

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

L. CARLITZ

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Rendiconti del Seminario Matematico della Università di Padova,
tome 62 (1980), p. 309-318

http://www.numdam.org/item?id=RSMUP_1980__62__309_0

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A Characterization of the Bernoulli and Euler Polynomials.

L. CARLITZ (*)

1. The following three multiplication formulas are well-known [5, pp. 18, 24]:

$$(1.1) \quad B_k(nx) = n^{k-1} \sum_{s=0}^{n-1} B_k\left(x + \frac{s}{n}\right)$$

$$(1.2) \quad E_k(nx) = n^k \sum_{s=0}^{k-1} (-1)^s E_k\left(x + \frac{s}{n}\right) \quad (n \text{ odd})$$

$$(1.3) \quad E_{k-1}(nx) = -\frac{2n^{k-1}}{k} \sum_{s=0}^{k-1} (-1)^s B_k\left(x + \frac{s}{n}\right) \quad (n \text{ even}),$$

where $B_k(x)$, $E_k(x)$ denote the Bernoulli and Euler polynomials in the standard notation,

$$(1.4) \quad \frac{xe^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!},$$

$$(1.5) \quad \frac{2e^{xz}}{e^z + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{z^k}{k!}.$$

Nielsen has observed [4, p. 54] that (1.1) and (1.2) characterize the respective polynomials. More precisely, if a monic polynomial of degree k satisfies (1.1) for a single value $n > 1$, then it is identical with

(*) Indirizzo dell'A.: Department of Mathematics - Duke University - Durham, N.C. 27706 U.S.A.

$B_k(x)$; similarly if a monic polynomial of degree k satisfies (1.2) for a single odd $n > 1$, then it is identical with $E_k(x)$. The present writer [1] has proved that if $f_k(x), g_{k-1}(x)$ are monic polynomials of degree $k, k-1$, respectively, that satisfy

$$(1.6) \quad g_{k-1}(nx) = -\frac{2n^{k-1}}{k} \sum_{s=0}^{n-1} (-1)^s f_k\left(x + \frac{s}{n}\right) \quad (n \text{ even})$$

for two distinct even k , then

$$(1.7) \quad f_k(x) = B_k(x) + c, \quad g_{k-1}(x) = E_{k-1}(x),$$

where c is an arbitrary constant.

The writer [2] has generalized (1.1), (1.2), (1.3) in the following way:

$$(1.8) \quad n^{k-1} \sum_{s=0}^{n-1} B_k\left(\frac{x}{n} + \frac{ms}{n}\right) = m^{k-1} \sum_{r=0}^{m-1} B_k\left(\frac{x}{m} + \frac{nr}{m}\right),$$

$$(1.9) \quad n^k \sum_{s=0}^{n-1} (-1)^s E_k\left(\frac{x}{n} + \frac{ms}{n}\right) = m^k \sum_{r=0}^{m-1} (-1)^r E_k\left(\frac{x}{m} + \frac{nr}{m}\right) \\ (m \equiv n \equiv 1 \pmod{2}),$$

$$(1.10) \quad n^k \sum_{s=0}^{n-1} (-1)^s B_{k+1}\left(\frac{x}{n} + \frac{ms}{n}\right) = -\frac{1}{2} (k+1) m^k \sum_{r=0}^{m-1} E_k\left(\frac{x}{m} + \frac{nr}{m}\right) \\ (n \text{ even}).$$

These results were suggested by the formula for the gamma function

$$(1.11) \quad \prod_{s=0}^{n-1} \Gamma\left(mx + \frac{ms}{n}\right) = \left(\frac{m}{n}\right)^{m^2 nx + (m^2 n - m^2 - n)/2} (2\pi)^{(n-m)/2} \prod_{r=0}^{m-1} \Gamma\left(nx + \frac{nr}{n}\right)$$

due to Schoblock [3, pp. 196-198]. For $m = 1$, (1.11) reduces to the familiar multiplication formula for the gamma function.

The purpose of the present note is to see to what extent the Bernoulli and Euler polynomials are characterized by (1.8), (1.9), (1.10). We show that (1.8) and (1.9) do indeed characterize the Bernoulli and Euler polynomials, respectively, if a monic polynomial of degree k satisfies (1.8) for two unequal values m, n , then it is identical with $B_k(x)$; if a monic polynomial of degree k satisfies (1.9) for two unequal odd values m, n , then it is identical with $E_k(x)$.

The situation for (1.10) is somewhat less simple. We show that if $f_{k+1}(x)$ and $g_k(x)$ are monic polynomials of degree $k+1$, and k , respectively, that satisfy

$$(1.12) \quad n^k \sum_{s=0}^{n-1} (-1)^s f_{k+1} \left(\frac{x}{n} + \frac{ms}{n} \right) = -\frac{1}{2} (k+1) m^k \sum_{r=0}^{m-1} (-1)^r g_k \left(\frac{x}{m} + \frac{nr}{m} \right)$$

for two pairs of m, n and m', n' , where n and n' are even and in addition

$$m'n - mn' \neq 0,$$

then

$$f_{k+1}(x) = B_{k+1}(x) + c, \quad g_k(x) = E_k(x),$$

where c is an arbitrary constant. If however (1.12) is assumed only for the single pair m, n with n even, then

$$(1.13) \quad f_{k+1}(x) = a_0 + \sum_{j=0}^k a_{j+1} m^{k-j} B_{j+1}(x)$$

if and only if

$$(1.14) \quad g_k(x) = \sum_{j=0}^k (j+1) a_{j+1} n^{k+j} E_j(x).$$

Conversely, if $g_k(x)$ is defined by (1.14) then $f_{k+1}(x)$ is determined by (1.13) with a_0 arbitrary.

We remark that the results concerning the Euler polynomial can be carried over to the Eulerian polynomials discussed in [1] and [2]; however we shall not do so in the present note.

2. We first prove

THEOREM 1. *Let the monic polynomial $f_k(x)$ of degree k satisfy*

$$(2.1) \quad n^{k-1} \sum_{s=0}^{n-1} f_k \left(\frac{x}{n} + \frac{ms}{n} \right) = m^{k-1} \sum_{r=0}^{m-1} f_k \left(\frac{x}{m} + \frac{nr}{m} \right)$$

for two distinct (positive) values m, n . Then

$$(2.2) \quad f_k(x) = B_k(x).$$

PROOF. Let

$$(2.3) \quad S_k(x; m, n) = n^{k-1} \sum_{s=0}^{n-1} B_k\left(\frac{x}{n} + \frac{ms}{n}\right),$$

so that, by (1.8),

$$(2.4) \quad S_k(x; m, n) = S_k(x; n, m) \quad (k = 0, 1, 2, \dots).$$

It is clear from (2.3) that $S_k(x; m, n)$ is a monic polynomial of degree k . Moreover, from the proof of (1.8), we have

$$(2.5) \quad \sum_{k=0}^{\infty} S_k(x; m, n) \frac{z^k}{k!} = \frac{ze^{xz}(e^{mnz} - 1)}{(e^{mz} - 1)(e^{nz} - 1)}.$$

Now put

$$(2.6) \quad f_k(x) = \sum_{j=0}^k a_j B_j(x) \quad (a_k = 1),$$

where the coefficients a_j are independent of x and are uniquely determined by $f_k(x)$. Thus (2.1) becomes

$$n^{k-1} \sum_{j=0}^k a_j \sum_{s=0}^{n-1} B_j\left(\frac{x}{n} + \frac{ms}{n}\right) = m^{k-1} \sum_{j=0}^k a_j \sum_{r=0}^{m-1} B_j\left(\frac{x}{m} + \frac{nr}{m}\right).$$

Hence, by (2.3),

$$\sum_{j=0}^k a_j n^{k-j} S_j(x; m, n) = \sum_{j=0}^k a_j m^{k-j} S_j(x; n, m),$$

so that, by (2.4),

$$(2.7) \quad \sum_{j=0}^k a_j (n^{k-j} - m^{k-j}) S_j(x; m, n) = 0.$$

Since $S_j(x; m, n)$ is of precise degree j in x , it follows from (2.7)

that

$$a_j = 0 \quad (j = 0, 1, 2, \dots, k-1)$$

and (2.6) reduces to $f_k(x) = B_k(x)$.

We remark that it follows from (2.5) that

$$(2.8) \quad mnkS_{k-1}(s; m, n) = (mB + nB + x + mn)^k - (mB + nB + x)^k,$$

where

$$(2.9) \quad (mB + nB + x)^k = \sum_{i+j \leq k} \frac{k!}{i!j!(k-i-j)!} m^i n^j B_i B_j x^{k-1-j}.$$

Alternatively, $(mB + nB + x)^k$ can be exhibited as a Bernoulli polynomial of higher order [5, Ch. 6].

3. Turning to (1.9) we shall prove

THEOREM 2. *Let the monic polynomial $g_k(x)$ of degree k satisfy*

$$(3.1) \quad n^k \sum_{s=0}^{n-1} (-1)^s g_k\left(\frac{x}{n} + \frac{ms}{n}\right) = m^k \sum_{r=0}^{m-1} (-1)^r g_k\left(\frac{x}{m} + \frac{nr}{m}\right)$$

for two distinct odd values of m, n . Then

$$(3.2) \quad g_k(x) = E_k(x).$$

PROOF. Let

$$(3.3) \quad T_k(x; m, n) = n^k \sum_{s=0}^{n-1} (-1)^s E_k\left(\frac{x}{n} + \frac{ms}{n}\right),$$

so that, by (1.9),

$$(3.4) \quad T_k(x; m, n) = T_k(x; n, m), \quad (k = 0, 1, 2, \dots),$$

at least for m, n both odd. It follows from (3.3) that, for n odd, $T_k(x; m, n)$ is a monic polynomial of degree k . From the proof of (1.9) we have

$$(3.5) \quad \sum_{k=0}^{\infty} T_k(x; m, n) \frac{z^k}{k!} = \frac{2e^{xz}(e^{mnz} + 1)}{(e^{mz} + 1)(e^{nz} + 1)}.$$

Now put

$$(3.6) \quad g_k(x) = \sum_{j=0}^k b_j E_j(x) \quad (b_k = 1),$$

where the coefficients b_j are independent of x and are uniquely determined by $g_k(x)$. Thus (3.1) becomes

$$n^k \sum_{j=0}^k b_j \sum_{s=0}^{n-1} (-1)^s E_j\left(\frac{x}{n} + \frac{ms}{n}\right) = m^k \sum_{j=0}^k b_j \sum_{t=0}^{m-1} (-1)^t E_j\left(\frac{x}{m} + \frac{nr}{m}\right).$$

Thus, by (3.3),

$$\sum_{j=0}^k b_j n^{k-1} T_j(x; m, n) = \sum_{j=0}^k b_j m^{k-j} T_j(s; n, m),$$

so that, by (3.4),

$$(3.7) \quad \sum_{j=0}^k b_j (n^{k-j} - m^{k-j}) T_j(x; m, n) = 0.$$

Since $T_j(x; m, n)$ is of degree j in x , it follows from (3.7) that

$$b_j = 0 \quad (j = 0, 1, 2, \dots, k-1)$$

and therefore (3.6) reduces to $g_k(x) = E_k(x)$.

It follows from (3.5) that

$$(3.8) \quad 2mnT_k(x; m, n) = \left(\frac{1}{2}mC + \frac{1}{2}nC + x + mn\right)^k + \left(\frac{1}{2}mC + \frac{1}{2}nC + x\right)^k \quad (m \equiv b \equiv 1 \pmod{\zeta}),$$

where

$$(3.9) \quad \left(\frac{1}{2}mC + \frac{1}{2}nC + x\right)^k = \sum_{i+j \leq k} \frac{k!}{i!j!(k-i-j)!} 2^{-i-j} m^i n^j C_i C_j x^{k-i-j}$$

and [5, p. 28]

$$(3.10) \quad E_k(x) = (x + \frac{1}{2}C)^k, \quad E_k(0) = 2^{-k}C_k.$$

For n even, it is proved in [2] that

$$\sum_{k=0}^{\infty} \frac{(nz)^k}{k!} \sum_{s=0}^{n-1} (-1)^s E_k \left(\frac{x}{n} + \frac{ms}{n} \right) = \frac{2e^{xz}(1 - e^{mnz})}{(e^{mz} + 1)(e^{nz} + 1)}.$$

Since the right hand side is symmetric in m, n , it follows that (1.9) holds provided only that m and n have the same parity. The definition (3.3) holds for arbitrary n and therefore

$$(3.11) \quad \sum_{k=0}^{\infty} \frac{z^k}{k!} T_k(x; m, n) = \frac{2e^{xz}(1 - e^{mnz})}{(e^{mz} + 1)(e^{nz} + 1)} \quad (n \text{ even}).$$

We accordingly get

$$(3.12) \quad 2mnT_k(x; m, n) = \\ = \left(\frac{1}{2}mC + \frac{1}{2}nC + x \right)^k - \left(\frac{1}{2}mC + \frac{1}{2}nC + x + mn \right)^k \quad (n \text{ even}).$$

Expanding the right member of (3.12) it is clear that, for n even, $T_k(x; m, n)$ is of degree $k - 1$; the coefficient of x^{k-1} is equal to $-mn$.

We now consider the equation (3.1) assuming that both m and n are even. The proof of Theorem 2 applies without change down to and including (3.7). In the present situation $T_j(x; m, n)$ is of degree $j - 1$ for $j \geq 1$. Hence we infer that

$$b_j = 0 \quad (j = 1, 2, \dots, k - 1).$$

Finally we may state

THEOREM 3. *Let the monic polynomial $g_k(x)$ satisfy (3.1) for two distinct even values of m, n . Then*

$$(3.9) \quad g_k(x) = E_k(x) + c,$$

where c is an arbitrary constant.

4. Let $f_{k+1}(x)$ be a monic polynomial of degree $k + 1$ and let $g_k(x)$ be a monic polynomial of degree k . Consider the equation

$$(4.1) \quad n^k \sum_{s=0}^{n-1} (-1)^s f_{k+1} \left(\frac{x}{n} + \frac{ms}{n} \right) = -\frac{1}{2} (k + 1) m^k \sum_{r=0}^{m-1} g_k \left(\frac{x}{m} + \frac{nr}{m} \right)$$

for fixed m and fixed even n .

Put

$$(4.2) \quad U_{k+1}(x; m, n) = n^k \sum_{s=0}^{n-1} (-1)^s B_{k+1} \left(\frac{x}{n} + \frac{ms}{n} \right)$$

and

$$(4.3) \quad V_k(x; m, n) = m^k \sum_{r=0}^{m-1} E_k \left(\frac{x}{m} + \frac{nr}{m} \right).$$

Then by (1.10)

$$(4.4) \quad U_{k+1}(x; m, n) = -\frac{1}{2}(k+1) V_k(x; m, n).$$

By (4.3) it is evident that $V_k(x; m, n)$ is monic of degree k . Hence $U_{k+1}(x; m, n)$ is of degree k and with highest coefficient equal to $-\frac{1}{2}(k+1)$.

Let

$$(4.5) \quad f_{k+1}(x) = \sum_{j=0}^{k+1} a_j B_j(x), \quad g_k(x) = \sum_{j=0}^k b_j E_j(x),$$

where the a_j, b_j are independent of x and are uniquely determined by $f_{k+1}(x)$ and $g_k(x)$, respectively; in particular, $a_{k+1} = b_k = 1$.

Substituting from (4.5) in (4.1), we get

$$n^k \sum_{j=0}^{k+1} a_j \sum_{s=0}^{n-1} (-1)^s B_j \left(\frac{x}{n} + \frac{ms}{n} \right) = -\frac{1}{2}(k+1) m^k \sum_{j=0}^k b_j \sum_{r=0}^{m-1} E_j \left(\frac{x}{m} + \frac{nr}{m} \right),$$

that is

$$(4.6) \quad \sum_{j=0}^{k+1} a_j n^{k-j+1} U_j(x; m, n) = -\frac{1}{2}(k+1) \sum_{j=0}^k b_j m^{k-j} V_j(x; m, n).$$

By (4.4) this reduces to

$$(4.7) \quad a_0 n^{k+1} U_0(x; m, n) + \frac{1}{2}(k+1) \sum_{j=0}^k \{b_j m^{k-j} - (j+1) a_{j+1} n^{k-j}\} V_j(x; m, n) = 0.$$

Note that

$$U_0(x; m, n) = n^{-1} \sum_{s=0}^{n-1} (-1)^s B_0 \left(\frac{x}{n} + \frac{ms}{n} \right) = 0.$$

Since $V_j(x; m, n)$ is of degree j it follows from (4.7) that

$$(4.8) \quad b_j m^{k-j} - (j+1) a_{j+1} n^{k-j} = 0 \quad (j = 0, 1, 2, \dots, k).$$

For $j = k$, (4.8) is automatically satisfied in view of $a_{k+1} = b_k = 1$.

We now assume that (4.1) is satisfied by a second pair of numbers m', n' , with n' even. Then by (4.8) we have also

$$(4.9) \quad b_j m'^{k-j} - (j+1) a_{j+1} n'^{k-j} = 0 \quad (j = 0, 1, 2, \dots, k).$$

It follows from (4.8) and (4.9) that

$$(4.10) \quad a_{j+1}((m'n)^{k-j} - (mn')^{k-j}) = 0 \quad (j = 0, 1, \dots, k-1).$$

For $j = k-1$, (4.10) reduces to

$$a_k(m'n - mn') = 0.$$

We therefore assume that

$$(4.11) \quad m'n - mn' \neq 0.$$

It is then clear that (4.10) implies

$$(4.12) \quad a_j = 0 \quad (j = 1, 2, \dots, k),$$

so that

$$(4.13) \quad b_j = 0 \quad (j = 0, 1, \dots, k-1).$$

This completes the proof of

THEOREM 4. *Let $f_{k+1}(x)$ and $g_k(x)$ be monic polynomials of degree $k+1$ and k ; respectively. Assume that*

$$(4.14) \quad n^k \sum_{s=0}^{n-1} (-1)^s f_{k+1}\left(\frac{x}{n} + \frac{ms}{n}\right) = -\frac{1}{2}(k+1)m^k \sum_{r=0}^{m-1} g_k\left(\frac{x}{m} + \frac{nr}{m}\right)$$

for two pairs of number m, n and m', n' , where n and n' are even and in addition

$$(4.15) \quad m'n - mn' \neq 0.$$

Then

$$(4.16) \quad f_{k+1}(x) = B_{k+1}(x) + c, \quad g_k(x) = E_k(x),$$

where c is an arbitrary constant.

If we assume only that (4.14) is satisfied for the pair m, n we get the following

COROLLARY. *Let $f_{k+1}(x)$ and $g_k(x)$ satisfy the hypothesis of Theorem 4. Assume that (4.14) holds for the pair m, n with n even. Let*

$$(4.17) \quad f_{k+1}(x) = a_0 + \sum_{j=0}^k a_{j+1} m^{k-j} B_{j+1}(x).$$

Then $g_k(x)$ is uniquely determined by

$$(4.18) \quad g_k(x) = \sum_{j=0}^k (j+1) a_{j+1} n^{k-j} E_j(x).$$

Conversely, if $g_k(x)$ is given by (4.18) then $f_{k+1}(x)$ is determined by (4.17) with a_0 arbitrary.

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Manoscritto pervenuto in redazione il 24 aprile 1979.