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On V -Rings and Unit-Regular Rings.

ROGER YUE CHI MING (*)

Introduction.

Rings whose simple right modules are injective, called right V -rings by C. Faith, are studied in [11] and by many other authors (cf. [6]). A ring A is called unit-regular if, for each $a \in A$, there exists an invertible element $u \in A$ such that $a = aua$. A is said to be of bounded index if the supremum of the indices of all nilpotent elements of A is finite. Unit-regular rings and regular rings of bounded index, which are intermediate between von Neumann regular rings and strongly regular rings, are interesting classes of regular rings. The purpose of this paper is to consider certain conditions for left V -rings to be unit-regular and for regular rings to have bounded index. In the first section, it is proved that regular rings and left V -rings, under conditions weaker than being left duo, are unit-regular. The results are related to questions raised by Fisher [6]. In the second section, semi-prime rings whose maximal left ideals and left annihilator ideals are quasi-injective are proved to be regular with bounded index (this extends [9, Theorem 2.9]). A generalization of [9, Theorem 2.13] is also obtained. If A is a left continuous regular ring whose maximal essential left ideals are ideals, then $A = B \oplus C$, where B is a left and right continuous strongly regular ring and C is a left and right self-injective regular ring of bounded index. Results in [9], [21], [23], [24] are improved.

Throughout, A represents an associative ring with identity and A -modules are unitary. Z will always denote the left singular ideal

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of A and A is called left non-singular if $Z = 0$. As usual, an ideal of A means a two-sided ideal of A and A is called left duo if every left ideal of A is an ideal. A left A -module M is called p -injective if, for any principal left ideal I of A and any left A -homomorphism $g: I \rightarrow M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in I$. A is von Neumann regular iff every left A -module is p -injective [18].

This result has an analogue in the theory of semi-groups [10, Theorem 2.1]. Since it is well-known that A is von Neumann regular iff every left A -module is flat, it is worthwhile to note that for any p -injective left ideal I of A , A/I is a cyclic flat left A -module. Also, a result of Ikeda and Nakayama [8, Theorem 1] asserts that A is a left p -injective ring iff every principal right ideal of A is a right annihilator. Following [13] and [14], A is called a left $p - V$ -ring (resp. $p - V'$ -ring) if every simple (resp. simple singular) left A -module is p -injective. Left $p - V$ -rings generalise both von Neumann regular rings and left V -rings. Since left $p - V'$ -rings need not be semi-prime, they generalise left $p - V$ -rings. In [23], characterisations of these rings in terms of maximal left subideals are given and the well-known characteristic property of left V -rings given by Villamayor [11, Theorem 2.1] is weakened. The importance of uniform, essential and complement one-sided ideals in ring theory is one motivation of the study of CTE and $ECTE$ rings defined as follows:

DEFINITIONS. A is called a CTE (resp. $ECTE$) ring if, for any proper essential left ideal E of A , any complement (resp. any essential or complement) left subideal is an ideal of E .

Obviously, CTE and $ECTE$ rings generalise left duo rings (in fact, they generalise ALD rings introduced in [24]). Left uniform rings (in particular, left Ore domains) are CTE . Left PCI rings have been studied by A. K. Boyle and C. Faith (cf. [5, p. 140]). R. F. Damiano (answering a question of Faith) proved that left PCI rings are either semi-simple Artinian or simple left Noetherian, left hereditary left Ore V -domains [4, Theorem 1]. Consequently, left PCI rings are also CTE . Thus CTE rings form quite an interesting class of rings and certain connections with $ECTE$ and unit-regular rings are considered. Since J. H. Cozzens has constructed a simple principal left and right ideal domain whose singular right modules are injective but which is not regular, it then follows that CTE left or right V -rings need not be regular. It is evident that CTE and $ECTE$ regular rings generalise effectively strongly regular rings.

1. Unit-regular rings.

Von Neumann regular rings are extensively studied for their own sake and for their connections with other rings (cf. the bibliographies of [6] and [7]). C. Faith and J. H. Cozzens have shown that, in general, there is no inclusion between the class of von Neumann regular rings and that of left V -rings. This leads to the important questions raised by J. W. Fisher [6, Queries (a), (b), (c)]. Since unit-regular rings and regular rings of bounded index generalise strongly regular rings, they have drawn the attention of many authors. Among the results proved in this section is the following: An ECTE regular or left V -ring is a unit-regular ring whose primitive factor rings are Artinian. As usual, A is called an I -ring if every non-nil left ideal of A contains a non-zero idempotent. Obviously, if A is von Neumann regular, then every factor ring of A is a semi-primitive I -ring.

LEMMA 1.1. *Let A be a prime CTE semi-primitive I -ring. Then A is simple Artinian.*

PROOF. Suppose A is not simple Artinian. Then there exists a maximal left ideal M which is essential in ${}_A A$. Since $M \neq 0$, M contains a non-zero idempotent e . Then $M = Ae \oplus B$, where $B \neq 0$ and since Ae is a complement left subideal of M , then $AeB \subseteq Ae \cap B = 0$, which contradicts the primeness of A .

LEMMA 1.2. *If A is an ECTE ring, then any prime factor ring of A is ECTE.*

PROOF. Let P be a prime ideal of A and set $B = A/P$. Let E/P be a proper essential left ideal of B and C/P a non-zero complement or essential left subideal of E/P . Suppose that ${}_A C$ is not essential in ${}_A E$. Then there exists a non-zero left subideal D of E such that $C \cap D = 0$. By Zorn's Lemma, there exists a complement left subideal K of E such that $C \subseteq K$ and $K \cap D = 0$. Since E is a proper essential left ideal of A , then $KD \subseteq K \cap D = 0$ which implies $(K/P)((D \oplus P)/P) = 0$, contradicting the primeness of B . Thus ${}_A C$ is essential in ${}_A E$ which implies that C is an ideal of E , whence C/P is an ideal of E/P in B which proves that B is ECTE.

LEMMA 1.3. *If A is a semi-prime ECTE ring, then A is left non-singular.*

PROOF. Suppose there exists $0 \neq z \in Z$ such that $z^2 = 0$. If M is a maximal left ideal of A containing $l(z)$, since A is ECTE, then $l(z)M \subseteq l(z)$ which implies $(Mz)^2 \subseteq AzMz \subseteq l(z)Mz \subseteq l(z)z = 0$, whence $Mz = 0$ (A being semi-prime). Therefore $M = l(z)$ and since Az ($\approx A/M$) is a minimal left ideal of A , then Az is a direct summand of $_A A$ which implies M a direct summand of $_A A$. This contradiction proves that Z contains no non-zero nilpotent element, whence $Z = 0$ by [21, Lemma 2.1].

PROPOSITION 1.4. *Let A be such that each factor ring is a semi-primitive I-ring. The following conditions are then equivalent:*

- (1) *A is a unit-regular, left and right V-ring whose primitive factor rings are Artinian;*
- (2) *Every primitive factor ring of A is CTE.*

PROOF. Obviously, (1) implies (2).

(2) implies (1) by [6, Theorems 4 and 14], [7, Theorem 6.10] and Lemma 1.1.

The next corollary is related to [6, Query (c)].

COROLLARY 1.5. *If A is an ECTE ring whose factor rings are semi-primitive I-rings, then A is a unit-regular left and right V-ring.*

PROOF. Apply Lemma 1.2 to Proposition 1.4.

It is well-known that a left self-injective strongly regular ring is right self-injective. [7, Corollary 6.22 and Theorem 7.20], Lemmas 1.2, 1.3 and Proposition 1.4, yield the following generalisation (compare with [9, Theorem 2.9]).

THEOREM 1.6. *Let A be a semi-prime ECTE left self-injective ring. Then A is a right self-injective regular ring with bounded index.*

Applying [7, Corollaries 6.4 and 6.22] and [25, Corollary 6] to Theorem 1.6, we get

COROLLARY 1.7. *If A is a semi-prime ECTE left self-injective ring and F a finitely generated non-singular left A -module, then $\text{End}_A(F)$ is a left and right self-injective regular ring of bounded index.*

The well-known result that left duo left V -rings are strongly regular (cf. for example [2]) is extended in the next theorem which also (a) answers the question raised in [24, Remark] and (b) shows that *CTE* rings generalise effectively *ECTE* rings (cf. also [6, Query (b)]).

THEOREM 1.8. *If A is an ECTE left V -ring, then A is a unit-regular right V -ring.*

PROOF. It is sufficient to prove that any prime factor ring B of A is Artinian. Then A is regular by [7, Corollary 1.18] and the theorem will follow from Corollary 1.5. Suppose B is not Artinian. Then there exists a maximal essential left ideal M of B . If B has non-zero socle, then M contains a minimal left ideal U of B and $M = U \oplus K$. Since $K \neq 0$ and $UK \subseteq U \cap K = 0$, this contradicts the primeness of B . Now suppose B has zero socle. If $0 \neq b \in B$ such that $b^2 = 0$, L a maximal left ideal of B containing $l(b)$ (in B), since $_B L$ is essential in $_B B$, the proof of Lemma 1.2 shows that $l(b)$ is an ideal of L , whence $(Lb)^2 \subseteq BbLb \subseteq l(b)Lb \subseteq l(b)b = 0$ which implies $L = l(b)$. This yields a minimal left ideal Bb ($\approx B/L$), a contradiction. Thus B is a reduced ring whence B is an integral domain [23, Proposition 6]. Since B is a left V -ring, then M contains a maximal left subideal D . If $0 \neq d \in D$, $f: Md \rightarrow M/D$ the left B -homomorphism defined by $f(bd) = b + D$ for all $b \in M$, since M/D is injective, there exists $v \in M$ such that $b + D = f(bd) = b + dv + D$ for all $b \in M$. Since D is an ideal of M , $b - dv \in D$ which implies $b \in D$, whence $M = D$, again a contradiction. This proves B Artinian.

If A is *ECTE*, then for any proper essential left ideal E , every maximal left subideal is an ideal of E . The next corollary then follows from [24, Theorem 1.2].

COROLLARY 1.9. *An ECTE left $p - V$ -ring is a unit-regular left and right V -ring.*

[7, Theorem 4.14, Corollaries 4.7, 6.12 and 6.16] and Corollary 1.9 then imply

COROLLARY 1.10. *Let A be an ECTE left $p - V$ -ring and P a finitely generated projective left A -module. Then*

- (1) $\text{End}_A(P)$ is unit-regular;

- (2) If F is a finitely generated left A -module, then F is directly finite and every injective or surjective endomorphism of F is an automorphism;
- (3) If M, N are left A -modules such that $P \oplus M \approx P \oplus N$, then $M \approx N$;
- (4) If B is an ECTE left $p - V$ -ring and if there exists a positive integer n such that $M_n(A) \approx M_n(B)$, then $A \approx B$.

Theorem 1.8 raises the following

QUESTION 1. Is an ECTE right V -ring von Neumann regular?

In this direction, we have the next

PROPOSITION 1.11. Let A be an ECTE right V -ring whose simple right modules are flat. Then A is a unit-regular left V -ring.

PROOF. It is sufficient to prove that any prime factor ring B of A is Artinian for then, the proposition follows from [7, Corollary 1.18] and Corollary 1.5. By Lemma 1.2, B is a CTE ring and the proof of Theorem 1.8 shows that B is either simple Artinian or B is a reduced ring with zero socle. In the latter case, since every simple right B -module is flat (this is because every simple right A -module is flat), then B is strongly regular by [21, Theorem 1.4], whence B is a division ring which contradicts B with zero socle.

The proof of [22, Proposition 4] shows that A is a left hereditary ring iff for any left A -module M , if \hat{M} is an injective hull of M , then \hat{M}/M is injective. V. C. Cateforis and F. L. Sandomierski have proved that a commutative ring whose singular modules are injective is regular, hereditary. [24, Theorem 1.2] and Theorem 1.8 imply the following non-commutative (and not necessarily reduced) version.

PROPOSITION 1.12. Let A be an ECTE ring whose singular left modules are injective. Then A is a unit-regular, left hereditary, left and right V -ring.

REMARK 1. Since any left annihilator in a left non-singular ring is a complement left ideal (cf. for example [3]), the proof of Theorem 1.8 together with [21, Lemma 2.1] show that a prime ECTE ring is either simple Artinian or a left Ore domain.

Since a direct sum of p -injective left A -modules is p -injective, the next result then follows from [5, Theorem 25.5.1].

THEOREM 1.13. *Let A be a prime ECTE ring whose p -injective left and right modules are injective. Then A/I is an Artinian serial ring for every non-zero ideal I of A .*

[5, Lemma 18.34 B], Corollary 1.9 and Remark 1 imply

PROPOSITION 1.14. *If A is an ECTE left Noetherian ring such that A/P is a left $p - V$ -ring for every non-zero prime ideal P of A , then A is either left Artinian or an integral domain.*

2. Regular rings of bounded index.

It is well-known that strongly regular rings are unit-regular, left and right V -rings. (An elementary proof that strongly regular rings are unit-regular is implicit in [18, Proposition 1]). In [7, Problem 10], Goodearl asks whether regular left and right V -rings are unit-regular. Following [23], A is called an *ELT* (resp. *MELT*) ring if every essential (resp. maximal essential) left ideal of A is an ideal of A . A result of Jain, Mohamed and Singh [9] asserts that A is an *ELT*, left self-injective ring iff every left ideal of A is quasi-injective (such rings are called left q -rings). Several main results of [9] will be generalised in this section. Our first result contains a partial answer to [7, Problem 10].

THEOREM 2.1. *Let A be a MELT regular left and right V -ring. Then A is unit-regular.*

PROOF. Let B be a primitive factor ring of A . Then B is also a *MELT* regular left and right V -ring. If we suppose that B has zero socle, then the proof of [18, Proposition 3] shows that B is strongly regular which implies that B is a division ring. This contradiction proves that B has non-zero socle. Then B is simple Artinian (cf. [7, p. 350 (Problem 52)]) whence A is unit-regular [7, Theorem 6.10].

[23, Proposition 8 (2)] may now be strengthened as follows:

COROLLARY 2.2. *If A is a semi-prime MELT, P.I. ring whose simple right modules are flat, then A is a unit-regular left and right V -ring whose primitive factor rings are Artinian.*

[6, Theorem 16] and [19, Lemma 1] imply

COROLLARY 2.3. *A MELT, P.I., left $p - V$ -ring is unit-regular.*

The proof of Theorem 2.1 yields the next corollary which is related to [22, Question].

COROLLARY 2.4. *A MELT left V -ring whose primitive factor rings are P.I. is a unit-regular, right V -ring.*

[6, Theorems 11 and 19] imply

COROLLARY 2.5. *Let A be a regular left and right V -ring and G a finite group whose order is a unit in A . If the group ring $A[G]$ is MELT, then $A[G]$ is a unit-regular ring whose primitive factor rings are Artinian.*

[6, Corollary 22] implies

COROLLARY 2.6. *Let A be a P.I. ring and G an Artinian group such that the group ring $A[G]$ is MELT regular. Then $A[G]$ is a unit-regular ring whose primitive factor are Artinian.*

Rings whose left singular ideals are zero abound. If $Z = 0$, then the maximal left quotient ring of A is a left self-injective regular ring. If M is a left non-singular quasi-injective module over an arbitrary ring A , then $\text{End}_A(M)$ is a left self-injective regular ring. These examples show that left self-injective regular rings play an important role in ring theory.

If A is a left self-injective regular ring such that A/T is Artinian for all maximal (two-sided) ideals T of A , then A has bounded index [7, p. 79]. Unit-regular rings whose primitive factor rings are Artinian need not be of bounded index [7, p. 77]. Since any factor ring of a MELT ring is MELT, the next result then follows from [7, Theorems 6.21, 7.20, 9.25 and Corollary 9.27] and the fact that a MELT simple ring is Artinian.

THEOREM 2.7. *Let A be a MELT left self-injective regular ring. Then A is a right self-injective, left and right V -ring of bounded index. In that case (1) A is isomorphic to a finite direct product of full matrix rings over strongly regular rings; (2) Every non-zero ideal of A contains a non-zero central idempotent; (3) The intersection of maximal ideals of A is zero.*

We now give two new characteristic properties of left self-injective regular rings in terms of quasi-injectivity ([9, Theorem 2.9] is thereby improved).

THEOREM 2.8. *The following conditions are equivalent:*

- (1) A is left self-injective regular;
- (2) A is a semi-prime ring whose left annihilator ideals are quasi-injective;
- (3) Every left annihilator ideal of A is quasi-injective and every simple right A -module is flat.

PROOF. If A is left self-injective regular, then any left annihilator (being a complement left ideal) is a direct summand of ${}_A A$ which shows that (1) implies (2) and (3).

(2) implies (1): ${}_A A$ is quasi-injective which implies A a left self-injective ring. Suppose there exists $0 \neq z \in Z$ (the left singular ideal of A) such that $z^2 = 0$. Since $l(z)$ is an essential left ideal which is quasi-injective, then $l(z) = l(z)A$ which implies $(Az)^2 \subseteq l(z)Az = l(z)z = 0$ and since A is semi-prime, then $z = 0$, a contradiction. Therefore $Z = 0$ by [21, Lemma 2.1] which proves A regular.

(3) implies (1): As before, if we suppose that there exists $0 \neq z \in Z$ such that $z^2 = 0$, then $l(z)$ is an ideal of A and if R is a maximal right ideal of A containing $l(z)$, then A/R is a simple flat right A -module, whence $z = wz$ for some $w \in R$ by [1, Proposition 2.1]. Therefore $1 - w \in l(z) \subseteq R$ which implies $1 \in R$, a contradiction. Thus $Z = 0$ by [21, Lemma 2.1] which implies A left self-injective regular.

The next result shows that semi-prime left q -rings [9] are, in fact, regular rings of bounded index.

THEOREM 2.9. *Let A be a semi-prime ring whose maximal left ideals and left annihilators are quasi-injective. Then A is a MELT left and right self-injective regular, left and right V -ring of bounded index.*

PROOF. Let M be a maximal essential left ideal of A . Since A is left self-injective regular (Theorem 2.8), then ${}_A A$ is the injective hull of ${}_A M$ and since M is quasi-injective, then $M = MA$ which proves that A is MELT. Theorem 2.7 then applies.

COROLLARY 2.10. *Let A be a ring whose maximal left ideals and left annihilators are quasi-injective and such that every simple right A -module is flat. Then A is a left and right self-injective regular ring of bounded index.*

PROOF. Apply Theorems 2.8 and 2.9.

Applying [17, Remark 5 (a)] and [25, Corollary 10] to Theorem 2.9, we get

COROLLARY 2.11. *Let A be a semi-prime ring whose maximal left ideals and left annihilators are quasi-injective. Then any quasi-injective or essentially finitely generated left (or right) A -module contains its singular submodule as a direct summand.*

Throughout, Q will always denote the maximal left quotient ring of A whenever A is left non-singular. The next result contains generalisations of [9, Theorem 2.13], [21, Theorem 2.4] and [23, Theorem 11 (4)].

THEOREM 2.12. *The following conditions are equivalent for a prime ring A :*

- (1) A is simple Artinian;
- (2) A is a MELT left self-injective regular ring;
- (3) A is a MELT right self-injective regular ring;
- (4) A is a MELT left continuous regular ring;
- (5) All maximal left ideals and left annihilators are quasi-injective;
- (6) A is a left self-injective left $p - V'$ -ring whose maximal left ideals are quasi-injective;
- (7) A is a left or right $p - V'$ -ring such that both A and Q are MELT;
- (8) A is a regular ring such that every maximal left ideal of Q is quasi-injective;
- (9) A is an ECTE left p -injective ring;
- (10) A is an ECTE right p -injective ring;
- (11) A is an ECTE ring whose simple left modules are flat.

PROOF. Obviously, (1) implies (2) through (11).

(2) implies (1): Since a MELT regular ring with zero socle is strongly regular (cf. the proof of [18, Proposition 3]), it then follows that A must have non-zero socle. Since A is a left and right V -ring (Theorem 2.7), then A is simple Artinian from [7, p. 350].

Similarly, (3) implies (1).

Since a prime left continuous regular ring is left self-injective [15, p. 604], then (4) implies (2).

The proof of Theorem 2.9 shows that (5) implies (2). Assume (6). Then A is *MELT* and $Z = 0$ by [23, Proposition 8 (1)]. Thus (6) implies (2).

The proof of [23, Theorem 11 (4)] shows that (7) implies (1).

(8) implies (1): Q is a prime *MELT* left self-injective regular ring and since (2) implies (1), then Q is simple Artinian. Now A is left Goldie by [12, Theorem 1.6], whence A is simple Artinian [19, Corollary 8].

By [8, Theorem 1] and Remark 1, either (9) or (10) implies (1).

Finally, (11) implies (1) by [1, Proposition 2.1] and Remark 1.

QUESTION 2. Is a prime *MELT* left or right self-injective ring Artinian?

If A is strongly regular, then Q is a left and right self-injective strongly regular ring, whence Q is both a left and right q -ring. (Arbitrary left q -rings need not be right q -rings.) The next result, which is related to [6, Query (c)], yields a class of regular rings of bounded index which strictly contains the class of strongly regular rings.

PROPOSITION 2.13. *Let A be a regular ring such that every maximal left ideal of Q is quasi-injective. Then A is a left and right V -ring of bounded index.*

PROOF. Q is a left self-injective regular ring of bounded index by Theorem 2.9. Then A is a left and right V -ring of bounded index by [7, Corollaries 7.4 and 7.10].

[12, Theorem 1.6] and Theorem 2.12 (5) imply

PROPOSITION 2.14. *If A is a prime left non-singular ring, then A is left Goldie iff every maximal left ideal of Q is quasi-injective.*

A theorem of I. Connell asserts that if A is left self-injective and G is a finite group, then the group ring $A[G]$ is left self-injective. Theorem 2.9 then implies

PROPOSITION 2.15. *Let A be a left self-injective regular ring and G a finite group such that every maximal left ideal of $A[G]$ is quasi-injective. Then $A[G]$ is a left and right V -ring of bounded index.*

Since a *CTE* regular ring is *ECTE*, [7, Corollary 7.4 and Theorem 7.20], Lemma 1.2 and Proposition 1.4 imply

PROPOSITION 2.16. *Let A be a regular ring such that Q is *CTE*. Then A and Q are left and right V -rings of bounded index.*

PROPOSITION 2.17. *Let A be a *CTE* regular ring. Then*

- (1) *Q is also the maximal right quotient ring of A ;*
- (2) *Every complement left (resp. right) ideal of A is a left (resp. right) annihilator.*

PROOF. (1) follows from Lemma 1.2, Proposition 1.4 and [7, Theorem 6.21]. Then (2) is a result of Y. Utumi.

[7, Corollary 7.13], [25, Corollary 6] and Theorem 2.9 yield

PROPOSITION 2.18. *Let A be a left self-injective regular ring whose maximal left ideals are quasi-injective. If F is a finitely generated non-singular left A -module, then $\text{End}_A(F)$ is a left and right self-injective regular ring of bounded index.*

[7, Corollary 9.4] and Theorem 2.12 (3) yield

PROPOSITION 2.19. *Let A be a prime right self-injective regular ring and M a non-zero right non-singular injective A -module. Then $\text{End}_A(M)$ is simple Artinian iff it is *MELT*.*

REMARK 2. If A is a prime P.I. ring, then Q is the simple Artinian classical left quotient ring of A iff every maximal left ideal of Q is quasi-injective. (This follows from a result of W. Martindale III and Theorem 2.12 (2).)

QUESTION 3. Is a semi-prime P.I. left p -injective ring von Neumann regular?

It is known that every (left and right) continuous regular ring is unit-regular [7, Corollary 13.23]. If A is a left continuous regular ring, then $A = B \oplus C$, where B and C are ideals of A (whence each of B, C is, if non-zero, a ring with identity) and B is a left continuous strongly regular ring while C is a left self-injective regular ring [15, p. 604]. The next decomposition result (which generalises [9, Theorem 2.13]) then follows from Theorem 2.9.

THEOREM 2.20. *Let A be a *MELT* left continuous regular ring. Then $A = B \oplus C$, where B is a left and right continuous strongly regular*

ring and C is a left and right self-injective regular ring of bounded index such that every maximal left ideal is quasi-injective.

Finally, we turn to strongly regular rings ([18, Proposition 3] is improved).

THEOREM 2.21. *The following conditions are equivalent:*

- (1) A is strongly regular;
- (2) A is a semi-prime left $p - V'$ -ring whose maximal left ideals are ideals;
- (3) A is a left duo ring whose simple left modules are flat.

PROOF. (1) implies (2) and (3) obviously.

(2) implies (1): Since A is semi-prime and every maximal left ideal is an ideal, then every minimal left ideal of A (if it exists) is injective. Then A becomes a left $p - V$ -ring and the proof of [18, Proposition 3] shows that A is strongly regular.

(3) implies (1): For any $b \in A$, $r(b) = r(Ab) = r(AbA)$ is an ideal of A and if we suppose that $Ab + r(b) \neq A$, let K be a maximal left ideal of A containing $Ab + r(b)$. Since A/K is a flat left A -module, then $b = bd$ for some $d \in K$ [1, Proposition 2.1] which implies $1 - d \in r(b) \subseteq K$, whence $1 \in K$, a contradiction. Thus $Ab + r(b) = A$ which implies $b = beb$ for some $e \in A$. Since A is left duo, then A is strongly regular.

In view of [10], we add a last

REMARK 3. It would be nice to know which of our results have semi-group analogues.

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