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Algebras of Real-Valued Uniform Maps.

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Introduction.

The main object of this paper is a particular full subcategory of the category of uniform spaces and uniformly continuous functions (which we shall sometimes call simply «maps»). Such category, which we denote by \( \mathcal{A} \), consists of those uniform spaces whose set of real-valued uniform maps turns out to be an algebra. It is easily shown that the category \( \mathcal{A} \) is closed under quotients, colimits and coproducts; on the contrary subspaces and products of objects in \( \mathcal{A} \) generally do not belong to \( \mathcal{A} \) any longer.

The investigation of certain classes of algebras of uniformly continuous functions has been tackled by J. R. Isbell in \([1.1]\) and A. W. Hager in the more general context of vector lattices \([H]\). However no condition has been given to characterize the objects of the category \( \mathcal{A} \). We can observe that there exist some classes of uniform spaces which trivially belong to the category \( \mathcal{A} \) such as fine spaces and precompact spaces; more generally we shall prove that locally fine spaces belong to \( \mathcal{A} \); this fact is obtained as an application of our main result \((\text{Theorem 1.3})\) which gives a characterization of the spaces \( \mu X \) belonging to \( \mathcal{A} \) in terms of a certain uniformity \( \varrho \) of \( R \). Such \( \varrho \) is the weak uniformity of the continuous polynomial dominated functions and it turns out that these functions are the only uniformly continuous functions from \( \varrho R \) to \( R \) equipped with the usual uniformity: so that \( \varrho R \) is in \( \mathcal{A} \). An accurate description of \( \varrho \) is given.

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The second paragraph deals with convexity of the algebras of real-valued uniform maps in \( C(X) \).

In [H] Hager points out that there exists a coreflection from the category of uniform spaces onto the category \( \mathfrak{A} \); in the third paragraph we give a direct construction of this coreflection by means of the uniformity \( q \).

The last section is devoted to the study of the algebras of real-valued uniform maps, namely of their prime ideals: we point out some analogies between such algebras and the algebras \( C(X) \). The similarity does not work on the order structure of these quotients: an example is given in which every hyper-real quotient field fails to be \( \eta_1 \).

1. **A characterization of the objects of \( \mathfrak{A} \).**

In this paper we are concerned with uniformities regarded as filters of coverings and all the proofs are obtained using the relative techniques; for axioms, terminology and notations we refer to [1.2]. If \( u \) is the usual uniformity on the real line, we shall briefly write \( R \) instead of \( uR \); and we shall write \( U(\mu X) \) instead of \( U(\mu X, R) \) to denote the set of all uniform maps from the uniform space \( \mu X \) to \( R \).

Let \( \mu \) be a uniformity on \( R \) finer than \( u \) such that \( U(\mu R) \) is an algebra: since \( U(\mu R) \) contains the identity function \( i \), it necessarily contains the polynomial functions (in one variable); moreover if a continuous function \( f \) is dominated by a polynomial \( p \) (i.e. \( |f| < p \)), we have

\[
   f = \frac{f}{1 + \nu^2 p} \cdot (1 + \nu^2 p)
\]

belongs to \( U(\mu R) \) because

\[
   \lim_{x \to \infty} \frac{f}{1 + \nu^2 p} = 0 \quad \text{and so} \quad \frac{f}{1 + \nu^2 p} \in U(R).
\]

The set of continuous polynomial dominated functions is plainly an algebra; we denote by \( q \) the weak uniformity induced by this family of functions.

**Definition.** A countable covering \( \mathfrak{U} = \{ U_i : i \in \mathbb{Z}, \ U_i \text{ open} \} \) of a topological space \( X \) is said to be a chain if \( U_i \cap U_j = \emptyset \) whenever \(|i - j| > 1|\).
1.1 Lemma. Let $\alpha$ be the set of the chains of $R$ which satisfy the condition:

- there exist natural numbers $n, m$ such that for every real $k > 0$ it is: $\text{card} \{i: [-k, k] \cap U_i \neq \emptyset\} < m + k^n$ (we denote by $\text{card} \ A$ the power of the set $A$).

$\alpha$ is a sub-basis of $\wp$.

Proof. For real $\varepsilon > 0$, let $\mathcal{U}_\varepsilon$ be the chain whose elements are the balls $B(ie, \varepsilon)$ with radius $\varepsilon$ and center $ie, i \in \mathbb{Z}$ and $f: R \to R$ continuous and dominated by a polynomial which can be thought of the form $m + x^n$ without lack of generality; the inverse image $f^{-1}(\mathcal{U}_\varepsilon)$ is of course a chain and satisfies the condition $\ast$ for:

\[
\text{card} \left\{i: f^{-1}B(ie, \varepsilon) \cap [-k, k] \neq \emptyset\right\} < \text{card} \left\{i: B(ie, \varepsilon) \cap \right.
\left.[-(m + k^n), m + k^n] \neq \emptyset\right\} < \frac{2(m + k^n)}{\varepsilon} + 3;
\]

then $f^{-1}(\mathcal{U}_\varepsilon) \in \alpha$. Now choose $\mathcal{U} \in \alpha$; every $U_i \in \mathcal{U}$ is a disjoint union of open intervals; we are going to shrink $\mathcal{U}$ taking in place of $U_i$ the set $U'_i$ made of the connected components of $U_i$ which are not contained in a different $U_j$; the family $\mathcal{U}' = \{U'_i: i \in \mathbb{Z}\}$ is still a covering: in fact if $x$ belongs to a single $U'_i$, then $x \in U'_i$; otherwise $x \in (a, b) \cap \cap (c, d)$ where $(a, b)$ is a component of $U_i$, $(c, d)$ a component of $U_j$; $a = c \in U_i$ would imply $U_i \cap U_j \cap U_k \neq \emptyset$ against the definition; so either $(a, b) \in U'_i$ or $(c, d) U'_j$. Trivially $\mathcal{U}'$ is a chain and a component $(a, b)$ of $U'_i$ borders on intervals of $U'_{i-1}$ or $U'_{i+1}$.

If $(a, b)$ is a component of $U'_i$, we define on $[a, b]$: $f(x) = i$ if $x$ belongs just to $U'_i$ (such points do exist since $R$ is connected); $f(a) = j$ if a belongs to $U'_j$ (hence $j = i \pm 1$) and similarly $f(b)$; elsewhere let $f$ be linear.

It is not hard to prove that the definition of $f$ is a good one; $f$ realizes $\mathcal{U}$ since $f^{-1}(n - 1, n + 1) = U'_n$, for $n \in \mathbb{Z}$. If $m, n$ are constants related to $\mathcal{U}$ by the property $\ast$, $f$ is dominated by the polynomial $m + x^n + |f(0)| + 1$: in fact, since the indexes $i$ for which $[0, x] \cap [x, 0]$ if $x < 0$ meets $U'_i$ form an interval in $\mathbb{Z}$ (again since $[0, x]$ is connected and $\mathcal{U}$ is a chain) and their number is less or equal to $m + x^n$, and since $x \in U'_i$ implies $i - 1 < f(x) < i + 1$, we have $|f(x) - f(0)| < m + x^n + 1$. This implies that $\alpha$ is contained in $\wp$, hence the thesis follows.
1.2 Theorem. $U(qR)$ coincides with the algebra of polynomial dominated continuous functions.

Proof. Trivially the polynomial dominated continuous functions belong to $U(qR)$. Then observe that a finite intersection of chains of $\alpha$, although it is not a chain, verifies the condition $*$: in fact let $\mathcal{U}_i$, $i = 1, \ldots, s$, be chains of $\alpha$, $m_i$ and $n_i$ suitable constants as in $*$, and $\mathcal{U} = \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_s$. Clearly for every positive real number $k$ we have:

$$\text{for suitable } m, n. \text{ Let } f \in U(qR) \text{ and call } \mathcal{W}_i \text{ the covering of the open balls with integer center and radius one: } f^{-1}(\mathcal{W}_0) \text{ is coarser than } \mathcal{W} = \mathcal{W}_1 \cap \cdots \cap \mathcal{W}_s \text{ for some } \mathcal{W}_i \in \alpha. \text{ It is not restrictive to suppose } f(0) = 0, f(x) > 0; \text{ if } k \text{ is the greatest integer which does not exceed } f(x), \text{ there exist } k + 1 \text{ points } x_0, \ldots, x_k \in [0, x] \text{ such that } f(x_i) = i; \text{ since } \mathcal{W} < f^{-1}(\mathcal{W}_1), \text{ the points } x_i, i = 0, \ldots, k, \text{ must belong to distinct elements of } \mathcal{W}, \text{ then by condition } * \text{ } k + 1 < x^n + m, \text{ which gives } f(x) < x^n + m. \numberqed$$

Remark 1. A trivial consequence of the characterization of $U(qR)$ is that $\varrho$ does not coincide with the fine uniformity on $R$: for instance $\exp(x) \notin U(qR)$.

1.3 Theorem. For a uniform space $\mu X$, the following are equivalent:

i) $U(\mu X)$ is an algebra,

ii) $U(\mu X) = U(\mu X, qR)$.

Proof. i) $\Rightarrow$ ii): Let $U(\mu X)$ be an algebra and $f \in U(\mu X)$. To see that $f \in U(\mu X, qR)$ it is enough to prove that if $g \in U(qR)$ then $g \circ f \in U(\mu X)$. Choose a polynomial $p > 0$ such that: $\lim_{x \to \infty} (g/p) = 0; g/p \in U(qR)$ and so $(g/p) \circ f \in U(\mu X)$; by hypothesis we obtain that $p \circ f \in U(\mu X)$ and also that $g \circ f = (g/p) \circ f \cdot (p \circ f) \in U(\mu X)$.

ii) $\Rightarrow$ i): $U(\mu X)$ is closed under sums, and the square (any power) of a map $f$ belonging to $U(\mu X, qR)$ is in $U(\mu X, qR)$. By hypothesis $U(\mu X) = U(\mu X, qR)$, at last the thesis follows from the equality $f \cdot g = \frac{1}{2} ((f + g)^2 - f^2 - g^2). \numberqed$
1.4 **Corollary.** $U(qR, qR) = U(qR)$.

**Proof.** Apply theorems 1.2 and 1.3. ■

The following remarks show that theorem 1.3 cannot be improved:

**Remark 2.** There exist uniform spaces $\mu X$ such that $U(\mu X, q)$ is not an algebra: let $\mu X = qR \times qR$; the projections $\pi_1, \pi_2$ belong to $U(qR \times qR)$, while their product is uniformly continuous for a uniformity such as $\sigma R \times \sigma R$ if and only if $\sigma$ is the discrete uniformity.

**Remark 3.** Denote by $U^*(\mu X)$ the set of the bounded uniform maps from $\mu X$ to $R$. $U^*(\mu X)$ is an algebra for every uniform space $\mu X$.

**Remark 4.** There exist uniform spaces $\mu X$ such that $U(\mu X, qR)$ is an algebra and is different from $U(\mu X)$: we take $\mu X = R$ and prove that $U(R, qR)$ does not contain unbounded functions. Suppose that it contains such a function $f$; trivially it is not restrictive to assume that $f(R) \subset [0, +\infty)$; we choose a sequence $x_n \in R$ such that $f(x_n) = n$ and $f$ is non-constant on every neighbourhood of $x_n$. Let $\varepsilon_n \in (0, \frac{1}{2})$ such that $f^{-1}B(n, \varepsilon_n) \not\subset B(x_n, 1/n)$. Let $U = R \setminus N$, $V = \bigcup_{n \in N} B(n, \varepsilon_n)$; hence $\{U, V\}$ is a chain belonging to $q$ while $\{f^{-1}(U), f^{-1}(V)\}$ does not belong to the usual uniformity since it has no Lebesgue number. Furthermore $U^*(R)$ is contained in $U(R, qR)$ which is therefore an algebra.

The same argument works changing $R$ with a connected metric uniform space.

We are going now to apply theorem 1.3 to locally fine spaces (the definition can be found in [I.2]).

We indicate by $\mu^{(1)}$ the derivative of the uniformity $\mu$ and by $\hat{\mu}$ the locally fine reflection of $\mu$ (see [I.2] chapter VII).

1.5. **Lemma.** Let $\mu X$ be a uniform space and assume that there is a uniform covering $\mathcal{U}$ whose elements are compact. Then $\mu^{(1)}$ is the fine uniformity and $X$ is paracompact.

**Proof.** Let $\mathcal{V}$ be an open covering; if $U \in \mathcal{U}$, $U \cap \mathcal{V}$ is uniform in $U$, hence $\mathcal{V}$ is uniformly locally uniform, i.e. $\mathcal{V} \in \mu^{(1)}$. Now we have $\mu^{(1)} \supset FT \mu \supset \hat{\mu} \supset \mu^{(1)}$ and so $FT \mu = \mu^{(1)}$. At last $X$ is paracompact since every open covering is normal ([E] P 8.B). ■
1.6 Theorem. If $\mu X$ is a locally fine uniform space, $U(\mu X)$ is an algebra.

Proof. Observe that

$$U(\mu X) \subseteq U(\lambda \mu X, \lambda R) \subseteq U(\mu X, FR) \subseteq U(\mu X, qR).$$

The first inclusion holds since $\lambda$ is a functor; the second follows from the hypothesis and lemma 1.5; the third is obvious. The result follows now from theorem 1.3.

2. Convexity of $U(\mu X)$.

Observe that owing to theorem 1.2, $U(qR)$ is an order-convex algebra in $C(R)$; furthermore it is trivial that $U(\mu X) = C(X)$ is a convex algebra if $\mu$ is the fine uniformity on $X$. However we have the following:

2.1 Proposition.

i) If $\mu X$ is a precompact space, $U(\mu X)$ is an algebra and $U(\mu X)$ is order-convex if and only if it coincides with $C^*(X)$, hence if and only if the completion of $\mu X$ is the Stone-Čech compactification.

ii) If $\mu X$ belong to $\mathcal{A}$, $U(\mu X)$ is order-convex if and only if the Samuel compactification is the Stone-Čech compactification.

Proof. i) Trivial by the equality: $U(\mu X) = U^*(\mu X)$.

ii) First observe that $U(\mu X)$ is convex if and only if $U^*(\mu X)$ is convex: indeed the necessity is trivial, conversely for $f \in C(X)$, $g \in U(\mu X)$ such that $0 < f < g$ we have $f = f/(1 + g) \cdot (1 + g)$ where the first factor is a bounded function. Furthermore if $p\mu X$ denotes the precompact reflection of $\mu X$, $U^*(\mu X) = U(p\mu X)$ (see [1.2] II.30); the conclusion follows from i).

Recall that for a topological space $X$ the following are equivalent:

i) The fine uniformity is precompact.

ii) Every admissible uniformity is precompact.

iii) $X$ is pseudocompact.
Hence we conclude that if $X$ is a pseudocompact space then $\mu X$ belongs to $\mathcal{U}$ for every admissible uniformity $\mu$ and $U(\mu X)$ is convex if and only if $\mu$ is the fine uniformity. Moreover for a topological space $X$ we can observe:

2.2 Corollary. Let $\beta X$ be the Stone-Čech compactification of the topological space $X$. For every $X$ such that $\text{card } (\beta X \setminus X) > 1$ there is an admissible uniformity $\mu$ for which $U(\mu X)$ is a non-convex algebra.

Proof. Trivial. ■

If in proposition 2.1 i) we add a completeness hypothesis, $X$ becomes compact, hence $\mu X$ is fine; however we can give an example of a complete space such that $U(\mu X)$ is a non-convex algebra.

Example. Take $X = \{x_i, y_i : i \in I, \text{card } I > \aleph_0\}$. For every subset $J$ of $I$ such that $\text{card } J < \aleph_0$, let:

$$\mathcal{U}_J = \{\{x_h\}, \{y_k\}, \{x_k, y_k\} : h \in J, k \in I \setminus J\}.$$ 

The coverings $\mathcal{U}_J$ form a basis for a uniformity $\mu$ on the set $X$. Clearly $\mu X$ is a complete space. Take $f \in U(\mu X)$ and, for $n \in \mathbb{N}$, let $U_n$ be a covering of $\mu$ such that the diameter of $f(U)$ is less than $1/n$ for every $U \in \mathcal{U}_n$, then say $J = \bigcup_{n \in \mathbb{N}} J_n$. We get that $\mathcal{U}_{J_n}$ refines $\mathcal{U}_{J_n}$ for every $n \in \mathbb{N}$, hence $f(x_k) = f(y_k)$ for every $k \in I \setminus J$ since $\text{diam } \{f(x_k), f(y_k)\} < 1/n$ for every $n \in \mathbb{N}$. Therefore $U(\mu X)$ consists of the functions $f \in \mathbb{R}^X$ such that $f(x_i) = f(y_i)$ out of a suitable countable set, hence it is an algebra; plainly it is not convex because the characteristic function of the set $\{x_i : i \in I\}$ is a bounded continuous function which is not a uniform map.

3. A coreflection on the category $\mathcal{U}$.

From the considerations at the beginning of section 1 we easily get that $\varrho$ is the coarsest uniformity finer than the usual one such that the set of the real-valued uniform maps is an algebra. The problem we are now going to investigate is to see if analogously for every uniform space $\mu X$ there exists a coarsest uniformity $\varphi$ among the finer ones than $\mu$ for which the set of real-valued maps is an algebra.
Denote by $\mu^*$ the uniformity generated by the sub-basis of the coverings $f^{-1}(\mathcal{U})$ with $f \in U(\mu X)$, $\mathcal{U} \in \mathcal{E}$. Put $\mu_0 = \mu$; for every ordinal number $\alpha$ take $\mu_{\alpha+1} = \mu^* \setminus \mu_\alpha$ and for every limit ordinal number $\alpha$ put $\mu_\alpha = \bigwedge_{\beta < \alpha} \mu_\beta$; clearly the set of the uniformities $\mu_\alpha$ is totally ordered.

3.1 LEMMA. If $\sigma < \mu$ and $\sigma X \in \mathcal{U}$, then $\sigma < \mu^*$ and as a consequence $\sigma < \mu_\alpha$ for every $\alpha$.

PROOF. $U(\mu X) \subset U(\sigma X) = U(\sigma X, \rho R)$. 

Indicate with $\mathcal{U}(\mu X)$ the set $\{v | vX \in \mathcal{U}, v < \mu\}$.

3.2 THEOREM. Let $\mu X$ be a uniform space. $\mathcal{U}(\mu X)$ has a minimum $a\mu$ (which necessarily induces the same topology as $\mu$).

PROOF. Defined $\mu_\alpha$ as above, there exists

$$\bar{\alpha} < \exp\left(\exp\left(\exp\left(\text{card } X\right)\right)\right)$$

such that $\mu_{\bar{\alpha}+1} = \mu_\bar{\alpha}$ since the set of $\mu_\alpha$ is totally ordered and there are less than $\exp\left(\exp\left(\exp\left(\text{card } X\right)\right)\right)$ uniformities. Put $a\mu = \mu_{\bar{\alpha}}$: we have $U(a\mu X) \subset U(\mu_{\bar{\alpha}+1}, \rho R) = U(a\mu X, \rho R)$, therefore $a\mu \in \mathcal{U}$ by theorem 1.3. Furthermore $a\mu$ is the smallest element of $\mathcal{U}(\mu X)$ by lemma 3.1. 

Later on we shall need the following interesting lemma:

3.3 LEMMA. Let $A$ be a commutative algebra over a field of characteristic $0$, $B$ a vector subspace of $A$. The vector space $C$ spanned by the powers of the elements of $B$ is an algebra.

PROOF. We prove that $C$ contains the elements of the type $x^r \cdot y^s$ for any $x, y \in B$, $r, s \in \mathbb{N}$. For $n$ fixed we consider, for $x, y \in B$ and $i = 0, 1, ... , n$ the following elements:

$$c_i = (2^i x + y)^n = \sum_{k=0}^{n} 2^i k \binom{n}{k} x^k \cdot y^{n-k};$$

$c_i \in C$ and the matrix $(n + 1) \times (n + 1)$ which in the entry $(i, k)$ has the element $a_{i,k} = 2^i k \binom{n}{k}$ is invertible: in fact

$$\det(a_{i,k}) = \left(\prod_{k=0}^{n} \binom{n}{k}\right) \det(a'_{i,k})$$
where \( a'_{i,k} = 2^{i^k} \) and \( \det (a'_{i,k}) \neq 0 \) because it is the determinant of Vandermonde of the numbers \( 1, 2, \ldots, 2^n \). If we denote by \( (\bar{a}_{i,k}) \) the inverse of \( (a_{i,k}) \), we have \( x^k \cdot y^{n-k} = \sum_{i=0}^{n} \bar{a}_{i,k} e_i \) hence \( x^k \cdot y^{n-k} \in C \). By an inductive argument it can now be easily shown that if \( x_1, \ldots, x_l \) are distinct elements of \( B \), then \( x_1^{n_1} \cdots x_l^{n_l} \) belongs to \( C \).

3.4 Corollary. The vector space spanned by \( U(\mu X, R) \) in \( C(X) \) is an algebra.

Proof. In fact \( U(\mu X, R) \) is closed under powers.

3.5 Corollary. The algebra spanned in \( C(X) \) by \( U(\mu X) \) is contained in \( U(\mu_1 X) \), then \( \mu_1 \) is the coarsest uniformity finer than \( \mu \) for which this situation occurs.

Proof. Trivial.

3.6 Proposition. Using the notations of theorem 3.2, \( \bar{\alpha} < \omega_1 \).

Proof. Take \( f, g \in U(\mu_{\omega_1} X) \) and call \( \{U_{1/n} \} \) the covering of \( R \) consisting of the open intervals with radius \( 1/n \), \( n \in N \). There exist \( U_{n} \in \mu_{\omega_1} \) which refines both \( f^{-1}(U_{1/n}) \) and \( g^{-1}(U_{1/n}) \) and there are ordinal numbers \( \alpha_n < \omega_1 \) such that \( U_{n} \in \mu_{\alpha_n} \). Let \( \alpha = \sup \alpha_n \); then \( \alpha < \omega_1 \) and we have that \( f, g \in U(\mu_{\alpha} X) \) and by corollary 3.5 \( f \cdot g \in U(\mu_{\alpha+1} X) \subset U(\mu_{\omega_1} X) \).

Remark 5. We are unable to say wether 3.6 may be strengthened: in all the examples we have tested, we have found \( \bar{\alpha} = 1 \).

The uniformity \( \mu \mu \) may be reached by an alternative construction of algebraic type: for every uniformity \( \mu \) let \( \mu^* \) be the weak uniformity of the algebra generated by \( U(\mu X) \) in \( C(X) \): then set \( \mu_0 = \mu \), \( \mu_{\alpha+1} = \mu^* \) for every ordinal number, and if \( \alpha \) is a limit ordinal number, \( \mu_{\alpha} = \bigwedge_{\beta < \alpha} \mu_{\beta} \). Corollary 3.5 shows that \( \mu_{\alpha} = \mu_{\alpha} \) for every ordinal number \( \alpha \), hence the two constructions proposed are quite equivalent. We think that perhaps Hager refers to this second construction in his paper [H]. In the same paper the author says that the assignation \( a: \mu X \to \alpha \mu X \) is a coreflection onto the category \( \mathcal{A} \): a direct proof of this fact can be easily given.

3.7 Lemma. Let \( f \) belong to \( U(\mu X, vY) \); then \( f \) belongs to \( U(\mu^* X, v^* Y) \), hence to \( U(\mu_1 X, v_1 Y) \).
3.8 Proposition. $a$ is a coreflection from the category of uniform spaces onto $\mathcal{U}$.

Proof. By transfinite induction it can be easily shown that if $f \in U(\mu X, \nu Y)$, then $f \in U(\alpha \mu X, \alpha \nu Y)$ for every ordinal number $\alpha$, namely for $\alpha = \omega_1$. Furthermore if $f \in U(a(\mu X, \nu Y)$ then $f \in U(\mu X, \alpha \nu Y)$, hence $a$ is a coreflection.

3.9 Proposition. The functor $a$ commutes with the completion.

Proof. First observe that, if we indicate by $(\mu X)^\wedge$ the completion of the uniform space $\mu X$, $(a\mu X)^\wedge$ is still an element of $\mathcal{U}$, hence the identity function of $X$ is a map between $a\mu X$ and $\mu X$ which extends to a map from $(\mu X)^\wedge$ into $a(\mu X)^\wedge$. On the other hand $a(\mu X)^\wedge$ induces over $X$ a uniformity $\nu$ belonging to $\mathcal{U}(\mu X)$ therefore the identity function $i : \nu X \to a\mu X$ is a map which extends to the completions.

We make now a consideration about $\mathbb{R}^n$ equipped with the usual uniformity $\mu$; employing techniques analogous to the ones used in the proof of 1.1 and 1.2, we can describe the uniformity $a\mu = \varphi_\mu$ which turns out to be the weak uniformity of the functions dominated by polynomials (with $n$ variables); in fact the fundamental topic is to show that a sub-basis for $a\mu$ is composed of the chains which satisfy the condition $\ast$, where the interval $[-k, k]$ is replaced by the closed ball with center in the origin and radius $k$.

4. Prime and maximal ideals in $U(\mu X)$.

Owing to the features of the uniformities studied in this paper, the problem of examining the properties of the algebras of uniformly continuous real-valued maps arises quite naturally. Such algebras being both $\varphi$-algebras and Riesz spaces, many results descend directly from those theories.

Definitions. A totally ordered set $X$ is said to be $\eta_1$ if for every pair of subsets $A, B$ such that $A < B$ and $\text{card} (A \cup B) < \aleph_1$ there exists $x \in X$ such that $A < x < B$; if for every pair of non-empty subsets $A, B$ satisfying the previous conditions there exists $x \in X$ such that $A < x < B$, the set $X$ is said to be $q \cdot \eta_1$ (=$\text{quasi} \eta_1$).
In the following proposition we list a number of known facts (see [HIJ] and [GJ]):

4.1 PROPOSITION. Let $\mu X$ belong to $\mathfrak{A}$, $P$ a prime ideal of $U(\mu X)$, $M$ a maximal one; then:

i) $P$ is absolutely convex in $U(\mu X)$;

ii) $U(\mu X)/P$ is a totally ordered domain;

iii) $P(f) > 0$ if and only if $f \equiv |f|$ (mod. $P$);

iv) $f \mapsto P(f)$ is a map of lattices;

v) $P$ is contained in a unique maximal ideal;

vi) $U(\mu X)/M$ is a real-closed field;

vii) $U(\mu X)/M$ is a $\eta_0$ field.

With regard to the quotients $U(\mu X)/M$ we may go on with the following:

4.2 THEOREM. Let $A$ be a lattice-ordered sub-algebra of $C(X)$ with 1, closed under bounded inversion, $M$ a maximal ideal of $A$. Given $s > 0$ belonging to $A$, put $Z_n = \{x: s(x) > n, n \in N\}$ and $I_s = \{f \in A: f(Z_n) = 0$ for some $n\}$. Then $M(s)$ is infinitely large if and only if $M \supset I_s$.

PROOF. If $s$ is a bounded function, the proposition follows trivially. Assume now $s$ unbounded so that $I_s$ is a proper ideal. If $M(s)$ is infinitely large, put $g_n = -(s - n) \land 0$ and observe that $g_n$ is a bounded function, so that $s - n + g_n$ cannot belong to $M$. Since $(s - n + g_n) \cdot g_n = 0$ it follows that $g_n \in M$ for every $n$. Given $f \in I_s$ there exists $n \in N$ such that $f(Z_n) = 0$, that is $f$ vanishes on the zero-set of $g_n$; by a simple calculation one sees that if $x \in X \setminus Z_n$ then $g_{n+1}(x) > 1$. By the equality $f = g_{n+1} \cdot f/(1 \lor g_{n+1})$ we conclude that $f$ belong to $M$.

Conversely let $g_n$ be the functions defined above: the functions $g_n$ belong to $I_s$, hence to $M$, and since $g_n + s > n$ we have $M(s) = M(s + g_n) > n$. ■

4.3 THEOREM. Let $\mu X \in \mathfrak{A}$. The following are equivalent:

i) $U(\mu X)/M = \mathbb{R}$ for every maximal ideal $M$;

ii) $U(\mu X) = U^*(\mu X)$;
iii) $\mathcal{q}\mu$ is precompact ($\mathcal{q}\mu$ is the weak uniformity induced by all the real-valued functions uniformly continuous on $\mu X$).

**Proof.** i) $\Rightarrow$ ii): if $U(\mu X)$ contains an unbounded function $f$, $I_f$ is a proper ideal and by theorem 4.2 $M(f)$ is infinitely large for every maximal ideal $M \supset I_f$.

ii) $\Rightarrow$ i): trivial.

ii) $\Rightarrow$ iii): if $\mathcal{q}\mu$ is not precompact, there exists a uniform covering $\mathcal{U}$ of $R$ and a map $f \in U(\mu X)$ such that $f^{-1}(\mathcal{U})$ cannot be refined by a finite uniform covering of $\mathcal{q}\mu$. This necessarily implies that $f$ is unbounded.

iii) $\Rightarrow$ ii): by definitions.

When $U(\mu X)/M$ is not the real field, it is an ordered field which contains a copy of $R$; naturally we wonder if, as it happens for the residue fields of rings of real-valued continuous functions, $U(\mu X)/M$ turns out to be $q_1$; in the general case this question has a negative answer: in fact in view of 4.3 there are hyper-real quotient fields of $U(qR)$, and all of them have cofinality $\aleph_0$: a cofinal countable set can be obtained by $\mathcal{M}(m + x^{2n})$ for a maximal ideal $M$ and natural numbers $m, n$.

Incidentally we point out the following consequences of this observation:

a) $U(qR)$ cannot be isomorphic to any $C(X)$; however this fact can be proved directly;

b) all the quotient fields of $U(qR)$ are complete in the uniformity canonically induced by the order: in fact it can be easily shown (see [M]) that if an ordered field is $q \cdot q_1$ and not $q_1$ then it is complete; hence in view of 4.1 vii) (see [HIJ]) if $M$ is a maximal ideal of a $\varphi$-algebra $A$ and $A/M$ is not $q_1$ then it is complete (in the order uniformity).

Using the techniques of [GJ] 13.7 we can prove the following partial result:

4.4 **Proposition.** If $\mu X$ is a locally fine uniform space, $M$ a maximal ideal of $U(\mu X)$ such that $U(\mu X)/M$ is not real, then $U(\mu X)/M$ is $q_1$.

**Proof.** We omit the details; observe that if $\mathcal{U}_\varepsilon$ is a uniform covering of $R$ of balls whose radius is $\varepsilon < 1/2$, with notations of [GJ] 13.7,
\(h^{-1}(\mathcal{U}_a)\) is uniformly locally uniform, hence \(h\) belongs to \(U(\mu X)\); then use 4.2 to see that \(M(h) > M(f_n)\).

We shall now make some considerations about the maximal spectrum \(\text{Max} \ (U(\mu X))\), for \(\mu X \in \mathfrak{X}\), equipped as usual with the hull-kernel topology whose closed sets are \(V(I) = \{M \in \text{Max} \ (U(\mu X)) : M \supset I\}\) where \(I\) is an ideal of \(U(\mu X)\). It is well known that \(\text{Max} \ (U(\mu X))\) a compact Hausdorff space; the natural function \(\vartheta : X \to \text{Max} \ (U(\mu X))\) defined by \(\vartheta(x) = \{f \in U(\mu X) : f(x) = 0\}\) is obviously \(1 - 1\) since \(U(\mu X)\) separates points; moreover \(\vartheta(X)\) is dense in \(\text{Max} \ (U(\mu X))\). Now recall that a function between uniform spaces is said to be a \(\delta\)-map if the preimages of two uniformly separated sets are uniformly separated and a \(\delta\)-isomorphism if its inverse is a \(\delta\)-map too. Recall also that in view of [1.2] II.35, a function between precompact spaces is a uniform isomorphism if and only if it is a \(\delta\)-isomorphism. Now we can state the following:

4.5. PROPOSITION. Let \(\mu X\) belong to \(\mathfrak{X}\); \(\vartheta : p\mu X \to \text{Max} \ (U(\mu X))\) (defined as above) is a uniform isomorphism onto its image; hence \(\text{Max} \ (U(\mu X))\) is the Samuel compactification of \(\mu X\).

PROOF. By the previous remarks we shall prove that \(\vartheta\) is a \(\delta\)-isomorphism. If \(A, B\) are (uniformly) separated in \(\text{Max} \ (U(\mu X))\), they are contained in disjoint closed sets \(V(I)\), \(V(J)\) respectively, hence there exist \(i \in I\), \(j \in J\) such that \(i + j = 1\); then the functions \(i, j\) vanish respectively on \(\vartheta^{-1}(V(I))\) and \(\vartheta^{-1}(V(J))\) so that \(\vartheta^{-1}(A)\), \(\vartheta^{-1}(B)\) are uniformly separated by either function.

Conversely if \(A, B\) are subsets of \(X\) and there exists \(i \in U(p\mu X)\) such that \(i(A) = 0\), \(i(B) = 1\), then \(\vartheta(A) \subset V(i)\), \(\vartheta(B) \subset V(1 - i)\) and \(V(i) \cap V(1 - i) = \emptyset\).

As a consequence \(\text{Max} \ (U(\mu X))\) is the completion of \(\vartheta(p\mu X)\) hence the Samuel compactification of \(\mu X\).

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