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G. Artico

A. Le Donne

R. Moresco

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Algebras of Real-Valued Uniform Maps.

G. ARTICO - A. LE DONNE - R. MORESCO (*)

Introduction.

The main object of this paper is a particular full subcategory of the category of uniform spaces and uniformly continuous functions (which we shall sometimes call simply «maps»). Such category, which we denote by $\mathfrak{A}$, consists of those uniform spaces whose set of real-valued uniform maps turns out to be an algebra. It is easily shown that the category $\mathfrak{A}$ is closed under quotients, colimits and coproducts; on the contrary subspaces and products of objects in $\mathfrak{A}$ generally do not belong to $\mathfrak{A}$ any longer.

The investigation of certain classes of algebras of uniformly continuous functions has been tackled by J. R. Isbell in [I.1] and A. W. Hager in the more general context of vector lattices [H]. However no condition has been given to characterize the objects of the category $\mathfrak{A}$. We can observe that there exist some classes of uniform spaces which trivially belong to the category $\mathfrak{A}$ such as fine spaces and precompact spaces; more generally we shall prove that locally fine spaces belong to $\mathfrak{A}$; this fact is obtained as an application of our main result (Theorem 1.3) which gives a characterization of the spaces μX belonging to $\mathfrak{A}$ in terms of a certain uniformity ϱ of $\boldsymbol{R}$. Such ϱ is the weak uniformity of the continuous polynomial dominated functions and it turns out that these functions are the only uniformly continuous functions from $\varrho\boldsymbol{R}$ to $\boldsymbol{R}$ equipped with the usual uniformity: so that $\varrho\boldsymbol{R}$ is in $\mathfrak{A}$. An accurate description of ϱ is given.

(*) Indirizzo degli AA.: Istituto di Matematica Applicata dell'Università - Via Belzoni, 7 - 35100 Padova.

The second paragraph deals with convexity of the algebras of real-valued uniform maps in $C(X)$.

In [H] Hager points out that there exists a coreflection from the category of uniform spaces onto the category $\mathfrak{A}$: in the third paragraph we give a direct construction of this coreflection by means of the uniformity ϱ.

The last section is devoted to the study of the algebras of real-valued uniform maps, namely of their prime ideals: we point out some analogies between such algebras and the algebras $C(X)$. The similarity does not work on the order structure of these quotients: an example is given in which every hyper-real quotient field fails to be η_1.

1. A characterization of the objects of $\mathfrak{A}$.

In this paper we are concerned with uniformities regarded as filters of coverings and all the proofs are obtained using the relative techniques; for axioms, terminology and notations we refer to [I.2]. If u is the usual uniformity on the real line, we shall briefly write $\boldsymbol{R}$ instead of $u\boldsymbol{R}$; and we shall write $U(\mu X)$ instead of $U(\mu X, \boldsymbol{R})$ to denote the set of all uniform maps from the uniform space μX to $\boldsymbol{R}$.

Let μ be a uniformity on $\boldsymbol{R}$ finer than u such that $U(\mu\boldsymbol{R})$ is an algebra: since $U(\mu\boldsymbol{R})$ contains the identity function ι, it necessarily contains the polynomial functions (in one variable); moreover if a continuous function f is dominated by a polynomial p (i.e. $|f| \leqslant p$), we have

$$f = \frac{f}{1 + \iota^2 p} \cdot (1 + \iota^2 p)$$

belongs to $U(\mu\boldsymbol{R})$ because

$$\lim_{x\to\infty} \frac{f}{1 + \iota^2 p} = 0 \qquad \text{and so} \qquad \frac{f}{1 + \iota^2 p} \in U(\boldsymbol{R})\ .$$

The set of continuous polinomial dominated functions is plainly an algebra; we denote by ϱ the weak uniformity induced by this family of functions.

DEFINITION. A countable covering $\mathcal{U} = \{U_i\colon i \in \boldsymbol{Z},\ U_i \text{ open}\}$ of a topological space X is said to be a chain if $U_i \cap U_j = \emptyset$ whenever $|i - j| > 1$.

1.1 LEMMA. Let α be the set of the chains of $\boldsymbol{R}$ which satisfy the condition:

$*$) there exist natural numbers n, m such that for every real $k > 0$ it is: $\operatorname{card}\{i\colon [-k, k] \cap U_i \neq \emptyset\} \leqslant m + k^n$ (we denote by card A the power of the set A).

α is a sub-basis of ϱ.

PROOF. For real $\varepsilon > 0$, let $\mathcal{U}_\varepsilon$ be the chain whose elements are the balls $B(i\varepsilon, \varepsilon)$ with radius ε and center $i\varepsilon$, $i \in \boldsymbol{Z}$ and $f\colon \boldsymbol{R} \to \boldsymbol{R}$ continuous and dominated by a polynomial which can be thought of the form $m + x^n$ without lack of generality; the inverse image $f^{-1}(\mathcal{U}_\varepsilon)$ is of course a chain and satisfies the condition $*$ for:

$$\operatorname{card}\{i\colon f^{-1}B(i\varepsilon, \varepsilon) \cap [-k, k] \neq \emptyset\} \leqslant \operatorname{card}\{i\colon B(i\varepsilon, \varepsilon) \cap \\ \cap [-(m + k^n), m + k^n] \neq \emptyset\} \leqslant \frac{2(m + k^n)}{\varepsilon} + 3;$$

then $f^{-1}(\mathcal{U}_\varepsilon) \in \alpha$. Now choose $\mathcal{U} \in \alpha$; every $U_i \in \mathcal{U}$ is a disjoint union of open intervals; we are going to shrink $\mathcal{U}$ taking in place of U_i the set U'_i made of the connected components of U_i which are not contained in a different U_j; the family $\mathcal{U}' = \{U'_i\colon i \in \boldsymbol{Z}\}$ is still a covering: in fact if x belongs to a single U_i, then $x \in U'_i$; otherwise $x \in (a, b) \cap \cap (c, d)$ where (a, b) is a component of U_i, (c, d) a component of U_j; $a = c \in U_k$ would imply $U_i \cap U_j \cap U_k \neq \emptyset$ against the definition; so either $(a, b) \in U'_i$ or $(c, d)\, U'_j$. Trivially $\mathcal{U}'$ is a chain and a component (a, b) of U'_i borders on intervals of U'_{i-1} or U'_{i+1}.

If (a, b) is a component of U'_i, we define on $[a, b]$: $f(x) = i$ if x belongs just to U'_i (such points do exist since $\boldsymbol{R}$ is connected); $f(a) = j$ if a belongs to U'_j (hence $j = i \pm 1$) and similarly $f(b)$; elsewhere let f be linear.

It is not hard to prove that the definition of f is a good one; f realizes $\mathcal{U}$ since $f^{-1}(n-1, n+1) = U'_n$, for $n \in \boldsymbol{Z}$. If m, n are constants related to $\mathcal{U}$ by the property $*$, f is dominated by the polynomial $m + x^n + |f(0)| + 1$: in fact, since the indexes i for which $[0, x]$ ($[x, 0]$ if $x < 0$) meets U'_i form an interval in $\boldsymbol{Z}$ (again since $[0, x]$ is connected and $\mathcal{U}$ is a chain) and their number is less or equal to $m + x^n$, and since $x \in U'_i$ implies $i - 1 < f(x) < i + 1$, we have $|f(x) - f(0)| \leqslant m + x^n + 1$. This implies that α is contained in ϱ, hence the thesis follows. ■

1.2 Theorem. $U(\varrho\boldsymbol{R})$ coincides with the algebra of polynomial dominated continuous functions.

Proof. Trivially the polynomial dominated continuous functions belong to $U(\varrho\boldsymbol{R})$. Then observe that a finite intersection of chains of α, although it is not a chain, verifies the condition $*$: in fact let $\mathcal{V}_i$, $i = 1, \dots, s$, be chains of α, m_i and n_i suitable constants as in $*$, and $\mathcal{V} = \mathcal{V}_i \wedge \dots \wedge \mathcal{V}_s$. Clearly for every positive real number k we have:

$$\operatorname{card}\{V \in \mathcal{V}\colon V \cap [-k, k] \neq \emptyset\} \leqslant \prod_{i=1}^{s} (m_i + k^{n_i}) \leqslant m + k^n$$

for suitable m, n. Let f belong to $U(\varrho\boldsymbol{R})$ and call $\mathcal{U}_1$ the covering of the open balls with integer center and radius one: $f^{-1}(\mathcal{U}_1)$ is coarser than $\mathcal{W} = \mathcal{W}_1 \wedge \dots \wedge \mathcal{W}_r$ for some $\mathcal{W}_i \in \alpha$. It is not restrictive to suppose $f(0) = 0$, $f(x) \geqslant 0$; if k is the greatest integer which does not exceed $f(x)$, there exist $k + 1$ points $x_0, \dots, x_k \in [0, x]$ such that $f(x_i) = i$; since $\mathcal{W} < f^{-1}(\mathcal{U}_1)$, the points x_i, $i = 0, \dots, k$, must belong to distinct elements of $\mathcal{W}$, then by condition $*$ $k + 1 \leqslant x^n + m$, which gives $f(x) \leqslant x^n + m$. ■

Remark 1. A trivial consequence of the characterization of $U(\varrho\boldsymbol{R})$ is that ϱ does not coincide with the fine uniformity on $\boldsymbol{R}$: for instance $\exp(x) \notin U(\varrho\boldsymbol{R})$.

1.3 Theorem. For a uniform space μX, the following are equivalent:

i) $U(\mu X)$ is an algebra,

ii) $U(\mu X) = U(\mu X, \varrho\boldsymbol{R})$.

Proof. i) $\Rightarrow$ ii): Let $U(\mu X)$ be an algebra and $f \in U(\mu X)$. To see that $f \in U(\mu X, \varrho\boldsymbol{R})$ it is enough to prove that if $g \in U(\varrho\boldsymbol{R})$ then $g \circ f \in U(\mu X)$. Choose a polynomial $p > 0$ such that: $\lim_{x \to \infty} (g/p) = 0$; $g/p \in U(\varrho\boldsymbol{R})$ and so $(g/p) \circ f \in U(\mu X)$; by hypothesis we obtain that $p \circ f \in U(\mu X)$ and also that $g \circ f = ((g/p) \circ f) \cdot (p \circ f) \in U(\mu X)$.

ii) $\Rightarrow$ i): $U(\mu X)$ is closed under sums, and the square (any power) of a map f belonging to $U(\mu X, \varrho\boldsymbol{R})$ is in $U(\mu X, \varrho\boldsymbol{R})$. By hypothesis $U(\mu X) = U(\mu X, \varrho\boldsymbol{R})$, at last the thesis follows from the equality $f \cdot g = \frac{1}{2}((f + g)^2 - f^2 - g^2)$. ■

1.4 COROLLARY. $U(\varrho \boldsymbol{R}, \varrho \boldsymbol{R}) = U(\varrho \boldsymbol{R})$.

PROOF. Apply theorems 1.2 and 1.3. ■

The following remarks show that theorem 1.3 cannot be improved:

REMARK 2. There exist uniform spaces μX such that $U(\mu X, \varrho)$ is not an algebra: let $\mu X = \varrho \boldsymbol{R} \times \varrho \boldsymbol{R}$; the projections π_1, π_2 belong to $U(\varrho \boldsymbol{R} \times \varrho \boldsymbol{R})$, while their product is uniformly continuous for a uniformity such as $\sigma \boldsymbol{R} \times \sigma \boldsymbol{R}$ if and only if σ is the discrete uniformity.

REMARK 3. Denote by $U^*(\mu X)$ the set of the bounded uniform maps from μX to $\boldsymbol{R}$. $U^*(\mu X)$ is an algebra for every uniform space μX.

REMARK 4. There exist uniform spaces μX such that $U(\mu X, \varrho \boldsymbol{R})$ is an algebra and is different from $U(\mu X)$: we take $\mu X = \boldsymbol{R}$ and prove that $U(\boldsymbol{R}, \varrho \boldsymbol{R})$ does not contain unbounded functions. Suppose that it contains such a function f; trivially it is not restrictive to assume that $f(\boldsymbol{R}) \subset [0, +\infty)$; we choose a sequence $x_n \in \boldsymbol{R}$ such that $f(x_n) = n$ and f is non-constant on every neighbourhood of x_n. Let $\varepsilon_n \in (0, \frac{1}{2})$ such that $f^{-1}B(n, \varepsilon_n) \not\supset B(x_n, 1/n)$. Let $U = \boldsymbol{R} \setminus \boldsymbol{N}$, $V = \bigcup_{n \in \boldsymbol{N}} B(n, \varepsilon_n)$; hence $\{U, V\}$ is a chain belonging to ϱ while $\{f^{-1}(U), f^{-1}(V)\}$ does not belong to the usual uniformity since it has no Lebesgue number. Furthermore $U^*(\boldsymbol{R})$ is contained in $U(\boldsymbol{R}, \varrho \boldsymbol{R})$ which is therefore an algebra.

The same argument works changing $\boldsymbol{R}$ with a connected metric uniform space.

We are going now to apply theorem 1.3 to locally fine spaces (the definition can be found in [I.2]).

We indicate by $\mu^{(1)}$ the derivative of the uniformity μ and by $\lambda\mu$ the locally fine reflection of μ (see [I.2] chapter VII).

1.5. LEMMA. Let μX be a uniform space and assume that there is a uniform covering $\mathcal{U}$ whose elements are compact. Then $\mu^{(1)}$ is the fine uniformity and X is paracompact.

PROOF. Let $\mathcal{V}$ be an open covering; if $U \in \mathcal{U}$, $U \cap \mathcal{V}$ is uniform in U, hence $\mathcal{V}$ is uniformly locally uniform, i.e. $\mathcal{V} \in \mu^{(1)}$. Now we have $\mu^{(1)} \supset FT\mu \supset \lambda\mu \supset \mu^{(1)}$ and so $FT\mu = \mu^{(1)}$. At last X is paracompact since every open covering is normal ([E] P 8.B). ■

1.6 THEOREM. If μX is a locally fine uniform space, $U(\mu X)$ is an algebra.

PROOF. Observe that

$$U(\mu X) \subset U(\lambda\mu X, \lambda\boldsymbol{R}) \subset U(\mu X, FT\boldsymbol{R}) \subset U(\mu X, \varrho\boldsymbol{R}) .$$

The first inclusion holds since λ is a functor; the second follows from the hypothesis and lemma 1.5; the third is obvious. The result follows now from theorem 1.3. ■

2. Convexity of $U(\mu X)$.

Observe that owing to theorem 1.2, $U(\varrho\boldsymbol{R})$ is an order-convex algebra in $C(\boldsymbol{R})$; furthermore it is trivial that $U(\mu X) = C(X)$ is a convex algebra if μ is the fine uniformity on X. However we have the following:

2.1 PROPOSITION.

i) If μX is a precompact space, $U(\mu X)$ is an algebra and $U(\mu X)$ is order-convex if and only if it coincides with $C^*(X)$, hence if and only if the completion of μX is the Stone-Čech compactification.

ii) If μX belong to $\mathfrak{A}$, $U(\mu X)$ is order-convex if and only if the Samuel compactification is the Stone-Čech compactification.

PROOF. i) Trivial by the equality: $U(\mu X) = U^*(\mu X)$.

ii) First observe that $U(\mu X)$ is convex if and only if $U^*(\mu X)$ is convex: indeed the necessity is trivial, conversely for $f \in C(X)$, $g \in U(\mu X)$ such that $0 \leqslant f \leqslant g$ we have $f = f/(1+g)\cdot(1+g)$ where the first factor is a bounded function. Furthermore if $p\mu X$ denotes the precompact reflection of μX, $U^*(\mu X) = U(p\mu X)$ (see [I.2] II.30); the conclusion follows from i). ■

Recall that for a topological space X the following are equivalent:

i) The fine uniformity is precompact.

ii) Every admissible uniformity is precompact.

iii) X is pseudocompact.

Hence we conclude that if X is a pseudocompact space then μX belongs to $\mathfrak{A}$ for every admissible uniformity μ and $U(\mu X)$ is convex if and only if μ is the fine uniformity. Moreover for a topological space X we can observe:

2.2 COROLLARY. Let βX be the Stone-Čech compactification of the topological space X. For every X such that card $(\beta X \setminus X) > 1$ there is an admissible uniformity μ for which $U(\mu X)$ is a non-convex algebra.

PROOF. Trivial. ■

If in proposition 2.1 i) we add a completeness hypothesis, X becomes compact, hence μX is fine; however we can give an example of a complete space such that $U(\mu X)$ is a non-convex algebra.

EXAMPLE. Take $X = \{x_i, y_i: i \in I, \text{card}\, I > \aleph_0\}$. For every subset J of I such that $\text{card}\, J \leqslant \aleph_0$, let:

$$\mathcal{U}_J = \Big\{\{x_h\}, \{y_h\}, \{x_k, y_k\}: h \in J, k \in I \setminus J\Big\}.$$

The coverings $\mathcal{U}_J$ form a basis for a uniformity μ on the set X. Clearly μX is a complete space. Take $f \in U(\mu X)$ and, for $n \in \boldsymbol{N}$, let U_{J_n} be a covering of μ such that the diameter of $f(U)$ is less than $1/n$ for every $U \in \mathcal{U}_{J_n}$, then say $\bar{J} = \bigcup_{n \in \mathbf{N}} J_n$. We get that $\mathcal{U}_{\bar{J}}$ refines $\mathcal{U}_{J_n}$ for every $n \in \boldsymbol{N}$, hence $f(x_k) = f(y_k)$ for every $k \in I \setminus \bar{J}$ since $\text{diam}\, \{f(x_k), f(y_k)\} < 1/n$ for every $n \in \boldsymbol{N}$. Therefore $U(\mu X)$ consists of the functions $f \in \boldsymbol{R}^X$ such that $f(x_i) = f(y_i)$ out of a suitable countable set, hence it is an algebra; plainly it is not convex because the characteristic function of the set $\{x_i: i \in I\}$ is a bounded continuous function which is not a uniform map.

3. A coreflection on the category $\mathfrak{A}$.

From the considerations at the beginning of section 1 we easily get that ϱ is the coarsest uniformity finer than the usual one such that the set of the real-valued uniform maps is an algebra. The problem we are now going to investigate is to see if analogously for every uniform space μX there exists a coarsest uniformity ν among the finer ones than μ for which the set of real-valued maps is an algebra.

Denote by μ^* the uniformity generated by the sub-basis of the coverings $f^{-1}(\mathcal{U})$ with $f \in U(\mu X)$, $\mathcal{U} \in \varrho$. Put $\mu_0 = \mu$; for every ordinal number α take $\mu_{\alpha+1} = \mu_\alpha^* \wedge \mu_\alpha$ and for every limit ordinal number α put $\mu_\alpha = \bigwedge_{\beta<\alpha} \mu_\beta$; clearly the set of the uniformities μ_α is totally ordered.

3.1 LEMMA. If $\sigma < \mu$ and $\sigma X \in \mathfrak{A}$, then $\sigma < \mu^*$ and as a consequence $\sigma < \mu_\alpha$ for every α.

PROOF. $U(\mu X) \subset U(\sigma X) = U(\sigma X, \varrho \boldsymbol{R})$. ■

Indicate with $\mathfrak{A}(\mu X)$ the set $\{\nu | \nu X \in \mathfrak{A}, \nu < \mu\}$.

3.2 THEOREM. Let μX be a uniform space. $\mathfrak{A}(\mu X)$ has a minimum $\underline{a}\mu$ (which necessarily induces the same topology as μ).

PROOF. Defined μ_α as above, there exists

$$\bar{\alpha} < \exp\big(\exp(\exp(\operatorname{card} X))\big)$$

such that $\mu_{\bar{\alpha}+1} = \mu_{\bar{\alpha}}$ since the set of μ_α is totally ordered and there are less than $\exp\big(\exp(\exp(\operatorname{card} X))\big)$ uniformities. Put $\underline{a}\mu = \mu_{\bar{\alpha}}$: we have $U(\underline{a}\mu X) \subset U(\mu_{\bar{\alpha}+1}, \varrho \boldsymbol{R}) = U(\underline{a}\mu X, \varrho\boldsymbol{R})$, therefore $\underline{a}\mu \in \mathfrak{A}$ by theorem 1.3. Furthermore $\underline{a}\mu$ is the smallest element of $\mathfrak{A}(\mu X)$ by lemma 3.1. ■

Later on we shall need the following interesting lemma:

3.3 LEMMA. Let A be a commutative algebra over a field of characteristic 0, B a vector subspace of A. The vector space C spanned by the powers of the elements of B is an algebra.

PROOF. We prove that C contains the elements of the type $x^r \cdot y^s$ for any $x, y \in B$, $r, s \in \boldsymbol{N}$. For n fixed we consider, for $x, y \in B$ and $i = 0, 1, \ldots, n$ the following elements:

$$c_i = (2^i x + y)^n = \sum_{k=0}^{n} 2^{i\cdot k} \binom{n}{k} x^k \cdot y^{n-k};$$

$c_i \in C$ and the matrix $(n+1)\times(n+1)$ which in the entry (i, k) has the element $a_{i,k} = 2^{i\cdot k}\binom{n}{k}$ is invertible: in fact

$$\det(a_{i,k}) = \left(\prod_{k=0}^{n}\binom{n}{k}\right)\det(a'_{i,k})$$

where $a'_{i,k} = 2^{i \cdot k}$ and $\det(a'_{i,k}) \neq 0$ because it is the determinant of Vandermonde of the numbers $1, 2, \ldots, 2^n$. If we denote by $(\bar{a}_{i,k})$ the inverse of $(a_{i,k})$, we have $x^k \cdot y^{n-k} = \sum_{i=0}^{n} \bar{a}_{i,k} c_i$ hence $x^k \cdot y^{n-k} \in C$. By an inductive argument it can now be easily shown that if $x_1, \ldots, x_j$ are distinct elements of B, then $x_1^{n_1} \ldots x_j^{n_j}$ belongs to C. ■

3.4 Corollary. The vector space spanned by $U(\mu X, \varrho \boldsymbol{R})$ in $C(X)$ is an algebra.

Proof. In fact $U(\mu X, \varrho \boldsymbol{R})$ is closed under powers. ■

3.5 Corollary. The algebra spanned in $C(X)$ by $U(\mu X)$ is contained in $U(\mu_1 X)$, then μ_1 is the coarsest uniformity finer than μ for which this situation occurs.

Proof. Trivial. ■

3.6 Proposition. Using the notations of theorem 3.2, $\bar{\alpha} \leqslant \omega_1$:

Proof. Take $f, g \in U(\mu_{\omega_1} X)$ and call $\mathfrak{U}_{1/n}$ the covering of $\boldsymbol{R}$ consisting of the open intervals with radius $1/n$, $n \in \boldsymbol{N}$. There exist $\mathfrak{V}_n \in \mu_{\omega_1}$ which refines both $f^{-1}(\mathfrak{U}_{1/n})$ and $g^{-1}(\mathfrak{U}_{1/n})$ and there are ordinal numbers $\alpha_n < \omega_1$ such that $\mathfrak{V}_n \in \mu_{\alpha_n}$. Let $\alpha = \sup \alpha_n$; then $\alpha < \omega_1$ and we have that $f, g \in U(\mu_\alpha X)$ and by corollary 3.5 $f \cdot g \in U(\mu_{\alpha+1} X) \subset U(\mu_{\omega_1} X)$. ■

Remark 5. We are unable to say wether 3.6 may be strengthened: in all the examples we have tested, we have found $\bar{\alpha} = 1$.

The uniformity $\underline{a}\mu$ may be reached by an alternative construction of algebraic type: for every uniformity μ let $\bar{\mu}^*$ be the weak uniformity of the algebra generated by $U(\mu X)$ in $C(X)$: then set $\bar{\mu}_0 = \mu$, $\bar{\mu}_{\alpha+1} = \bar{\mu}^*_\alpha \wedge \bar{\mu}_\alpha$ for every ordinal number, and if α is a limit ordinal number, $\bar{\mu}_\alpha = \bigwedge_{\beta<\alpha} \bar{\mu}_\beta$. Corollary 3.5 shows that $\bar{\mu}_\alpha = \mu_\alpha$ for every ordinal number α, hence the two constructions proposed are quite equivalent. We think that perhaps Hager refers to this second construction in his paper [H]. In the same paper the author says that the assignation $\underline{a}\colon \mu X \to \underline{a}\mu X$ is a coreflection onto the category $\mathfrak{A}$: a direct proof of this fact can be easily given.

3.7 Lemma. Let f belong to $U(\mu X, \nu Y)$; then f belongs to $U(\mu^* X, \nu^* Y)$, hence to $U(\mu_1 X, \nu_1 Y)$.

PROOF. Trivial. ■

3.8 PROPOSITION. $\underline{a}$ is a coreflection from the category of uniform spaces onto $\mathfrak{A}$.

PROOF. By transfinite induction it can be easily shown that if $f \in U(\mu X, \nu Y)$, then $f \in U(\mu_\alpha X, \nu_\alpha Y)$ for every ordinal number α, namely for $\alpha = \omega_1$. Furthermore if $f \in U(\underline{a}\mu X, \nu Y)$ then $f \in U(\underline{a}\mu X, \underline{a}\nu Y)$, hence $\underline{a}$ is a coreflection. ■

3.9 PROPOSITION. The functor $\underline{a}$ commutes with the completion.

PROOF. First observe that, if we indicate by $(\mu X)\hat{}$ the completion of the uniform space μX, $(\underline{a}\mu X)\hat{}$ is still an element of $\mathfrak{A}$, hence the identity function of X is a map between $\underline{a}\mu X$ and μX which extends to a map from $(\underline{a}\mu X)\hat{}$ into $\underline{a}(\mu X)\hat{}$. On the other hand $\underline{a}(\mu X)\hat{}$ induces over X a uniformity ν belonging to $\mathfrak{A}(\mu X)$ therefore the identity function $\iota: \nu X \to \underline{a}\mu X$ is a map which extends to the completions. ■

We make now a consideration about $\boldsymbol{R}^n$ equipped with the usual uniformity u: employing techniques analogous to the ones used in the proof of 1.1 and 1.2, we can describe the uniformity $au = \varrho_n$ which turns out to be the weak uniformity of the functions dominated by polynomials (with n variables): in fact the fundamental topic is to show that a sub-basis for $\underline{a}u$ is composed of the chains which satisfy the condition $*$, where the interval $[-k, k]$ is replaced by the closed ball with center in the origin and radius k.

4. Prime and maximal ideals in $U(\mu X)$.

Owing to the features of the uniformities studied in this paper, the problem of examinating the properties of the algebras of uniformly continuous real-valued maps arises quite naturally. Such algebras being both φ-algebras and Riesz spaces, many results descend directly from those theories.

DEFINITIONS. A totally ordered set X is said to be η_1 if for every pair of subsets A, B such that $A < B$ and card $(A \cup B) < \aleph_1$ there exists $x \in X$ such that $A < x < B$; if for every pair of non-empty subsets A, B satisfying the previous conditions there exists $x \in X$ such that $A \leqslant x < B$, the set X is said to be $q \cdot \eta_1$ (= quasi η_1).

In the following proposition we list a number of known facts (see [HIJ] and [GJ]):

4.1 PROPOSITION. Let μX belong to $\mathfrak{A}$, P a prime ideal of $U(\mu X)$, M a maximal one; then:

i) P is absolutely convex in $U(\mu X)$;

ii) $U(\mu X)/P$ is a totally ordered domain;

iii) $P(f) \geqslant 0$ if and only if $f \equiv |f|$ (mod. P);

iv) $f \hookrightarrow P(f)$ is a map of lattices;

v) P is contained in a unique maximal ideal;

vi) $U(\mu X)/M$ is a real-closed field;

vii) $U(\mu X)/M$ is a $q \cdot \eta_1$ field.

With regard to the quotients $U(\mu X)/M$ we may go on with the following:

4.2 THEOREM. Let A be a lattice-ordered sub-algebra of $C(X)$ with 1, closed under bounded inversion, M a maximal ideal of A. Given $s \geqslant 0$ belonging to A, put $Z_n = \{x \colon s(x) \geqslant n, n \in \boldsymbol{N}\}$ and $I_s = \{f \in A \colon f(Z_n) = 0$ for some $n\}$. Then $M(s)$ is infinitely large if and only if $M \supset I_s$.

PROOF. If s is a bounded function, the proposition follows trivially. Assume now s unbounded so that I_s is a proper ideal. If $M(s)$ is infinitely large, put $g_n = -\big((s - n) \wedge 0\big)$ and observe that g_n is a bounded function, so that $s - n + g_n$ cannot belong to M. Since $(s - n + g_n) \cdot g_n = 0$ it follows that $g_n \in M$ for every n. Given $f \in I_s$ there exists $n \in \boldsymbol{N}$ such that $f(Z_n) = 0$, that is f vanishes on the zero-set of g_n; by a simple calculation one sees that if $x \in X \setminus Z_n$ then $g_{n+1}(x) \geqslant 1$. By the equality $f = g_{n+1} \cdot f/(1 \vee g_{n+1})$ we conclude that f belong to M.

Conversely let g_n be the functions defined above: the functions g_n belong to I_s hence to M, and since $g_n + s \geqslant n$ we have $M(s) = M(s + g_n) \geqslant n$. ∎

4.3 THEOREM. Let $\mu X \in \mathfrak{A}$. The following are equivalent:

i) $U(\mu X)/M = \boldsymbol{R}$ for every maximal ideal M;

ii) $U(\mu X) = U^*(\mu X)$;

iii) $c\mu$ is precompact ($c\mu$ is the weak uniformity induced by all the real-valued functions uniformly continuous on μX).

PROOF. i) $\Rightarrow$ ii): if $U(\mu X)$ contains an unbounded function f, I_f is a proper ideal and by theorem 4.2 $M(f)$ is infinitely large for every maximal ideal $M \supset I_f$.

ii) $\Rightarrow$ i): trivial.

ii) $\Rightarrow$ iii): if $c\mu$ is not precompact, there exists a uniform covering $\mathcal{U}$ of $\boldsymbol{R}$ and a map $f \in U(\mu X)$ such that $f^{-1}(\mathcal{U})$ cannot be refined by a finite uniform covering of $c\mu$. This necessarily implies that f is unbounded.

iii) $\Rightarrow$ ii): by definitions. ∎

When $U(\mu X)/M$ is not the real field, it is an ordered field which contains a copy of $\boldsymbol{R}$; naturally we wonder if, as it happens for the residue fields of rings of real-valued continuous functions, $U(\mu X)/M$ turns out to be η_1; in the general case this question has a negative answer: in fact in view of 4.3 there are hyper-real quotient fields of $U(\varrho\boldsymbol{R})$, and all of them have cofinality $\aleph_0$: a cofinal countable set can be obtained by $M(m + x^{2n})$ for a maximal ideal M and natural numbers m, n.
Incidentally we point out the following consequences of this observation:

a) $U(\varrho\boldsymbol{R})$ cannot be isomorphic to any $C(X)$; however this fact can be proved directly;

b) all the quotient fields of $U(\varrho\boldsymbol{R})$ are complete in the uniformity canonically induced by the order: in fact it can be easily shown (see [M]) that if an ordered field is $q\cdot\eta_1$ and not η_1 then it is complete; hence in view of 4.1 vii) (see [HIJ]) if M is a maximal ideal of a φ-algebra A and A/M is not η_1 then it is complete (in the order uniformity).

Using the techniques of [GJ] 13.7 we can prove the following partial result:

4.4 PROPOSITION. If μX is a locally fine uniform space, M a maximal ideal of $U(\mu X)$ such that $U(\mu X)/M$ is not real, then $U(\mu X)/M$ is η_1.

PROOF. We omit the details; observe that if $\mathcal{U}_\varepsilon$ is a uniform covering of $\boldsymbol{R}$ of balls whose radius is $\varepsilon < 1/2$, with notations of [GJ] 13.7,

$h^{-1}(\mathcal{U}_\varepsilon)$ is uniformly locally uniform, hence h belongs to $U(\mu X)$; then use 4.2 to see that $M(h) \geqslant M(f_n)$. ■

We shall now make some considerations about the maximal spectrum Max $(U(\mu X))$, for $\mu X \in \mathfrak{A}$, equipped as usual with the hull-kernel topology whose closed sets are $V(I) = \{M \in \text{Max}\,(U(\mu X)) \colon M \supset I\}$ where I is an ideal of $U(\mu X)$. It is well known that Max $(U(\mu X)$ a compact Hausdorff space; the natural function $\vartheta\colon X \to \text{Max}\,(U(\mu X))$ defined by $\vartheta(x) = \{f \in U(\mu X) \colon f(x) = 0\}$ is obviously $1-1$ since $U(\mu X)$ separates points; moreover $\vartheta(X)$ is dense in Max $(U(\mu X))$. Now recall that a function between uniform spaces is said to be a δ-map if the preimages of two uniformly separated sets are uniformly separated and a δ-isomorphism if its inverse is a δ-map too. Recall also that in view of [I.2] II.35, a function between precompact spaces is a uniform isomorphism if and only if it is a δ-isomorphism. Now we can state the following:

4.5. Proposition. Let μX belong to $\mathfrak{A}$; $\vartheta\colon p\mu X \to \text{Max}\,(U(\mu X))$ (defined as above) is a uniform isomorphism onto its image; hence Max $(U(\mu X))$ is the Samuel compactification of μX.

Proof. By the previous remarks we shall prove that ϑ is a δ-isomorphism. If A, B are (uniformly) separated in Max $(U(\mu X))$, they are contained in disjoint closed sets $V(I)$, $V(J)$ respectively, hence there exist $i \in I$, $j \in J$ such that $i + j = 1$; then the functions i, j vanish respectively on $\vartheta^{-1}(V(I))$ and $\vartheta^{-1}(V(J))$ so that $\vartheta^{-1}(A)$, $\vartheta^{-1}(B)$ are uniformly separated by either function.

Conversely if A, B are subsets of X and there exists $i \in U(p\mu X)$ such that $i(A) = 0$, $i(B) = 1$, then $\vartheta(A) \subset V((i))$, $\vartheta(B) \subset V((1-i))$ and $V((i)) \cap V((1-i)) = \emptyset$.

As a consequence Max $(U(\mu X))$ is the completion of $\vartheta(p\mu X)$ hence the Samuel compactification of μX. ■

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