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Existence Theorems for Moving Boundary Optimization Problems.

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SUMMARY - We prove the existence of a minimum for an integral functional whose domain of integration varies in the class of closed convex subsets of a bounded closed convex set in $\mathbb{R}^n$.

1. Introduction.

Let $\mathbb{R}^n$ be the usual n-dimensional euclidean space; we are given T, a bounded closed convex subset of $\mathbb{R}^n$, $L: T\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, a normal proper integrand and l, an extended real valued function defined on the space $\mathbb{K}(T)$ of all closed convex subsets of T, which is lower semicontinuous with respect to a suitable sort of convergence in $\mathbb{K}(T)$.

For any $\Omega \in \mathbb{K}(T)$ we denote by Ω^0 the interior of Ω and we consider $H^{1,1}(\Omega^0)$, the usual Sobolev space of real valued functions.

For any given pair (Ω, x), $\Omega \in \mathbb{K}(T)$, $x \in H^{1,1}(\Omega^0)$, we define

$$F(\Omega, x) = \int_\Omega L(t, x(t), \nabla x(t))\, dt + l(\Omega)$$

and if $u \in H^{1,1}(T^0)$ is a fixed function and γ_Ω is the trace operator in

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$H^{1,1}(\Omega^0)$, we consider the problem

$$\textit{Minimize } \{F(\Omega, x)\colon\ \Omega \in \mathbb{K}(T),\ x \in H^{1,1}(\Omega^0),\ \gamma_\Omega(x) = \gamma_\Omega(u)\}\ .$$

In this paper we shall prove a theorem which assures that it has at least one solution.

Our problem has been already treated by Ioffe in [10] in the one-dimensional case, while the problem to minimize $\int_\Omega L(t, x(t), \nabla x(t))\, dt$ is also treated in [8] when Ω is a fixed set in $\mathbb{R}^n$.

We also recall that one-dimensional case in a fixed interval is studied in [1-2-3-4-5] also when the arcs take values in an infinite dimensional space and when the integrand depends upon higher order derivatives.

We wish to express our thanks to Professor J. P. Cecconi for his interest in this work.

2. Notations.

Throughout all of this work $\mathbb{R}^n$ is the usual n-dimensional euclidean space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$; T is a fixed bounded closed convex subset of $\mathbb{R}^n$ and $\mathbb{K}(T)$ is the set of all closed convex subsets of T, equipped with the Kuratowski convergence [12].

For any fixed $\Omega \in \mathbb{K}(T)$, Ω^0 is the interior of Ω, $H^{1,1}(\Omega^0)$ is the space of all real functions $x \in L^1(\Omega)$ such that $\partial x/\partial t_i$, in the sense of distributions, is in $L^1(\Omega)$, for every $i = 1, 2, \ldots, n$, while $H_0^{1,1}(\Omega^0)$ is the closure in $H^{1,1}(\Omega^0)$ of $C_0^\infty(\Omega^0)$, the space of all real functions, which have compact support contained in Ω^0 and derivatives of any order in Ω^0; the vector

$$(\partial x/\partial t_1,\ \partial x/\partial t_2,\ \ldots.,\ \partial x/\partial t_n)$$

will be indicated briefly by ∇x.

Since Ω^0 is a bounded convex set, we may assert that there is a linear continuous trace operator:

$$\gamma_\Omega\colon H^{1,1}(\Omega^0) \to L^1(\partial\Omega)\ ,$$

and it can be proved, [13-14], that

$$H_0^{1,1}(\Omega^0) = \{x \in H^{1,1}(\Omega^0)\colon \gamma_\Omega(x) = 0\}\ .$$

$L\colon T\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a normal proper integrand, [15-16-17-18], *such that $L(t, x, \cdot)$ is convex for every $(t, x)\in T\times\mathbb{R}$* and

$$H\colon T\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$$

is defined as

$$H(t, x, p) = \sup\{\langle p, v\rangle - L(t, x, v)\colon v\in\mathbb{R}^n\}$$

and is usually called the Hamiltonian function associated to L.

$l\colon\mathbb{K}(T)\to\mathbb{R}\cup\{+\infty\}$ *is a lower semicontinuous function*.

For every pair (Ω, x), $\Omega\in\mathbb{K}(T)$, $x\in H^{1,1}(\Omega^0)$

$$F(\Omega, x) = \int_\Omega L\big(t, x(t), \nabla x(t)\big)\, dt + l(\Omega)$$

is a well defined real extended valued functional, as soon as we adopt the usual conventions about infinite values [1-2-3-4-5].

To avoid trivial cases we always suppose that there is at least one pair (Ω_1, x_1) such that $F(\Omega_1, x_1)$ is a real number.

3. Statement of the problem.

The problem we are dealing with can be posed in the following form: let $u\in H^{1,1}(T^0)$ and let

$$\beta = \inf\{F(\Omega, x)\colon \Omega\in\mathbb{K}(T),\ x\in H^{1,1}(\Omega^0),\ \gamma_\Omega(x) = \gamma_\Omega(u)\}\,.$$

Are there $\Omega_0\in\mathbb{K}(T)$, $x_0\in H^{1,1}(\Omega_0^0)$ such that $\gamma_{\Omega_0}(x_0) = \gamma_{\Omega_0}(u)$ and $F(\Omega_0, x_0) = \beta$?

Here we prove an existence theorem which gives an adfirmative answer to this question as soon as the following conditions are satisfied:

(1) *$\inf\{l(\Omega)\colon \Omega\in\mathbb{K}(T)\} = k > -\infty$;*

(2) *there is a normal integrand $g\colon T\times\mathbb{R}_+\to\mathbb{R}$ such that $g(\cdot, p)$ is summable for every p and there is $\alpha\in\mathbb{R}$ such that*

$$H(t, x, p)\leqslant g\big(t, |p|\big) + \alpha|x|\,.$$

REMARKS. We may always suppose, with no loss of generality, that g assumes positive values.

Let $\Omega^0 \in \mathbb{K}(T)$, if we choose

$$l(\Omega) = \begin{cases} 0\,, & \Omega = \Omega_0 \\ +\infty\,, & \Omega \neq \Omega_0 \end{cases}$$

our problem reduces to an usual fixed boundary problem.

We wish to remark that, in this case, we give an existence result which is better than the ones proved in [8]. In particular condition (2) is weaker than the « growth condition » used in [8] (for instance $L(t, x, v) = v^2 - |x|$ satisfies ((2) and does not satisfies the growth condition of [8]).

4. Some preliminar results.

We begin recalling the concept of Kuratowski convergence.

DEFINITION 1. *Let $\Omega_n \in \mathbb{K}(T)$, we say that Ω_n converges to Ω in the sense of Kuratowski and we simply write $\Omega_n \to \Omega$ when the following facts are verified*:

(i) *for every $x \in \Omega$ there is $x_n \in \Omega_n$ such that $x_n \to x$*;

(ii) *let $x_n \in \Omega_n$, for any subsequence $x_{n_k} \to x$, we have $x \in \Omega$.*

PROPOSITION 2. *Let $\Omega_n \in \mathbb{K}(T)$, $\Omega_n \to \Omega$, then $\Omega \in \mathbb{K}(T)$.*

The proof is an easy verification.

For every subset $\Omega \in \mathbb{R}^n$ we write $\partial\Omega$ for the boundary of Ω and Ω^c for $\mathbb{R}^n \setminus \Omega$.

DEFINITION 3. *Let $\Omega \in \mathbb{K}(T)$, $\varepsilon \in \mathbb{R}_+$, we define*

$$\Omega_{\varepsilon+} = \{x \in \mathbb{R}^n \colon d(x, \Omega) \leqslant \varepsilon\}$$

$$\Omega_{\varepsilon-} = \{x \in \mathbb{R}^n \colon d(x, \Omega^c) \geqslant \varepsilon\}\,.$$

PROPOSITION 4. *Let $\Omega_n \in \mathbb{K}(T)$, $\Omega_n \to \Omega$; then for every $\varepsilon \in \mathbb{R}_+$ there is $n_\varepsilon \in \mathbb{N}$ such that, for $n \geqslant n_\varepsilon$*

(i) $\Omega_n \subset \Omega_{\varepsilon+}$;

(ii) $\Omega \subset (\Omega_n)_{\varepsilon+}$.

Proof. Suppose that (i) is not true, then there are $x_n \in \Omega_n$ and $\varepsilon_0 \in \mathbf{R}_+$ such that $d(x_n, \Omega) > \varepsilon_0$. Since we may suppose $x_n \to x_0$, we have $x_0 \in \Omega$ and $d(x_0, \Omega) \geqslant \varepsilon_0$ which is absurd.

To prove (ii) let $B(x_i, \varepsilon/2) = \{x \in \mathbf{R}^n: |x_i - x| \leqslant \varepsilon/2\}$; then we may find a finite subset $N \subset \mathbf{N}$ such that

$$\Omega \subset \bigcup_{i \in N} B(x_i, \varepsilon/2), \qquad \text{with } x_i \in \Omega .$$

Since $\Omega_n \to \Omega$, there is $n_\varepsilon \in \mathbf{N}$ such that $d(x_i, \Omega_n) \leqslant \varepsilon/2$ for every $i \in N$ and for $n > n_\varepsilon$. So, if $x_0 \in \Omega$, there is $i_0 \in N$ such that

$$d(x_0, \Omega_n) \leqslant |x_0 - x_{i_0}| + d(x_{i_0}, \Omega_n) \leqslant \varepsilon \qquad \text{for } n > n_\varepsilon . \qquad \square$$

Lemma 5. *Let $\Omega \in \mathbf{K}(T)$, then $\Omega = (\Omega_{\varepsilon+})_{\varepsilon-}$.*

Proof. Let $x \in \Omega$, if $\inf\{|x - y|: y \in (\Omega_{\varepsilon+})^c\} < \varepsilon$ there is $x_0 \in (\Omega_{\varepsilon+})^c$ such that $|x - x_0| < \varepsilon$ and so we have $x_0 \in (\Omega_{\varepsilon+})^c$ and $x_0 \in \Omega_{\varepsilon+}$; this is absurd.

So $d(x, (\Omega_{\varepsilon+})^c) \geqslant \varepsilon$ and $x \in (\Omega_{\varepsilon+})_{\varepsilon-}$.

Conversely let $x \notin \Omega$, then there is $x_0 \in \Omega$ such that $0 < |x - x_0|$, $x - x_0 = \nu$, where ν is in the cone of the normals to Ω at x_0.

Now two cases are possible:

(i) $|\nu| > \varepsilon$;

(ii) $0 < |\nu| \leqslant \varepsilon$.

In the first case $d(x, \Omega) = |x - x_0| = |\nu| > \varepsilon$; so $x \notin \Omega_{\varepsilon+}$ *i.e.* $x \in (\Omega_{\varepsilon+})^c$ and, since $d(x, (\Omega_{\varepsilon+})^c) = 0$, $x \notin (\Omega_{\varepsilon+})_{\varepsilon-}$.

In the second case let $\lambda = \varepsilon + |\nu|/2$ and $y_0 = x_0 + \lambda\nu/|\nu|$; we have: $d(y_0, \Omega) = |y_0 - x_0| = \lambda > \varepsilon$ and $y_0 \in (\Omega_{\varepsilon+})^c$. So

$$|y_0 - x| = \big|\lambda - |\nu|\big| < \varepsilon, d(x, (\Omega_{\varepsilon+})^c) < \varepsilon \text{ and } x \notin (\Omega_{\varepsilon+})_{\varepsilon-} . \qquad \square$$

Theorem 6. *Let $\Omega_n \in \mathbf{K}(T)$, $\Omega_n \to \Omega$, then for every $\varepsilon \in \mathbf{R}_+$ there is $n_\varepsilon \in \mathbf{N}$ such that*

$$\Omega_{\varepsilon-} \subset \Omega_n \subset \Omega_{\varepsilon+} \qquad \textit{for every } n > n_\varepsilon .$$

Proof. It follows from proposition 4 and lemma 5 as soon as we observe that if $\Omega_1, \Omega_2 \in \mathbf{K}(T)$, $\Omega_1 \subset \Omega_2$ then $(\Omega_1)_{\varepsilon-} \subset (\Omega_2)_{\varepsilon-}$. $\square$

When $\Omega_1, \Omega_2 \in \mathbb{K}(T)$, we define the symmetric difference between Ω_1 and Ω_2 as

$$\Omega_1 \triangle \Omega_2 = (\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1) .$$

THEOREM 7. *Let* $\Omega_n \in \mathbb{K}(T)$, $\Omega_n \to \Omega$, *then*

$$\lim_n \operatorname{meas} (\Omega_n \triangle \Omega) = 0 .$$

PROOF. We have $\Omega_n \Delta \Omega \subset \Omega_{\varepsilon^+} \setminus \Omega_{\varepsilon^-}$ for $n > n_\varepsilon$; so it is enough to prove that meas $(\Omega_{\varepsilon^+} \setminus \Omega_{\varepsilon^-}) \to 0$ as $\varepsilon \to 0$.

Fixed any real positive number δ, there is an open set $G \supset \Omega$ and a closed set $F \subset \Omega^0$ such that meas $(G \setminus \Omega) < \delta/2$ and meas $(\Omega \setminus F) =$ $=$ meas $(\Omega^0 \setminus F) < \delta/2$.

Since it is easily seen that there is $\varepsilon_0 \in \mathbb{R}_+$ such that when $\varepsilon < \varepsilon_0$ $G \supset \Omega_{\varepsilon^+}$ and $F \subset \Omega_{\varepsilon^-}$, we may assert that

$$\operatorname{meas} (\Omega_{\varepsilon^+} \setminus \Omega_{\varepsilon^-}) \leqslant \operatorname{meas} (G \setminus \Omega) + \operatorname{meas} (\Omega \setminus F) < \delta . \qquad \square$$

COROLLARY 8. *Let* $\Omega_n \in \mathbb{K}(T)$, $\Omega_n \to \Omega$, *then*

$$\lim_\varepsilon \lim_n \operatorname{meas} (\Omega_n \setminus \Omega_{\varepsilon^-}) = 0 .$$

PROOF. When n is sufficiently large we have

$$\Omega_n \setminus \Omega_{\varepsilon^-} = [(\Omega_n \setminus \Omega) \cup (\Omega \setminus \Omega_{\varepsilon^-})] \setminus (\Omega \setminus \Omega_n) ,$$

$$(\Omega_n \setminus \Omega) \cap (\Omega \setminus \Omega_{\varepsilon^-}) = \emptyset ,$$

$$(\Omega \setminus \Omega_n) \subset [(\Omega_n \setminus \Omega) \cup (\Omega \setminus \Omega_{\varepsilon^-})] ,$$

and so

$$\operatorname{meas} (\Omega_n \setminus \Omega_{\varepsilon^-}) = \operatorname{meas} (\Omega_n \setminus \Omega) + \operatorname{meas} (\Omega \setminus \Omega_{\varepsilon^-}) - \operatorname{meas} (\Omega \setminus \Omega_n) \qquad \square$$

5. Main theorem.

In this part we state and prove the existence result for our problem.

THEOREM 9. *Suppose that conditions* (1) *and* (2) *are satisfied and let* $u \in H^{1,1}(T^0)$ *be a fixed function*; *then there is* $\Omega_0 \in \mathbb{K}(T)$ *and there is*

$x_0 \in H^{1,1}(\Omega_0^0)$ such that $\gamma_{\Omega_0}(x_0) = \gamma_{\Omega_0}(u)$ and

$$F(\Omega_0, x_0) = \inf\{F(\Omega, x)\colon\ \Omega \in \mathbb{K}(T),\ x \in H^{1,1}(\Omega^0),\ \gamma_\Omega(x) = \gamma_\Omega(u)\}\ .$$

PROOF. Unless to substitute $L(t, x, v)$ with $L(t, x + u(t), v + \nabla u(t))$, we may reduce ourselves to prove that there is $\Omega_0 \in \mathbb{K}(T)$ and there is $x_0 \in H_0^{1,1}(\Omega^0)$ such that

$$F(\Omega_0, x_0) = \inf\{F(\Omega, x)\colon\ \Omega \in \mathbb{K}(T),\ x \in H_0^{1,1}(\Omega^0)\}\ .$$

(Observe that $L(t, x + u(t), v + \nabla u(t))$ satisfies condition (2) as soon as it holds for $L(t, x, v)$).

For any $x \in H_0^{1,1}(\Omega^0)$ we define:

$$y(t) = \begin{cases} x(t)\ , & t \in \Omega^0 \\ 0\ , & t \in T^0 \setminus \Omega\ . \end{cases}$$

Obviously it results $y \in H_0^{1,1}(T^0)$.

Now, let $\Omega_n \in \mathbb{K}(T)$, $x_n \in H_0^{1,1}(\Omega_n^0)$ be a minimizing sequence. Since T is a bounded set, we may suppose, [9], that $\Omega_n \to \Omega_0$ and $\Omega_0 \in \mathbb{K}(T)$ (see also proposition 2); moreover, for some constant M, we have, for every $\lambda \in \mathbb{R}_+$,

$$\begin{aligned}(3)\qquad M &\geqslant \int_{\Omega_n} L(t, x_n(t), \nabla x_n(t))\,dt + l(\Omega_n) \geqslant \\ &\geqslant \int_{\Omega_n} \sup\{\langle p, \nabla x_n\rangle - H(t, x_n(t), p)\colon\ p \in \mathbb{R}^n\}\,dt + k \geqslant \\ &\geqslant \int_{\Omega_n} \{\lambda|\nabla x_n(t)| - g(t, \lambda) - \alpha|x_n(t)|\}\,dt + k \geqslant \\ &\geqslant \lambda\int_T |\nabla y_n(t)|\,dt - \int_T g(t, \lambda)\,dt - \alpha\int_T |y_n(t)|\,dt + k\ .\end{aligned}$$

Now, by Poincaré's inequality [10-11], we deduce that:

$$M \geqslant \lambda C\int_T |y_n(t)|\,dt - \int_T g(t, \lambda)\,dt - \alpha\int_T |y_n(t)|\,dt + k \leqslant (\lambda C - \alpha)\|y_n\|_{L^1(T)} + H$$

and $M - H \geqslant (\lambda C - \alpha)\|y_n\|_{L^1(T)}$.

So, since λ may be chosen arbitrarily large, we conclude that

$$\|y_n\|_{L^1(T)} \leqslant N .$$

If we put

$$f(t, s) = \sup \{rs - g(t, r) \colon \; r \in \mathbb{R}_+\}$$

by (3) we have

$$M - k \geqslant \int_T \sup \{\lambda |\nabla y_n(t)| - g(t, \lambda) \colon \; \lambda \in \mathbb{R}_+\} \, dt - \alpha \|y_n\|_{L^1(T)} \geqslant$$
$$\geqslant \int_T f(t, |\nabla y_n(t)|) \, dt - \alpha N .$$

So using (2) and [14], we deduce that ∇y_n is weakly compact in $(L^1(T))^n$, and in particular we obtain that

$$\|\nabla y_n\|_{(L^1(T))^n} \leqslant R .$$

Now, since y_n is bounded in $L^1(T)$ while ∇y_n is weakly compact in $(L^1(T))^n$, we may find a subsequence, which we call y_n again, such that $y_n \to x_0$ weakly in $H_0^{1,1}(T^0)$.

Let $t \in T^0 \setminus \Omega_0$, then, since $\Omega_n \to \Omega_0$, for n sufficiently large $t \notin \Omega_n$ and $y_n(t) = 0$; so by [8], $x_0(t) = 0$ for almost every $t \in T^0 \setminus \Omega_0$. Moreover

$$\int_{\partial(T^0 \setminus \Omega_0)} |\gamma_{(T^0 \setminus \Omega_0)}(x_0)| \leqslant A \int_{T^0 \setminus \Omega_0} \{|x_0| + |\nabla x_0|\} = 0$$

and we have $x_0 \in H_0^{1,1}(\Omega_0^0)$.

Finally let

$$\beta = \inf \{F(\Omega, x) \colon \; \Omega \in \mathbb{K}(T), \; x \in H_0^{1,1}(\Omega_0^0)\} ;$$

we have:

$$\beta = \lim_n \int_{\Omega_n} L(t, x_n(t), \nabla x_n(t)) \, dt + l(\Omega_n) \geqslant \liminf_n \int_{\Omega_{\varepsilon-}} L(t, y_n(t), \nabla y_n(t)) \, dt +$$
$$+ \int_{\Omega_n \setminus \Omega_{\varepsilon-}} L(t, y_n(t), \nabla y_n(t)) \, dt + l(\Omega_0) \geqslant \int_{\Omega_{\varepsilon-}} L(t, x_0(t), \nabla x_0(t)) \, dt + l(\Omega_0) +$$
$$+ \liminf_n \int_{\Omega_n \setminus \Omega_{\varepsilon-}} \{- g(t, 0) - \alpha |y_n(t)|\} \, dt .$$

for every $\varepsilon \in \mathbb{R}_+$, as soon as we remember the semicontinuity result of [11].

Now, by monotone convergence theorem, as $L(t, x_0(t), \nabla x_0(t))$ is minorated by a summable function (see (2)),

$$\lim_{\varepsilon} \int_{\Omega_{\varepsilon^-}} L(t, x_0(t), \nabla x_0(t)) dt = \int_{\Omega} L(t, x_0(t), \nabla x_0(t))\, dt;$$

on the other hand, using corollary 8 and the Dunford-Pettis theorem, we deduce that

$$\lim_{\varepsilon} \liminf_{n} \int_{\Omega_n \setminus \Omega_{\varepsilon^-}} \{-g(t, 0) - \alpha |y_n(t)|\}\, dt = 0 .$$

So when $\varepsilon \to 0$ we obtain

$$\beta = F(\Omega_0, x_0) . \qquad \square$$

6. Some particular cases.

Let's now briefly consider some possible forms for l, which are of particular interest.

Let $\Omega \in \mathbb{K}(T)$ and let

$$\chi_\Omega(t) = \begin{cases} 1, & t \in \Omega \\ 0, & t \notin \Omega \end{cases}$$

be its characteristic function.

Let moreover π_Ω be the vector valued measure defined on Borel sets in $\mathbb{R}^n$ as the distributional derivative of χ_Ω.

It is well known that the perimeter of Ω may be defined as

$$p(\Omega) = |\pi_\Omega|$$

where $|\pi_\Omega|$ denotes the total variation of π_Ω in $\mathbb{R}^n$, [6-7-9].

Using the results of [6-7] and taking into account theorem 7 we may also state the following

PROPOSITION 10. *Let $\Omega_n \in \mathbb{K}(T)$, $\Omega_n \to \Omega$, then $\liminf_n p(\Omega_n) \geqslant p(\Omega)$; moreover if $p(\Omega_n) \leqslant k$, for some $k \in \mathbb{R}_+$, then for every function $f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\lim_n \int_{\mathbf{R}^n} f\, d\pi_{\Omega_n} = \int_{\mathbf{R}^n} f\, d\pi_\Omega .$$

Therefore we may assert that if we define

$$l_1(\Omega) = p(\Omega)\,, \qquad l_2(\Omega) = \begin{cases} 0\,, & p(\Omega) \leqslant k \\ +\infty\,, & p(\Omega) > k \end{cases},$$

$$l_3(\Omega) = \begin{cases} \int_{\mathbf{R}^n} f\, d\pi_\Omega\,, & p(\Omega) \leqslant k \\ +\infty\,, & p(\Omega) > k \end{cases}$$

where $k \in \mathbb{R}_+$ and $f \in C_0^\infty(\mathbb{R}^n)$, then l_i, $i = 1, 2, 3$, is lower semicontinuous in $\mathbb{K}(T)$ and satisfies condition (1). This fact allows us to apply theorem 9 with such functions.

REFERENCES

[1] O. CALIGARIS - P. OLIVA, *Un teorema di esistenza per problemi di Lagrange in spazi di Banach,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **61** (1976), pp. 571-579.

[2] O. CALIGARIS - P. OLIVA, *An existence theorem for Lagrange problems in Banach spaces,* J. Optimization Theory Appl., **27** (1978), pp. 83-91.

[3] O. CALIGARIS - P. OLIVA, *Non convex control problems in Banach spaces,* Appl. Math. Optim. **5** (1979) pp. 315-329.

[4] O. CALIGARIS - P. OLIVA, *Some remarks about Bolza and Lagrange problems in Hilbert spaces,* **15**-B (1978) pp. 956-967, Boll. Un. Mat. Ital.

[5] O. CALIGARIS - P. OLIVA, *Existence theorems for higher order problems in calculus of variations,* to appear.

[6] E. DE GIORGI, *Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni,* Ann. Mat. Pura Appl., **36** (1954), pp. 191-213.

[7] E. DE GIORGI, *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni,* Ricerche Mat., **4** (1955), pp. 95-113.

[8] F. Ferro, *Optimization theorems in the n-dimensional calculus of variations*, **16**-B (1979) pp. 225-265, Boll. Un. Mat. Ital.
[9] W. H. Fleming, *Functions whose partial derivatives are measures*, Illinois J. Math., **4** (1960), pp. 452-478.
[10] A. D. Ioffe, *An existence theorem for a general Bolza problem*, SIAM J. Control Optimization, **14** (1976), pp. 458-466.
[11] A. D. Ioffe, *On lower semicontinuity of integral functional I*, SIAM J. Control Optimization, **15** (1977), pp. 521-538.
[12] C. Kuratowski, *Topologie*, Panstwowe Wydawnictwo Naukowe, Warszawa, 1958.
[13] C. B. Morrey Jr., *Multiple integrals in the calculus of variations*, Spiinger, Berlin, 1966.
[14] J. Nečas, *Les métodes directes en théorie des équations elliptiques*, Masson et C.ie, éditeurs, Paris; Academia, éditeurs, Prague, 1967.
[15] R. T. Rockafellar, *Integral functionals, normal integrands and measurable selections*, in *Nonlinear Operators and the Calculus of variations*, L. Waelbroeck editor, pp. 157-207, Springer-Verlag, 1976.
[16] R. T. Rockafellar, *Optimal arcs and the minimum value function in problems of Lagrange*, Trans. Amer. Math. Soc., **180** (1973), pp. 53-83.
[17] R. T. Rockafellar, *Existence theorems for general control problems of Bolza and Lagrange*, Advances in Math., **15** (1975), pp. 312-333.
[18] R. T. Rockafellar, *Convex integral functionals and duality*, in *Contribution to Nonlinear Analysis*, E. H. Zarantonello editor, pp. 215-236, Academic Press, New York, 1971.

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