{\gamma_1} Q_{a^*} (= a^\nu_t) \quad \text{for all} \quad \gamma_1 \in \mathrm{des}_{\mathscr{S}_0}(\sim (\exists_1 x)p)\,.$$

Now let $\gamma_2 \in \mathrm{des}_{\mathscr{S}_0}(q)$. By the inductive hypothesis $\gamma_2 \in Q_q$, i.e. $(\exists_1 x)p \in \gamma_2$. Then by A5.17 $(a)$, (2.1), and $(5.1)_2$,

$$(8.7) \qquad (y)(p[x/y] \supset y = (\imath x)p) \in \gamma_2 \quad \text{and} \quad (\exists x)p \in \gamma_2\,.$$

Since $\gamma_2$ is $\mathscr{T}$-$ML^\nu$-saturated, by $(8.7)_2$ $p[x/a] \in \gamma_2$ for some constant $a$; furthermore by $(8.7)_1$ $(p[x/a] \supset a = (\imath x)p) \in \gamma_2$: Hence $(a = (\imath x)p) \in \gamma_2$ which is equivalent to

$$(8.8) \qquad Q_\Delta =^t_{\gamma_2} Q_a \quad \text{for some } a \text{ and} \quad \gamma_2 \in \mathrm{des}_{\mathscr{S}_0}((\exists_1 x)p) .$$

By the inductive hypothesis, $\mathrm{des}_{\mathscr{S}_0}(p[x/a]) = Q_{p[x/a]}$, so that by (8.3), $\gamma_2 \in \mathrm{des}_{\mathscr{S}_0}(p[x/a])$ which by Theor. 4.2 is equivalent to

$$(8.9) \qquad \gamma_2 \in \mathrm{des}_{\mathscr{M}_0 V'}(p) \quad \text{for} \quad V' = V\begin{pmatrix} x \\ \mathrm{des}_{\mathscr{S}_0}(a) \end{pmatrix} = V\begin{pmatrix} x \\ Q_a \end{pmatrix}.$$

The thesis follows now by (8.6), (8.8), and (8.9). Q.E.D.

## 9. Accomplishment of the proof of the completeness theorems.

Let us return now to our completeness theorem. It remains to prove that $H$ (and hence $K$) is $\gamma$-satisfiable—see Def. 4.1—in $\mathscr{M}_0$ for some $\gamma \in \Gamma$. Recalling how the set $\Gamma_H$ of the elementary possible cases was constructed—i.e. (7.1)—we note that $H$ itself is an element, $\bar{\gamma}$, of $\Gamma_H$. Hence, by (8.5) applied to $\bar{\gamma}$, we have

$$(9.1) \qquad p \in H \quad \text{iff} \quad \bar{\gamma} \in \mathrm{des}_{\mathscr{M}_0 V_0}(p) , \quad \text{for every wff } p ,$$

that is, $H$ is $\bar{\gamma}$-satisfiable in $\mathscr{M}_0$.

We now prove that $\mathscr{M}_0$ is a general $\mathscr{T}$-model. $\mathscr{M}_0$ is a $\mathscr{T}$-model; indeed, every $\gamma \in \Gamma_H$ (being maximal $\mathscr{T}$-consistent) contains the axioms of $\mathscr{T}$ and their extensional closure, and hence, by (8.5), $\mathrm{des}_{\mathscr{M}_0 V}(p) = \Gamma$ for every $\mathscr{M}_0$-valuation $V$ and every axiom $p$ of $\mathscr{T}$. Since $\mathscr{T}$ is an $MC^\nu$-theory, by Theor. 6.1, $\mathscr{M}_0$ is weakly general. Furthermore, let $\xi = d(p, \emptyset, \mathscr{M}_0, V)$ for some wff $p$ and $\mathscr{M}_0$-valuation $V$. Since, by (8.4) and Theor. 8.3, every $QI$ (in the domain of $\mathscr{M}_0$) has the form $\mathrm{des}_{\mathscr{S}_0}(\Delta)$ for some $\Delta$, we may use Theor. 6.2 to derive $\xi = d(p', \emptyset, \mathscr{M}_0, V_0)$, for some wff $p'$. But $d(p', \emptyset, \mathscr{M}_0, V_0) = \mathrm{des}_{\mathscr{S}_0}(p')$. Therefore by Theor. 8.3, $\xi = Q_{p'}$ and $\xi \in \mathscr{Q}\mathscr{I}_0$; that is, $\mathscr{M}_0$ is general. Thus Theor. 7.1 has been proved.

We can now briefly show how a completeness theorem for contingent $MC^\nu$-theories can be proved.

The proof of such a theorem can be wholly analogue to the one relative to modally closed $MC^\nu$-theories; that is, it consists in the construction (from a given contingent $MC^\nu$-theory $\mathscr{T}_C$) of a contingent $\mathscr{T}_C$-model $\langle \mathscr{M}, \gamma_R \rangle$ in which a given $\mathscr{T}_C$-consistent set $K_0$ of formulas is $\gamma_R$-satisfiable. However, in this case, we cannot use every result relative to modally closed theories; for instance, (5.2) does not hold when $\mathscr{T}$ is (properly) contingent and hence the proof of Lemma 7.1—in which (5.2) is applied—fails to be valid.

In any case, it is not necessary to repeat the whole proof of the completeness theorem; indeed, by an easy device, we can use the preceding proof for our present goals.

Let $\mathscr{T}_C$ be a contingent $MC^\nu$-theory and let $K_0$ be a $\mathscr{T}_C$-consistent set of formulas. Let us denote by $C$ the set of contingent proper axioms of $\mathscr{T}_C$ (that is, the proper axioms of $\mathscr{T}_C$ that are not modally closed) and by $\mathscr{T}$ the modally closed part of $\mathscr{T}_C$ (that is the $MC^\nu$-theory obtained from $\mathscr{T}_C$ by subtracting $C$ from the set of its axioms). Furthermore, let

(9.2) $K = K_0 \cup \{(..)p : p \in C\}$, where $(..)p$ denotes the extensional closure of $p$.

Obviously, $K$ is $\mathscr{T}_C$-consistent.

We may now build up a $\mathscr{T}$-$ML^\nu$-saturated extension $H$ of $K$ and the general $ML^\nu$-system $\mathscr{S}_0$ just as in N. 8. Of course, Theor. 8.3 holds and, in particular, (9.1) holds too.

If we consider the contingent $ML^\nu$-interpretation $\mathscr{M}_C = \langle \mathscr{M}_0, \gamma_R \rangle$ where $\gamma_R$ is $H$, then, by (9.1), $H$ is $\gamma_R$-satisfiable in $\mathscr{M}_C$.

It remains to prove that $\mathscr{M}_C$ is a $\mathscr{T}_C$-model. We already know that $\mathscr{M}_C$ is a $\mathscr{T}$-model, then let $p$ be a contingent axiom of $\mathscr{T}_C$; by (9.2) the extensional closure $p'$ of $p$ belongs to $H$ and hence, by (9.1), $\gamma_R \in \mathrm{des}_{\mathscr{M}_0 V_0}(p')$ and $\gamma_R \in \mathrm{des}_{\mathscr{M}_0 V}(p)$ for all $\mathscr{M}_0$-valuation $V$. That is, $\mathscr{M}_C$ is a general $\mathscr{T}_C$-model.

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