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compacts for systems like : $(P(D_x, D_y)u = f, Qu = 0)$**

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**A link between global solvability
and solvability over compacts
for systems like: $(P(D_x, D_y)u = f, Qu = 0)$.**

GIULIANO BRATTI (*)

0. Introduction.

Let A be an open subset of R^3 such that:

if A_0 is its intersection with plane xy , every point $p \in A$ can be connected by an «orthogonal segment», (in A), with some point $p_0 \in A_0$. Then we can show the following:

THEOREM 1. Let $P = P(D_x, D_y)$ be a partial differential operator with constant coefficients; Q_2 e Q_3 will be, respectively, the Laplace's operators in two and three variables.

Then:

i) if A_0 is P -convex,
we have: the global solvability of the overdetermined system:
 $(Pu = f, Q_3u = 0)$ is equivalent to the solvability of the same system over compact subsets of A .

REMARK 1. Without loss generality, we can always think P and Q_2 are prime between them; it depends upon the global solvability, in A , of the system: $(Q_2u = f, Q_3u = 0)$.

REMARK 2. Connections of the above theorem with some E. De Giorgi's conjecture [2], are evident.

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1. Better then everything it's to make directly the proof of Theorem 1; for symbols and terminology, look at [3].

PROOF.

a) In A , the system: $(Pu = f, D_t u = 0)$ is globally solvable, with C^∞ -solutions (of course for every f such that: $D_t f = 0$). a) depends upon the i) hypothesis.

As a consequence: if $E'(A)$ is the space of distribution with compact support in A , ${}^tPE'(A) + {}^tD_t E'(A)$ is closed in $E'(A)$. In fact: if we pose: $\ker D_{t/A} = (f \in C^\infty(A): D_t f = 0)$, the solvability of the above system means that: $P(\ker D_{t/A}) = \ker D_{t/A}$. By the theorem about the «surjections between Fréchet's spaces» we have:

if: ${}^tPm_j + {}^tD_t n_j$ is convergent, in $E'(A)$, to m_0 , there exists $m_1 \in E'(A)$ such that: $m_0 = {}^tPm_1$ over functions of $\ker D_{t/A}$.

By [3], pagg. 77-78, we have: $m_0 = {}^tPm_1 + {}^tD_t n_1$, with $n_1 \in E'(R^3)$; for the fact that: $\text{supp}(m_0 - {}^tPm_1) \subseteq A$, also $\text{supp}(n_1)$ is in A .

b) In A , the system: $(Pu = f, Q_3 u = 0, D_t u = 0)$ is globally solvable; with the same arguments, like in a), it comes out:

$${}^tPE'(A) + Q_3 E'(A) + {}^tD_t E'(A) \quad \text{is closed in } E'(A).$$

b') In A , we'll consider the system: $(Pu = f_1, Q_3 u = f_2, D_t u = f_3)$; we suppose the data: (f_1, f_2, f_3) compatible and in $C^\infty(A)$.

We like to show the above system is solvable in $C^\infty(A)$.

Call D_3 the subspace, of $C^\infty(A)^3$, of the compatible data; call: $(P, Q_3, D_t): C^\infty(A) \rightarrow D_3$, the (continuous and linear) map:

$$(P, Q_3, D_t)u = (Pu, Q_3 u, D_t u);$$

let ${}^t(P, Q_3, D_t)$ be its transposed between the dual space.

Of course: if (m_1, m_2, m_3) is a functional over D_3 and:

$${}^t(P, Q_3, D_t)(m_1, m_2, m_3) = 0,$$

which means: $-Q_3 m_2 = {}^tPm_1 + {}^tD_t m_3$, we have:

- 1) $Q_3 m_2$ is orthogonal to the space: $\ker P_{/A} \wedge \ker D_{t/A}$; then:
- 2) m_2 is orthogonal to

$$Q_3(\ker P_{/A} \wedge \ker D_{t/A}) = (\ker P_{/A} \wedge \ker D_{t/A}).$$

Last equality comes out from the solvability, over A_0 , of the system: $(Pu = f, Q_2u = 0)$ over simply connected open subset of R^2 .

Because $a)$ above, $m_2 = {}^tPh + {}^tD_ik$, with $(h, k) \in E'(A)^2$. Now, it's easy to see that: $m_1 = -Q_3h + {}^tD_ip$, and $m_3 = -Q_3k - {}^tPp$; again: $p \in E'(A)$.

This show that: (P, Q_3, D_i) is injective; by $b)$, its image, in $E'(A)$, is closed; then: the system: $(Pu = f_1, Q_3u = f_2, D_iu = f_3)$ is solvable in $C^\infty(A)$ for every data, in $C^\infty(A)$, compatible.

$c)$ From $b')$ and from the theorem about « *surjections between Fréchet's spaces* » we have:

1) if $\text{supp}({}^tPm_1 + Q_3m_2 + {}^tD_im_3) \subseteq K \subseteq A$, and if: $\text{ord.}({}^tPm_1 + {}^tQ_3m_2 + {}^tD_im_3) \leq n$, $n \in N$, there exist:

2) a compact subset of A , $K(n)$, and three distributions, $(h, k, p) \in E'(A)^3$, such that:

$$\text{supp}(m_1 - Q_3h + {}^tD_ip, m_2 + {}^tPh + {}^tD_ik, m_3 - Q_3k - {}^tPp) \subseteq K(n)^3.$$

$d)$ Suppose: ${}^tPm_1 + Q_3m_2$ with support in $K \subseteq A$ and with order less than n . By $c)$ above:

$$\text{supp}(m_1 - Q_3h + {}^tD_ip, m_2 + {}^tPh + {}^tD_ik, -Q_3k - {}^tPp) \subseteq K(n)^3;$$

in $E'(A)$ we can solve the system:

$$1) Q_3h + {}^tD_ip = Q_3h_1, {}^tPh + {}^tD_ik = {}^tPh_1 + {}^tD_ik_1,$$

$$2) Q_3k - {}^tPp = -Q_3k_1 - {}^tPp_1,$$

(it's simple exercise).

So we have:

$$\text{supp}(m_1 - Q_3h_1) \subseteq K(n); \quad \text{supp}(m_2 + {}^tPh_1 + {}^tD_ik_1) \subseteq K(n).$$

Because we can suppose:

$$K \subseteq K(n),$$

we have:

$${}^tPm_1 + Q_3m_2 + Q_3{}^tD_ik_1$$

has its support in $K(n)$; this implies, choosing $K'(n)$ a little bigger than $K(n)$, that:

$$\text{supp } (m_1 - Q_3 h_1, m_2 + {}^t P h_1) \subseteq K'(n)^2.$$

e) $\bar{d}$) above shows: if ${}^t P m_1 + Q_3 m_2$ is continuous in relation with a semi-norma p over $C^\infty(A)$, there exists a semi-norm q over the compatible data $D_2 = ((f, g): Pg = Q_3 f)$, in relation to which (m_1, m_2) are continuous over D_2 .

Solvability of the system: $(Pu = f, Q_3 u = 0)$ over compact subsets of A , and the ellipticity of Q_3 , from which comes out the fact that A is Q -convex, shows the thesis of the Theorem 1.

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