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M. Welleda Baldoni Silva

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Branching Theorems for Semisimple Lie Groups of Real Rank One.

M. WELLEDA BALDONI SILVA (*)

1. Introduction.

Let $G_{\mathbf{C}}$ be a connected, simply connected, simple complex Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Let $\mathfrak{g} \subset \mathfrak{g}_{\mathbf{C}}$ be a real form of $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{g} \neq sl(2, \mathbf{R})$ and let G be the analytic subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and K be the analytic subgroup of G corresponding to $\mathfrak{k}$. We assume that $\operatorname{rk} K = \operatorname{rk} G$ and G has split rank one, i.e. the symmetric space G/K has rank one. Under these assumption, the Cartan classification of the real forms implies that G is, up to isomorphism, one of the following groups:

(1) $Spin(2n, 1)$, $n \geqslant 2$,

(2) $SU(n, 1)$, $n \geqslant 2$,

(3) $Sp(n, 1)$ $n \geqslant 2$,

(4) F_4 the analytic group corresponding to the real form $\mathfrak{g} = f_4(-20)$ of $\mathfrak{g} = f_4$, with character -20.

The restrictions on the indices are set in order to avoid overlappings. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, then $\dim \mathfrak{a} = 1$. Let $\mathfrak{m}$ (resp. M) be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ (resp. in K).

In this paper we study the problem of computing the multiplicities with which finite dimensional irreducible (complex) representa-

(*) Indirizzo dell'A.: Dipartimento di Matematica - Libera Università di Trento - 38050 Povo (Trento), Italy.

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tions of M occur in the restriction to M of finite dimensional irreducible (complex) representations of K. The idea is to introduce a connected subgroup K_1 of K in such a way that modulo an outer automorphism of the Lie algebra of K_1, the branching theorem from K to K_1 and from K_1 to M is classical or known.

We do this by means of a case by case analysis, defining K_1 differently in each situation. It would be possible to define K_1 in general, independent of the class of groups we are considering, but this is not in the spirit of this paper. This approach can be found in [1].

2. Preliminaries.

We need some more notation. If $\mathfrak{s}$ is a real semisimple Lie algebra, we denote by $\mathfrak{s}_{\mathbf{C}}$ its complexification and by $\mathfrak{z}_s$ its center. Let $\mathfrak{h} \subset \mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$. Let B denote the Killing form of $\mathfrak{g}$. For each $\alpha \in \mathfrak{h}_{\mathbf{C}}^*$, let h_α be the unique element of $\mathfrak{h}_{\mathbf{C}}$ so that $B(H, h_\alpha) = \alpha(H)$, $\forall H \in \mathfrak{h}_{\mathbf{C}}$. If α is a root, we call $H_\alpha = 2h_\alpha/B(h_\alpha, h_\alpha)$ the root normal of α. Let $\mathfrak{h}^- \subset \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{m}$, then $\mathfrak{h}_0 = \mathfrak{h}^- \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by $(\,,\,)$ the dual of the killing form restricted to $i\mathfrak{h}$ or to $i\mathfrak{h}^- + a$.

Now let U be a compact connected Lie group and let T be a maximal torus of U. Denote by $\mathfrak{t}$ and $\mathfrak{u}$ the Lie algebras of T and U, respectively. Then $\mathfrak{t} = \mathfrak{z}_u \oplus \mathfrak{t}_1$, where $\mathfrak{t}_1$ is a Cartan subalgebra of $[\mathfrak{u}, \mathfrak{u}]$.

It is well known (c.f. [6], Theorem 4.6.12) that $\hat{U}$, the set of all equivalence classes of irreducible finite dimensional (complex) representations of U, is in bijective correspondence with

$D_U = \{\lambda$ linear form on $\mathfrak{t}_{\mathbf{C}}$ such that

1) $\lambda(\Gamma_U) \subset 2\pi i\mathbb{Z}$ for $\Gamma_U = \{X \in \mathfrak{t}\colon \exp X = e\}$,

2) $\lambda|_{(t_1)_{\mathbf{C}}}$ is dominant integral relative to some choice of positive roots$\}$.

If $\lambda \in D_U$, we denote by (π_λ, V_λ) the U representation parametrized by λ, and by $((\pi_\lambda)_*, V_\lambda)$ the differential of π. If H is a compact connected subgroup of U and $S \subset T$ is a maximal torus of H, then for $\lambda \in D_U$, $\mu \in D_H$, we define $m_\lambda(\mu)$ to be the multiplicity with which the finite dimensional (complex) representation of H, π_μ, appears in $\pi_\lambda|_H$.

The following lemma is obvious. Since we will encounter the situation of the lemma many times in the course of this paper, we state it here.

LEMMA 2.1. *Let* $\mathfrak{g}_1 \subset \mathfrak{g}_2$ *be two complex reductive Lie algebras. Let* $\mathfrak{k}_1 \subset \mathfrak{g}_1$ *be a subalgebra and let* Φ *be an isomorphism of* $\mathfrak{g}_1$ *onto* $\mathfrak{g}_2$. *If* (π, V) *is a finite dimensional representation of* $\mathfrak{g}_1$ *such that*

$$(\pi\circ\Phi^{-1}|_{\Phi(\mathfrak{k}_1)}, V) = \left(\sum_{j=1}^{k} \Lambda_j, \sum_{j=1}^{k} V_j\right)$$

$\{(\Lambda_j, V_j)$ *irreducible representation of* $\Phi(\mathfrak{k}_1)\}$, *then*

$$(\pi|_{\mathfrak{k}_1}, V) = \left(\sum_{j=1}^{k} \Lambda_j\circ\Phi, \sum_{j=1}^{k} V_j\right).$$

3. Branching theorem for $Spin(2n, 1)$, $n \geqslant 2$.

3.1. Let $G = Spin(2n, 1)$, then $\mathfrak{g} = so(2n, 1)$.

Let $\mathfrak{k} = \left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right),\ A \in so(2n)\right\}$ and $\mathfrak{p} = \left\{\left(\begin{array}{c|c} 0 & X \\ \hline X^t & 0 \end{array}\right),\ X \text{ real } 2n\times 1 \text{ matrix}\right\}$.

Let $H_0 = \left(\begin{array}{c|c} 0 & X \\ \hline X^t & 0 \end{array}\right)$ with $X = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$. Set $\mathfrak{a} = \mathbf{R}H_0$, then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$ and $\mathfrak{m} = \left\{\begin{pmatrix} A & & \\ & 0 & \\ & & 0 \end{pmatrix},\ A \in so(2n-1)\right\}$.

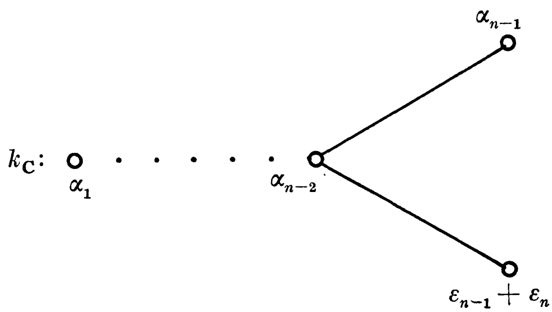
Let $\mathfrak{h} = \left\{\left(\begin{array}{cc|c} 0 & A & \\ -A^t & 0 & \\ \hline & & 0 \end{array}\right) \in sp(2n, 1), \text{ where } A \text{ is an } n\times n \text{ diagonal matrix}\right\}$.

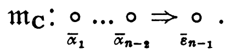
Write $H = (a_1, \ldots, a_n)$ for $H \in \mathfrak{h}_{\mathbf{C}}$, $a_i \in \mathbf{C}$. Let ε_i, $i = 1, \ldots, n$, be the linear functional on $\mathfrak{h}_{\mathbf{C}}$ defined by $\varepsilon_i(H) = a_i$ for $H = (a_1, \ldots, a_n)$. Then the roots of the complex Lie algebra $\mathfrak{g}_{\mathbf{C}}$ are, relative to $\mathfrak{h}_{\mathbf{C}}$, $\pm\varepsilon_i \pm \varepsilon_j$ $(1 \leqslant i < j \leqslant n)$ and $\pm\varepsilon_i$, $i = 1, \ldots, n$. The roots of $\mathfrak{k}_{\mathbf{C}}$, relative to $\mathfrak{h}_{\mathbf{C}}$, are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqslant i < j \leqslant n$.

Let $\mathfrak{h}^- = \{H \in \mathfrak{h} : H = (a_1, \ldots, a_{n-1}, 0)\}$. Then the roots of $\mathfrak{m}_C$ relative to $\mathfrak{h}_C^-$ are $\pm \bar{\varepsilon}_i \pm \bar{\varepsilon}_j$ $(1 \leqslant i < j \leqslant n-1)$ and $\bar{\varepsilon}_i$ $(i = 1, \ldots, n-1)$, where, the bar means the restriction.

Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \ldots, n-1$, $\alpha_n = \varepsilon_n$ and $\bar{\alpha}_i = \bar{\varepsilon}_i - \bar{\varepsilon}_{i+1}$, $i = 1, \ldots, n-2$.

In what follows the notion of dominance for $\mathfrak{k}_C$ and $\mathfrak{m}_C$ will be relative to the following choices:





We recall the following well known results (cf. e.g. [2] or [5]). Note that $K \simeq Spin(2n)$, $M \simeq Spin(2n-1)$.

LEMMA 3.2.

$$D_K = \left\{\lambda = \sum_{i=1}^{n} a_i \varepsilon_i,\ a_1 \geqslant \ldots \geqslant a_{n-1} \geqslant |a_n| \geqslant 0,\right.$$
$$\left. a_i - a_j \in \mathbf{Z} \text{ and } 2a_i \in \mathbf{Z},\ i, j = 1, \ldots, n\right\}.$$

LEMMA 3.3.

$$D_M = \left\{\mu = \sum_{i=1}^{n-1} b_i \bar{\varepsilon}_i,\ b_1 \geqslant \ldots \geqslant b_{n-1} \geqslant 0,\ b_i - b_j \in \mathbf{Z}\right.$$
$$\left. \text{and } 2b_i \in \mathbf{Z},\ i, j = 1, \ldots, n-1\right\}.$$

Theorem 3.4. *Let*

$$\lambda = \sum_{i=1}^{n} a_i \varepsilon_i \in D_K \qquad \text{and} \qquad \mu = \sum_{i=1}^{n-1} b_i \bar{\varepsilon}_i \in D_M .$$

Then $m_\lambda(\mu) = 0$ *or* 1.

If $a_i - b_j \notin \mathbb{Z}$, then $m_\lambda(\mu) = 0$. If $a_i - b_j \in \mathbb{Z}$, then $m_\lambda(\mu) = 1$ if and only if $a_1 \geqslant b_1 \geqslant \ldots \geqslant a_{n-1} \geqslant b_{n-1} \geqslant |a_n|$.

4. Branching theorem for $SU(n, 1)$, $n \geqslant 2$.

4.1. Let $G = SU(n, 1)$, $n \geqslant 2$, then $\mathfrak{g} = su(n, 1)$. Fix

$$\mathfrak{k} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & d \end{array} \right),\ A \in u(n),\ d \in u(1) \text{ and } \operatorname{tr} A + d = 0 \right\},$$

$\mathfrak{a} = \mathbf{R}H$ for $H = (h_{ij}) \in su(n, 1)$ defined by $h_{1,n+1} = h_{n+1,1} = 1$ and $h_{ij} = 0$ for all the other indices.

Then $$\mathfrak{m} = \left\{ \left(\begin{array}{c|c|c} d & & \\ \hline & A & \\ \hline & & d \end{array} \right) : A \in u(n-1),\ d \in u(1) \text{ and } 2d + \operatorname{tr} A = 0 \right\}.$$

Let $\mathfrak{h}$ be the diagonal matrices in $su(n, 1)$, then $\mathfrak{h} = \mathfrak{z}_{\mathfrak{k}} \oplus \mathfrak{h}_1$ where $\mathfrak{h}_1$ is a Cartan subalgebra of $[\mathfrak{k}, \mathfrak{k}] \simeq su(n)$. Let $\mathfrak{h}^-$ be the diagonal matrices of $\mathfrak{m}$, then $\mathfrak{h}^- = \mathfrak{z}_{\mathfrak{m}} \oplus \mathfrak{h}_1^-$, $\mathfrak{h}_1^-$ a Cartan subalgebra of $[\mathfrak{m}, \mathfrak{m}] \simeq \simeq su(n-1)$. Let X_i $(1 \leqslant i \leqslant n+1)$ be the $(n+1) \times (n+1)$ diagonal matrix which is 1 in the i-th diagonal entry and zero elsewhere. Then $\{X_i\}_{i=1}^{n+1}$ is a basis for the complex vector space $\tilde{\mathfrak{h}}$ of complex diagonal matrices. Let $\{\varepsilon_i\}_{i=1}^{n+1}$ be the dual basis. If ν is a linear functional on $\tilde{\mathfrak{h}}$, let $\bar{\nu}$ denote its restriction to $\mathfrak{h}_{\mathbf{C}}$ and $\bar{\bar{\varepsilon}}$ its restriction to $\mathfrak{h}_{\mathbf{C}}^-$. The roots of the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ relative to $\mathfrak{h}_{\mathbf{C}}$ are $\pm(\bar{\varepsilon}_i - \bar{\varepsilon}_j)$, $(1 \leqslant i < j \leqslant \leqslant n+1)$. The roots of the reductive Lie algebra $\mathfrak{k}_{\mathbf{C}}$ with respect to $\mathfrak{h}_{\mathbf{C}}$ are $\pm(\bar{\varepsilon}_i - \bar{\varepsilon}_j)$ $(1 \leqslant i < j \leqslant n)$ and finally the roots of $\mathfrak{m}_{\mathbf{C}}$ relative to $\mathfrak{h}_{\mathbf{C}}^-$ are $\pm(\bar{\bar{\varepsilon}}_i - \bar{\bar{\varepsilon}}_j)$, $(2 \leqslant i < j \leqslant n)$. Set $\alpha_i = \bar{\varepsilon}_i - \bar{\varepsilon}_{i+1}$ $(1 \leqslant i \leqslant n)$ and $\bar{\alpha}_i = = \bar{\bar{\varepsilon}}_i - \bar{\bar{\varepsilon}}_{i+1}$, $(2 \leqslant i \leqslant n-1)$. We fix as fundamental Weyl chambers for $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{m}_{\mathbf{C}}$ the ones determined by the following choice of simple roots:

$$\mathfrak{k}_{\mathbf{C}}:\ \underset{\alpha_1}{\circ} \text{———} \underset{\alpha_2}{\circ} \ldots \underset{\alpha_{n-2}}{\circ} \text{———} \underset{\alpha_{n-1}}{\circ}$$

$$\mathfrak{m}_{\mathbf{C}}:\ \underset{\bar{\alpha}_2}{\circ} \text{———} \underset{\bar{\alpha}_3}{\circ} \ldots \underset{\bar{\alpha}_{n-2}}{\circ} \text{———} \underset{\bar{\alpha}_{n-1}}{\circ}\ .$$

The notation of dominance will always be intended relative to this particular choice.

The root normal for α_i is $H_{\alpha_i} = X_i - X_{i+1}$ and the root normal for $\bar{\alpha}_i$ is $H_{\bar{\alpha}_i} = X_i - X_{i+1}$.

LEMMA 4.2.

$$D_K = \left\{ \lambda = \sum_{i=1}^{n+1} a_i \bar{\varepsilon}_i,\ a_1 \geqslant \ldots \geqslant a_n,\ a_i \in \mathbb{Z}\ (1 \leqslant i \leqslant n+1) \right\}.$$

PROOF. Let $\mu \in \mathfrak{h}_{\mathbf{C}}^*$, then $\mu = \sum_{i=1}^{n+1} a_i \bar{\varepsilon}_i$.

(1) μ is dominant integral relative to $\underset{\alpha_1}{\circ} \cdots \underset{\alpha_{n-1}}{\circ}$ if and only if $a_i - a_{i+1} \in \mathbb{Z}_+$ $(1 \leqslant i \leqslant n-1)$.

Indeed

$$\frac{2(\mu, \alpha_i)}{(\alpha_i, \alpha_i)} = \mu(H_{\alpha_i}) = \sum_{j=1}^{n+1} a_j \bar{\varepsilon}_j (X_i - X_{i+1}) = a_i - a_{i+1}.$$

(2) $\mu(\Gamma_K) \subset 2\pi i \mathbb{Z}$ if and only if $a_i - a_{i+1} \in \mathbb{Z}$, $1 \leqslant i \leqslant n$.

(3) If $b_i, c_i \in \mathbf{C}$ $(1 \leqslant i \leqslant n+1)$, then $\sum_{i=1}^{n+1} b_i \bar{\varepsilon}_i = \sum_{i=1}^{n+1} c_i \bar{\varepsilon}_i$ iff there exists a complex constant d such that $b_i = c_i + d$.

It follows from the fact that $\{\bar{\varepsilon}_i\}_{i=1}^n$ are linearly independent on $\mathfrak{h}_{\mathbf{C}}$.

(4) Now let $\lambda = \sum_{i=1}^{n+1} a_i \bar{\varepsilon}_i \in D_K$. Using (3) $\lambda = \sum_{i=1}^{n+1} (a_i - a_n) \bar{\varepsilon}_i$ and by (1) and (2) the coefficients of λ have the required properties.

The converse follows immediately.

LEMMA 4.3.

$$D_M = \left\{ \mu = b_1(\bar{\bar{\varepsilon}}_1 + \bar{\bar{\varepsilon}}_{n+1}) + \sum_{i=2}^{n} b_i \bar{\bar{\varepsilon}}_i :\ b_2 \geqslant \ldots \geqslant b_n,\ 2b_1 \in \mathbb{Z},\ b_i \in \mathbb{Z},\ i = 2, \ldots, n \right\}.$$

PROOF. Let $\mu \in (\mathfrak{h}_{\mathbf{C}}^-)^*$. Then μ is the restriction of a linear functional on $\mathfrak{h}$, i.e., $\mu = \sum_{i=1}^{n+1} a_i \bar{\bar{\varepsilon}}_i$. Since $\bar{\bar{\varepsilon}}_1 = \bar{\bar{\varepsilon}}_{n+1}$, then we can rewrite

$$\mu = \frac{a_1 + a_{n+1}}{2} (\bar{\bar{\varepsilon}}_1 + \bar{\bar{\varepsilon}}_{n+1}) + \sum_{i=1}^{n} a_i \bar{\bar{\varepsilon}}_i.$$

(1) If μ is dominant integral with respect to $\underset{\bar\alpha_2}{\circ}\cdots\underset{\bar\alpha_{n-1}}{\circ}$ then $\mu(H_{\alpha_i}) = a_i - a_{i+1} \in \mathbb{Z}_+$, $i = 2, \ldots, n-1$.

(2) Let

$$X = 2\pi i \begin{pmatrix} k & & & & \\ & k_2 & & & \\ & & \ddots & & \\ & & & k_n & \\ & & & & k \end{pmatrix}$$

with $k, k_i \in \mathbb{Z}$ and $\sum_2^n k_i + 2k = 0$, then $X \in \Gamma_M$, hence if $\mu(X) \subset 2\pi i\mathbb{Z}$, we have $(a_1 + a_{n+1})k + a_2 k_2 + \ldots + a_n k_n \in \mathbb{Z}$. For $k = 1$, $k_n = -2$ and $k_i = 0$ $(2 \leqslant i \leqslant n-1)$ we obtain $a_1 + a_{n+1} - 2a_n \in \mathbb{Z}$. If $\mu \in D_M$, since

$$\sum_{i=2}^{n} (a_i - a_n)\bar{\bar{\varepsilon}}_i + \left(\frac{a_1 + a_{n+1}}{2} - a_n\right)(\bar{\bar{\varepsilon}}_1 + \bar{\bar{\varepsilon}}_{n+1}) = \mu ,$$

the result follows.

The converse is immediate.

THEOREM 4.4. *Let*

$$\lambda = \sum_{i=1}^{n+1} a_i \bar{\varepsilon}_i \in D_K \qquad \textit{and} \qquad \mu = b_0(\bar{\bar{\varepsilon}}_1 + \bar{\bar{\varepsilon}}_{n+1}) + \sum_{i=2}^{n} b_i \varepsilon_i \in D_M .$$

Let $b_1 \in \mathbb{Z}$ *be defined by* $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i$. *Then* $m_\lambda(\mu) = 1$ *iff* $a_1 \geqslant b_2 \geqslant \ldots \ldots \geqslant b_n \geqslant a_n$ *and* $b_0 = (b_1 + a_{n+1})/2$. *Otherwise* $m_\lambda(\mu) = 0$.

PROOF. Let $\varphi_1: \mathfrak{k} \to su(n) \times \mathbb{R}$ be the Lie algebra isomorphism of $\mathfrak{k}$ onto $su(n) \times \mathbb{R}$, defined by

$$\varphi_1\begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} = \left(A + \frac{d}{n} I_n, id\right) \quad \text{for } A \in u(n),\ d \in u(1) \text{ and } \operatorname{tr} A + d = 0 .$$

Let

$$s\big(u(1) \times u(n-1)\big) = \left\{\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, a \in u(1), B \in u(n-1), \operatorname{tr} B + a = 0\right\}$$

and $\mathfrak{k}_1 = s\big(u(1) \times u(n-1)\big) \times \mathbb{R}$. Then $\mathfrak{k}_1$ is a subalgebra of $\varphi_1(\mathfrak{k})$.

Let $\varphi: s(u(1)\times u(n-1))\times\mathbf{R}\to su(n-1)\times\mathbf{R}\times\mathbf{R}$ be the Lie algebra isomorphism defined by

$$\varphi\left\{\begin{pmatrix} b & \\ & A\end{pmatrix}, c\right\} = \left(A + \frac{b}{n-1} I_{n-1}, ib - \frac{c}{n} + c,\ ib - \frac{c}{n} - c\right),$$

for $b\in u(1)$, $A\in u(n-1)$, $\operatorname{tr} A + b = 0$ and $c\in\mathbf{R}$. Then:

$$(1) \qquad \varphi\varphi_1(\mathfrak{m}) = su(n-1)\times\mathbf{R}\times 0 .$$

Let $\check{\mathfrak{h}}$ be the diagonal matrices in $sl(n, \mathbf{C})$ and $\{\check{\varepsilon}_i\}_{i=1}^n$ be the linear functional on $\check{\mathfrak{h}}$ defined by

$$\check{\varepsilon}_i\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix} = a_i \quad \text{for} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}\in\check{\mathfrak{h}} .$$

Fix $\underset{\check{\alpha}_1}{\circ}\cdots\underset{\check{\alpha}_{n-1}}{\circ}$ as fundamental Weyl chamber, where $\check{\alpha}_i = \check{\varepsilon}_i - \check{\varepsilon}_{i+1}$.

Let $\hat{\mathfrak{h}}$ be the diagonal matrices in $sl(n-1, \mathbf{C})$ and let $\{\hat{\varepsilon}_i\}_{i=1}^{n-1}$ and $\{\hat{\alpha}_i\}_{i=1}^{n-2}$ be defined similarly to the $\check{\varepsilon}_i$'s and $\check{\alpha}_i$'s. Fix $\underset{\hat{\alpha}_1}{\circ}\cdots\underset{\hat{\alpha}_{n-2}}{\circ}$ as fundamental Weyl chamber.

The notion of dominance for $sl(n, \mathbf{C})$ and $sl(n-1, \mathbf{C})$ will always be intended with respect to this particular choice of simple roots.

We will use in the proof of the theorem the branching laws suggested by the following diagram:

$$su(n)\times\mathbf{R}\supset s(u(1)\times u(n-1))\times\mathbf{R}\xrightarrow{\varphi} su(n-1)\times\mathbf{R}\times\mathbf{R}\supset su(n-1)\times\mathbf{R}\times 0 .$$

(1) Considering the action of $(\pi_\lambda)_*$ on the center of $\mathfrak{k}$ it follows easily that:

$$\left((\pi_\lambda)_*\circ\varphi_1^{-1}, V_\lambda\right)$$

as $su(n)\times\mathbf{R}$ representation is equivalent to

$$\left((\pi_{\lambda_1})_*\otimes\pi, V_{\lambda_1}\otimes\mathbf{C}\right)$$

where $(\pi_{\lambda_1}, V_{\lambda_1})$ is the irreducible $SU(n)$ representation of highest weight $\lambda_1 = \sum_{i=1}^n a_i\check{\varepsilon}_i$ and π is the translation of $\mathbf{R}$ over $\mathbf{C}$ given by $i\left((1/n)(a_1 + \ldots + a_n) - a_{n+1}\right)$.

$$(2) \quad D_{S(U(1)\times U(n-1))} = \left\{\lambda = \sum_{i=1}^{n} c_i \check{\varepsilon}_i,\ c_2 \geqslant \ldots \geqslant c_n,\ c_i \in \mathbb{Z},\ i = 1, \ldots, n\right.$$

and

$$(\pi_{\lambda_1})_*|_{s(u(1)\times u(n-1))} \qquad \sum_{\substack{\mu = \sum_{i=1}^{n} c_i \check{\varepsilon}_i \in D_{S(U(1)\times U(n-1))} \\ \sum_1^n c_i = \sum_1^n a_i : a_1 \geqslant c_2 \geqslant \ldots \geqslant c_n \geqslant a_n}} (\pi_\mu)_* .$$

For a proof of (2), cf. [5], Theorem 3.

(3) For each π_μ appearing in the above sum, let V_μ be the representation space (complex). Then $\big((\pi_\mu)_* \otimes \pi \circ \varphi^{-1}, V_\mu \otimes \mathbf{C}\big)$ is equivalent as $su(n-1)\times \mathbf{R} \times \mathbf{R}$ representation to

$$\big((\pi_{\mu_1})_* \otimes \pi_{\mu_2} \otimes \pi_{\mu_3}, V_{\mu_1} \otimes \mathbf{C} \otimes \mathbf{C}\big)$$

where (π_{μ_1}, V_{μ_1}) is the irreducible $SU(n-1)$ representation of highest weight $\mu_1 = \sum_{i=1}^{n-1} c_{i+1} \hat{\varepsilon}_i$ and π_{μ_2}, π_{μ_3} are the translations of $\mathbf{R}$ over $\mathbf{C}$ by

$$i\left(\frac{c_2 + \ldots + c_n}{n-1} - \frac{c_1 + a_{n+1}}{2}\right) \quad \text{and} \quad i\left(\frac{a_{n+1} - c_1}{2}\right)$$

respectively.

Indeed let $\tilde{\pi}_\mu$ be the $SU(n-1)$ representation defined by

$$(\tilde{\pi}_\mu)_*(A) = (\pi_\mu)_* \begin{pmatrix} 0 & \\ & A \end{pmatrix},$$

for $A \in su(n-1)$. ($SU(n-1)$ is simply connected). Then $(\tilde{\pi}_\mu)_* \simeq (\pi_{\mu_1})_*$. Let $B: V_{\mu_1} \to V_\mu$ be the interwining operator. Define $T: V_{\mu_1} \otimes \mathbf{C} \to V_\mu \otimes \mathbf{C}$ by $T(v \otimes x \otimes y) = Bv \otimes xy$, for $x, y \in \mathbf{C}$ and $v \in V_{\mu_1}$. If $A \in su(n-1)$ and $b, c \in \mathbf{R}$, then

$$(\pi_\mu)_* \otimes \pi \circ \varphi^{-1}(A, b, c)\, T(v \otimes x \otimes y) =$$

$$= (\pi_\mu)_* \otimes \pi \left(\left.\begin{array}{c|c} -i\left(\dfrac{b+c}{2} + \dfrac{b-c}{2n}\right) & \\ \hline & A + \dfrac{i\left(\dfrac{b+c}{2} + \dfrac{b-c}{2n}\right)}{n-1} I_{n-1} \end{array}\right., \frac{b-c}{2} \right).$$

$$T(v\otimes x\otimes y) = (\pi_\mu)_*\begin{pmatrix}0 & \\ & A\end{pmatrix} Bv\otimes xy +$$

$$+ (\pi_\mu)_*\begin{pmatrix} -i\left(\frac{b+c}{2}+\frac{b-c}{2n}\right) & \\ & \frac{i\left(\frac{b+c}{2}+\frac{b-c}{2n}\right)}{n-1} I_{n-1}\end{pmatrix} Bv\otimes xy +$$

$$+ Bv\otimes\pi\left(\frac{b-c}{2}\right)xy = (\tilde\pi_\mu)_*(A)Bv\otimes xy +$$

$$+\left(-i\left(\frac{b+c}{2}+\frac{b-c}{2n}\right)c_1+\frac{c_2+\ldots+c_n}{n-1}\cdot i\left(\frac{b+c}{2}+\frac{b-c}{2n}\right)\right)\cdot$$

$$\cdot Bv\otimes xy + Bv\otimes\frac{b-c}{2}i\left(\frac{a_1+\ldots+a_n}{n}-a_{n+1}\right)xy\,.$$

Since $a_1+\ldots+a_n = c_1+\ldots+c_n$, thus

$$-i\left(\frac{b+c}{2}+\frac{b-c}{2n}\right)c_1+\frac{c_2+\ldots+c_n}{n-1}i\left(\frac{b+c}{2}+\frac{b-c}{2n}\right)+$$

$$+i\left(\frac{b-c}{2}\right)\left(\frac{a_1+\ldots+a_n}{n}-a_{n+1}\right)=i\left(\frac{b+c}{2}\right)\left(-c_1+\frac{c_2+\ldots+c_n}{n-1}\right)+$$

$$+i\left(\frac{b-c}{2n}\right)\left(-c_1+\frac{c_2+\ldots+c_n}{n-1}+c_1+c_2+\ldots c_n-na_{n+1}\right)=$$

$$=i\,\frac{(b+c)}{2}\left(-c_1+\frac{c_2+\ldots+c_n}{n-1}\right)+i\,\frac{b-c}{2n}\left(n\left(\frac{c_2+\ldots+c_n}{n-1}\right)-na_{n+1}\right)=$$

$$=b\cdot i\left(\frac{(c_1+a_{n+1})}{2}+\frac{c_2+\ldots+c_n}{n-1}\right)+c\cdot i\left(\frac{-c_1+a_{n+1}}{2}\right).$$

Thus $(\pi_\mu)_*\otimes\pi\circ\varphi^{-1}(A, b, c)\circ T = T(\pi_{\mu_1})_*\otimes\pi_{\mu_2}\otimes\pi_{\mu_3}(A, b, c)$ and (3) is proved.

(4) $(\pi_{\mu_1})_*\otimes\pi_{\mu_2}\otimes\pi_{\mu_3}|_{su(n-1)\times\mathbf{R}\times 0}\simeq(\pi_{\mu_1})_*\otimes\pi_{\mu_2}\otimes 0$.

(5) $((\pi_{\mu_1})_*\otimes\pi_{\mu_2}\otimes 0\circ\varphi\circ\varphi_1, V_{\mu_1}\otimes\mathbf{C}\otimes\mathbf{C})\simeq((\pi_{\tilde\mu})_*, V_{\tilde\mu})$ as $\mathfrak{m}$ representation, where $\pi_{\tilde\mu}$ is the M-representation parametrized by

$$\tilde\mu = \frac{c_1+a_{n+1}}{2}(\bar{\bar\varepsilon}_1+\bar{\bar\varepsilon}_{n+1})+c_2\bar{\bar\varepsilon}_2+\ldots+c_n\bar{\bar\varepsilon}_n\,.$$

Indeed, let $\mathfrak{m}_0 = \left\{ X \in \mathfrak{m} : \begin{pmatrix} 0 & & \\ & A & \\ & & 0 \end{pmatrix}, A \in su(n-1) \right\}$, then $(\pi_{\tilde{\mu}})_*|_{\mathfrak{m}_0} \simeq$ $\simeq (\pi_{\mu_1})_*$. Let $B: V_{\mu_1} \to V_{\tilde{\mu}}$ be the intertwining operator. Define $T: V_{\mu_1} \otimes \mathbf{C} \otimes \mathbf{C} \to V_{\tilde{\mu}}$ by $T(v_1 \otimes x \otimes y) = xyBv_1$. Then for $v \in V_{\mu_1}$, $x, y \in \mathbf{C}$, $\begin{pmatrix} a & & \\ & A & \\ & & a \end{pmatrix} \in \mathfrak{m}$,

$$T\left((\pi_{\mu_1})_* \otimes \pi_{\mu_2} \otimes 0 \circ \varphi\varphi_1 \begin{pmatrix} a & & \\ & A & \\ & & a \end{pmatrix} v \otimes x \otimes y\right) =$$

$$= T\left((\pi_{\mu_1})_* \otimes \pi_{\mu_2} \otimes 0 \left(A + \frac{2a}{n-1} I_{n-1}, 2ia, 0\right) v \otimes x \otimes y\right) =$$

$$= T(\pi_{\mu_1})_* \left(A + \frac{2a}{n-1} I_{n-1}\right) v \otimes x \otimes y + Tv \otimes \pi_{\mu_2}(2ia) x \otimes y =$$

$$= xy(\pi_{\tilde{\mu}})_* \begin{pmatrix} 0 & & \\ & A + \frac{2a}{n-1} I_{n-1} & \\ & & 0 \end{pmatrix} Bv +$$

$$+ 2ia \cdot i \left(\frac{c_2 + \ldots + c_n}{n-1} - \frac{c_1 + a_{n+1}}{2}\right) xyBv =$$

$$= xy(\pi_{\tilde{\mu}})_* \begin{pmatrix} 0 & & \\ & A + \frac{2a}{n-1} I_{n-1} & \\ & & 0 \end{pmatrix} Bv +$$

$$+ xy(\pi_{\tilde{\mu}})_* \begin{pmatrix} a & & \\ & -\frac{2a}{n-1} I_{n-1} & \\ & & a \end{pmatrix} Bv =$$

$$= xy(\pi_{\tilde{\mu}})_* \begin{pmatrix} a & & \\ & A & \\ & & a \end{pmatrix} Bv = (\pi_{\tilde{\mu}})_* \begin{pmatrix} a & & \\ & A & \\ & & a \end{pmatrix} T(v \otimes x \otimes y) .$$

Hence T is an intertwining operator and (5) is true.

(6) By Lemma 2.1 and (1)-(5) we thus have:

$$
\begin{aligned}
(\pi_\lambda)_*|_{\mathfrak{m}} &\simeq \left((\pi_\lambda)_* \circ \varphi_1^{-1}|_{\varphi_1(\mathfrak{m})}\right) \circ \varphi_1 \simeq \left((\pi_{\lambda_1})_* \otimes \pi|_{\varphi_1(\mathfrak{m})}\right) \circ \varphi_1 \simeq \\
&\simeq \left((\pi_{\lambda_1})_* \otimes \pi|_{\mathfrak{k}_1}\right)|_{\varphi_1(\mathfrak{m})} \circ \varphi_1 \simeq \left(\sum_{\text{as in (2)}} (\pi_\mu)_* \otimes \pi\right)\Bigg|_{\varphi_1(\mathfrak{m})} \circ \varphi_1 \simeq \\
&\simeq \left(\sum (\pi_\mu)_* \otimes \pi \circ \varphi^{-1}|_{\varphi\circ\varphi_1(\mathfrak{m})}\right) \circ \varphi \circ \varphi_1 = \\
&= \left(\sum (\pi_{\mu_1})_* \otimes \pi_{\mu_2} \otimes 0\right) \circ \varphi \circ \varphi_1 = \sum_{\substack{\tilde{\mu} = \frac{c_1 + a_{n+1}}{2}(\bar{\bar{\varepsilon}}_1 + \bar{\bar{\varepsilon}}_{n+1}) + c_2\bar{\bar{\varepsilon}}_2 + \ldots + c_n\bar{\bar{\varepsilon}}_n \\ c_2 \geqslant \ldots \geqslant c_n,\ c_i \in \mathbf{Z},\ i = 1, \ldots, n \\ \sum_{i=1}^n c_i = \sum_{i=1}^n a_i,\ a_1 \geqslant c_2 \geqslant \ldots \geqslant c_n \geqslant a_n}} (\pi_{\tilde{\mu}})_* .
\end{aligned}
$$

(7) By the last equivalence of (6) it follows: $\pi_\lambda|_M \simeq \sum \pi_{\tilde{\mu}}$ and hence the theorem is proved.

5. Branching theorem for $Sp(n, 1)$, $n \geqslant 2$.

5.1. Let $G = Sp(n, 1)$, $n \geqslant 2$, then $\mathfrak{g} = sp(n, 1)$. Let

$$
\mathfrak{k} = \left\{ \begin{pmatrix} C & 0 & D & 0 \\ 0 & t & 0 & s \\ -\bar{D} & 0 & \bar{C} & 0 \\ 0 & -\bar{s} & 0 & \bar{t} \end{pmatrix} \begin{matrix} C, D \text{ complex } n \times n \text{ matrices, } C \in u(n) \\ D \text{ symmetric, } t \in u(1) \text{ and } s \in \mathbf{C} \end{matrix} \right\}
$$

and

$$
\mathfrak{p} = \left\{ \begin{pmatrix} 0 & C & 0 & D \\ \bar{C}^t & 0 & D^t & 0 \\ 0 & \bar{D} & 0 & -\bar{C} \\ \bar{D}^t & 0 & -C^t & 0 \end{pmatrix} C, D \text{ complex } n \times 1 \text{ matrices} \right\}.
$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition for $\mathfrak{g}$. Let $\mathfrak{a} = \mathbf{R}H$, where $H \in \mathfrak{p}$ has $D = 0$ and $C = \begin{pmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$, then $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and

$$\mathfrak{m} = \left\{ \begin{pmatrix} t & 0 & 0 & -s & 0 & 0 \\ 0 & C & 0 & 0 & D & 0 \\ 0 & 0 & t & 0 & 0 & s \\ \bar{s} & 0 & 0 & \bar{t} & 0 & 0 \\ 0 & -\bar{D} & 0 & 0 & \bar{C} & 0 \\ 0 & 0 & -\bar{s} & 0 & 0 & \bar{t} \end{pmatrix} \begin{array}{l} C\in u(n-1), D\leftrightarrow(n-1)\times(n-1) \\ \text{symmetric} \\ t\in u(1),\ s\in \mathbf{C} \end{array} \right\}.$$

Let $\mathfrak{h}$ be the set of the diagonal matrices in $\mathfrak{g}$. Let X_i $(1\leqslant i\leqslant n+1)$ be the $2(n+1)\times 2(n+1)$ diagonal matrix (a_{kl}) so that $a_{kk}=1$ for $k=i$ and $a_{kk}=-1$ for $k=n+1+i$ and $a_{kk}=0$ for all the other k's. $\{X_i\}_{i=1}^{n+1}$ is a basis for the complex space $\mathfrak{h}_{\mathbf{C}}$. Let $\{\varepsilon_i\}_{i=1}^{n+1}$ be the dual basis. The roots of $\mathfrak{g}_{\mathbf{C}}$ relative to $\mathfrak{h}_{\mathbf{C}}$ are $\pm\varepsilon_i\pm\varepsilon_j$ $(1\leqslant i<j\leqslant n+1)$ and $\pm 2\varepsilon_i$ $(1\leqslant i\leqslant n+1)$. The roots of $\mathfrak{k}_{\mathbf{C}}$ relative to $\mathfrak{h}_{\mathbf{C}}$ are $\pm\varepsilon_i\pm\varepsilon_j$ $(1\leqslant i<j\leqslant n)$ and $\pm 2\varepsilon_i$ $(1\leqslant i\leqslant n+1)$. Let $\mathfrak{h}^-$ be the diagonal matrices in $\mathfrak{m}$, then the roots of $\mathfrak{m}_{\mathbf{C}}$ with respect to $\mathfrak{h}^-_{\mathbf{C}}$ are $\bar{\varepsilon}_1+\bar{\varepsilon}_{n+1}$, $\pm\bar{\varepsilon}_i\pm\bar{\varepsilon}_j$ $(2\leqslant i<j\leqslant n)$, and $\pm 2\bar{\varepsilon}_i$ $(i=2,\ldots,n)$, where the bar means the restriction of the ε_i's to $\mathfrak{h}^-_{\mathbf{C}}$. Set $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$, $1\leqslant i\leqslant n$, $\alpha_{n+1}=2\varepsilon_{n+1}$ and $\bar{\alpha}_i=\bar{\varepsilon}_i-\bar{\varepsilon}_{i+1}$, $i=1,\ldots,n-1$ In what follows the notion of dominance for $\mathfrak{k}$ and $\mathfrak{m}$ will be relative to the following system of simple roots:

$$\mathfrak{k}_{\mathbf{C}}:\ \underset{\alpha_1}{\circ}\ \ldots\ \underset{\alpha_{n-1}}{\circ} \Leftarrow \underset{2\varepsilon_n}{\circ}\quad \underset{\alpha_{n+1}}{\circ}$$

$$\mathfrak{m}_{\mathbf{C}}:\ \underset{\bar{\alpha}_2}{\circ}\ \ldots\ \underset{\bar{\alpha}_{n-1}}{\circ} \Leftarrow \underset{2\bar{\varepsilon}_n}{\circ}\quad \underset{\bar{\varepsilon}_1+\bar{\varepsilon}_{n+1}}{\circ}\ .$$

The root normals for $\mathfrak{g}_{\mathbf{C}}$ are $H_{\pm\varepsilon_i\pm\varepsilon_j}=\pm X_i\pm X_j$ $(1\leqslant i<j\leqslant n+1)$ and $H_{\pm 2\varepsilon_i}=\pm X_i$ $(1\leqslant i\leqslant n+1)$.

The root normals for $\mathfrak{k}_{\mathbf{C}}$ are $H_{\pm\varepsilon_i\pm\varepsilon_j}=\pm X_i\pm X_j$ $(1\leqslant i<j\leqslant n)$ and $H_{\pm 2\varepsilon_i}=\pm X_i$ $(1\leqslant i\leqslant n+1)$.

The root normals for $\mathfrak{m}_{\mathbf{C}}$ are $H_{\pm\bar{\varepsilon}_i\pm\bar{\varepsilon}_j}=\pm X_i\pm X_j$ $(2\leqslant i<j\leqslant n)$ and $H_{\pm 2\bar{\varepsilon}_i}=\pm X_i$ $(2\leqslant i\leqslant n)$; $H_{\bar{\varepsilon}_1+\bar{\varepsilon}_{n+1}}=X_1+X_{n+1}$ and $H_{-\bar{\varepsilon}_1-\bar{\varepsilon}_{n+1}}=-X_1-X_{n+1}$

LEMMA 5.2.

$$D_K=\left\{\lambda=\sum_{i=1}^{n+1}a_i\varepsilon_i,\ a_1\geqslant\ldots\geqslant a_n\geqslant 0,\ a_{n+1}\geqslant 0,\ a_i\in\mathbf{Z}\ \textit{for}\ i=1,\ldots,n+1\right\}.$$

PROOF. Let $\lambda = \sum_{i=1}^{n+1} a_i \varepsilon_i \in D_K$. Since λ is dominant integral with respect to $\underset{\alpha_1}{\circ} \cdots \underset{\alpha_{n-1}}{\circ} \Rightarrow \underset{2\varepsilon_n}{\circ} \quad \underset{\alpha_{n+1}}{\circ}$ we have that

$$\lambda(H_{\alpha_i}) = \lambda(X_i - X_{i+1}) = a_i - a_{i+1} \in \mathbf{Z}_+ \qquad \text{for } i = 1, \ldots, n-1\,,$$

$$\lambda(H_{2\varepsilon_n}) = \lambda(X_n) = a_n \in \mathbf{Z}_+$$

and

$$\lambda(H_{\alpha_{n+1}}) = \lambda(X_{n+1}) = a_{n+1} \in \mathbf{Z}_+\,.$$

Hence $a_1 \geqslant \ldots \geqslant a_n \geqslant 0$, $a_{n+1} \geqslant 0$ and $a_i \in \mathbf{Z}$, $i = 1, \ldots, n+1$. Because the converse is obviously true, the lemma is proved.

LEMMA 5.3.

$$D_M = \Big\{\mu = b_1(\bar{\varepsilon}_1 + \bar{\varepsilon}_{n+1}) + \sum_{i=2}^{n} b_i \bar{\varepsilon}_i,\ b_2 \geqslant \ldots \geqslant b_n \geqslant 0, \quad 2b_1 \in \mathbf{Z}_+ \textit{ and } b_i \in \mathbf{Z} \textit{ for } i = 2, \ldots, n\Big\}\,.$$

PROOF. Let $\mu \in D_M$, then:

1) $\mu = \sum_{i=1}^{n+1} c_i \bar{\varepsilon}_i$

2) $2 \leqslant i \leqslant n-1$ $\mu(H_{\bar{\alpha}_i}) = c_i - c_{i+1} \in \mathbf{Z}_+$

$$\mu(H_{2\bar{\varepsilon}_n}) = c_n \in \mathbf{Z}_+$$

$$\mu(H_{\bar{\varepsilon}_1 + \bar{\varepsilon}_{n+1}}) = c_1 + c_{n+1} \in \mathbf{Z}_+$$

3) $\sum_{i=2}^{n} c_i \bar{\varepsilon}_i + \dfrac{c_1 + c_{n+1}}{2}(\bar{\varepsilon}_1 + \bar{\varepsilon}_{n+1}) = \sum_{i=1}^{n+1} c_i \bar{\varepsilon}_i = \mu$

1)-3) give the result.

We recall the following well known fact (cf. [4]).

LEMMA 5.4. *Let* $\mathfrak{g} = sp(1)$ *and* α *be a positive root for* $\mathfrak{g}_{\mathbf{C}}$, *relative to the diagonal matrices of* $\mathfrak{g}_{\mathbf{C}}$. *Set* $\lambda = (1/2)\alpha$ *and* $\mu_1 = k\lambda$, $\mu_2 = l\lambda$ *for* $k, l \in \mathbf{Z}_+$.

Let V_{μ_1}, V_{μ_2} be the irreducible $\mathfrak{g}_{\mathbf{C}}$ modules of highest weight μ_1 and μ_2 respectively. Then

$$V_{\mu_1} \otimes_{\mathbf{C}} V_{\mu_2} = \sum_{j=0}^{\min(k,l)} V_{(k+l-2j)\lambda}\,,$$

where $V_{(k+l-2j)\lambda}$ is the irreducible $\mathfrak{g}_{\mathbf{C}}$ module of highest weight $(k+l-2j)\lambda$.

THEOREM 5.5. *Let*

$$\lambda = \sum_{i=1}^{n+1} a_i \varepsilon_i \in D_K \qquad \textit{and} \qquad \mu = b_0(\bar{\varepsilon}_1 + \bar{\varepsilon}_{n+1}) + \sum_{i=2}^{n} b_i \bar{\varepsilon}_i \in D_M .$$

Define:

$$\begin{aligned} A_1 &= a_1 - \max(a_2, b_2) , \\ A_2 &= \min(a_2, b_2) - \max(a_3, b_3) , \\ &\vdots \\ A_{n-1} &= \min(a_{n-1}, b_{n-1}) - \max(a_n, b_n) , \\ A_n &= \min(a_n, b_n) . \end{aligned}$$

Then $m_\lambda(\mu) = 0$ *unless*:

1) $a_i \geqslant b_{i+1} \quad i = 1, \ldots, n-1$

2) $b_i \geqslant a_{i+1} \quad i = 2, \ldots, n-1$ and

3) $b_0 = \dfrac{a_{n+1} + b_1 - 2_j}{2}$ for some $j = 0, \ldots, \min(a_{n+1}, b_1)$

where b_1 *satisfies* $b_1 \in \mathbb{Z}_+$ *and* $\sum_{i=1}^{n}(a_i + b_i) \in 2\mathbb{Z}$. *If these conditions hold then*: $m_\lambda(\mu) = \sum_{b_1 \text{ satisfying } 3)} \tilde{m}_\lambda(\mu)$ *where*

$$\tilde{m}_\lambda(\mu) = \sum_{L \subset \{1 \ldots n\}} (-1)^{|L|} \binom{n - 2 - |L| + \frac{1}{2}\left(-b_1 + \sum_{1}^{n} A_i\right) - \sum_{i \in L} A_i}{n-2}$$

$\left(|L| \textit{ is the cardinality of } L \textit{ and } \binom{x}{y} \textit{ is defined to be } = 0 \textit{ if } x - y \notin \mathbb{Z}_+\right)$.

PROOF. Let $\varphi_1 \colon \mathfrak{k} \to sp(n) \times sp(1)$ be the Lie algebra isomorphism defined by:

$$\varphi_1 \begin{pmatrix} A & 0 & B & 0 \\ 0 & t & 0 & s \\ -\bar{B} & 0 & \bar{A} & 0 \\ 0 & -\bar{s} & 0 & \bar{t} \end{pmatrix} = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \begin{pmatrix} t & s \\ -\bar{s} & \bar{t} \end{pmatrix} \right\} .$$

Define

$$\mathfrak{k}_1 = \left\{ (X,Y); Y \in sp(1) \text{ and } X = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & A & 0 & B \\ -\bar{\beta} & 0 & \bar{\alpha} & 0 \\ 0 & -\bar{B} & 0 & \bar{A} \end{pmatrix} \in sp(1) \times sp(n-1) \right\} .$$

Then $\mathfrak{k}_1$ is a subalgebra of $\varphi_1(\mathfrak{k})$.

Let $\varphi: \mathfrak{k}_1 \to sp(n-1) \times sp(1) \times sp(1)$ be the Lie algebra isomorphism defined by

$$\varphi(X, Y) = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, Y \right\} \text{ for } X = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & A & 0 & B \\ -\bar{\beta} & 0 & \bar{\alpha} & 0 \\ 0 & -\bar{B} & 0 & \bar{A} \end{pmatrix}$$

in $sp(1) \times sp(n-1)$.

Then $\varphi = \psi_2 \circ \psi_1$ where $\psi_1: \mathfrak{k}_1 \to sp(n-1) \times sp(1) \times sp(1)$ is the Lie algebra isomorphism defined by

$$\psi_1(X, Y) = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, Y \right\}$$

and ψ_2 is the automorphism of $sp(n-1) \times sp(1) \times sp(1)$ defined by

$$\psi_2\left(Z, \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, Y \right) = \left(Z, \begin{pmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, Y \right)$$

for $Z \in sp(n-1)$, $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ and $Y \in sp(1)$.

Note that φ is defined so that $\varphi(\varphi_1(\mathfrak{m})) = sp(n-1) \times \Delta(sp(1))$. Hence « modulo φ » we can use Lepowsky's multiplicity theorem. Let $\check{\mathfrak{h}}$ be the diagonal matrices in $sp(n, \mathbf{C})$ and $\check{\varepsilon}_i$ $(1 \leqslant i \leqslant n)$ be the linear functional on $\check{\mathfrak{h}}$ defined by

$$\check{\varepsilon}_i \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & -a_1 & & \\ & & & & \ddots & \\ & & & & & -a_n \end{pmatrix} = a_i \text{ for } a_j \in \mathbf{C} .$$

Fix as a system of simple roots for $(sp(n, \mathbf{C}), \check{\mathfrak{h}})$

$$\{\check{\alpha}_i = \check{\varepsilon}_i - \check{\varepsilon}_{i+1},\ i = 1, \dots, n-1,\ \check{\alpha}_n = 2\varepsilon_n\}.$$

Define $\hat{\varepsilon}_i$ $(i = 1, \dots, n-1)$ in a similar way as linear functional on $\hat{\mathfrak{h}}$, the diagonal matrices of $sp(n-1, \mathbf{C})$. Fix

$$\{\hat{\alpha}_i = \hat{\varepsilon}_i - \hat{\varepsilon}_{i+1},\ i = 1, \dots, n-2,\ \hat{\alpha}_{n-1} = 2\hat{\varepsilon}_{n-1}\}$$

as simple roots for $(\mathrm{sp}(n-1, \mathbf{C}), \hat{\mathfrak{h}})$.

The notion of dominance for $sp(n, \mathbf{C})$ and $sp(n-1, \mathbf{C})$ will be relative to this choice of simple roots. We use in the proof of the theorem the branching laws suggested by the following diagram:

$$sp(n)\times sp(1) \supset (\mathfrak{k}_1) \xrightarrow{\varphi} sp(n-1)\times sp(1)\times sp(1) \supset sp(n-1)\times \Delta(sp(1)).$$

(1) $\big((\pi_\lambda)_* \circ \varphi_1^{-1}, V_\lambda\big) \simeq \big((\pi_{\lambda_1})_* \otimes \pi_{a_{n+1}}, V_{\lambda_1} \otimes V_{a_{n+1}}\big)$ as $sp(n)\times sp(1)$ representation, where $(\pi_{\lambda_1}, V_{\lambda_1})$ is the irreducible $Sp(n)$ representation of highest weight $\lambda_1 = \sum_{i=1}^{n} a_i \check{\varepsilon}_i$ and $(\pi_{a_{n+1}}, V_{a_{n+1}})$ is the irreducible $sp(1, \mathbf{C})$ modulo of dimension $a_{n+1} + 1$.

In fact the two representations have the same highest weight.

(2) $$D_{Sp(1)\times Sp(n-1)} = \Big\{\mu = \sum_{i=1}^{n} c_i \check{\varepsilon}_i,\ c_2 \geqslant \dots \geqslant c_n \geqslant 0,\ c_1 \geqslant 0,\ c_i \in \mathbf{Z},\ i = 1, \dots, n\Big\}.$$

and

$$(\pi_{\lambda_1})_*|_{\mathfrak{k}_1} \simeq \sum_{\substack{\mu = \sum_{i=1}^{n} b_i \check{\varepsilon}_i \in D_{Sp(1)\times Sp(n-1)} \\ \sum_{i=1}^{n} (a_i + b_i) \in 2\mathbf{Z} \\ a_i \geqslant b_{i+1} \quad i = 1, \dots, n-1 \\ b_i \geqslant a_{i+1} \quad i = 2, \dots, n-1}} \tilde{m}_\lambda(\mu) (\pi_\mu)_*$$

where $\tilde{m}_\lambda(\mu)$ is as in the statement of the theorem. For a proof of (2), cf. [5], Theorem 6.

(3) For each π_μ in the above sum, let V_μ as usual denote the representation space, then

$$\big((\pi_\mu)_* \otimes \pi_{a_{n+1}} \circ \varphi^{-1}, V_\mu \otimes V_{a_{n+1}}\big)$$

is equivalent as $sp(n-1)\times sp(1)\times sp(1)$ representation to

$$\big((\pi_{\mu_1})_*\otimes\pi_{b_1}\otimes\pi_{a_{n+1}},\ V_{\mu_1}\otimes V_{b_1}\otimes V_{a_{n+1}}\big)$$

where (π_{μ_1}, V_{μ_1}) is the irreducible representation of $Sp(n-1)$ of highest weight $\mu_1=\sum_{i=1}^{n-1} b_{i+1}\hat{\varepsilon}_i$ and (π_{b_1}, V_{b_1}) is the irreducible $sp(1,\mathbf{C})$ module of dimension b_1+1. In fact

$$(\pi_\mu)_*\otimes\pi_{a_{n+1}}\circ\varphi^{-1}=(\pi_\mu)_*\otimes\pi_{a_{n+1}}\circ\psi_1^{-1}(\circ\psi_2^{-1}\simeq(\pi_{\mu_1})^*\otimes\pi_{b_1}\otimes\pi_{a_{n+1}}\circ\psi_2^{-1}\,.$$

On the other hand $\pi_{b_1}\circ\psi_2^{-1}|_{sp(1,\mathbf{C})}\simeq\pi_{b_1}$ since ψ_2 doesn't change the highest weight.

$$(4)\quad (\pi_{\mu_1})_*\otimes\pi_{b_1}\otimes\pi_{a_{n+1}}|_{sp(n-1)\times\Delta(sp(1))}\simeq(\pi_{\mu_1})_*\otimes\sum_{j=0}^{\min(b_1,a_{n+1})}\pi_{b_1+a_{n+1}-2j}$$

where $\pi_{b_1+a_{n+1}-2j}$ is the irreducible $sp(1)$ module of dimension $b_1+ + a_{n+1}-2j+1$. (Cf. Lemma 5.4.)

$$(5)\quad (\pi_{\mu_1})_*\otimes\pi_{b_1+a_{n+1}-2j}\circ\varphi\circ\varphi_1\simeq(\pi_{\mu_j})_*\ \text{where}$$

$$\mu_j=\frac{b_1+a_{n+1}-2j}{2}(\bar{\varepsilon}_1+\bar{\varepsilon}_{n+1})+\sum_{i=2}^{n} b_i\bar{\varepsilon}_i\in D_M\,.$$

(5) is clear, since they have the same highest weight and are irreducible.

Finally, by Lemma 2.1 and (1)-(5), we have

$$(\pi_\lambda)_*|_{\mathfrak{m}}\simeq\big((\pi_\lambda)_*\circ\varphi_1^{-1}|_{\varphi_1(\mathfrak{m})}\big)\circ\varphi_1\simeq\big((\pi_{\lambda_1})_*\otimes\pi_{a_{n+1}}|_{\varphi_1(\mathfrak{m})}\big)\circ\varphi_1\simeq$$

$$\big\{\big((\pi_{\lambda_1})_*\otimes\pi_{a_{n+1}}|_{\mathfrak{k}_1}\big)_{\varphi_1(\mathfrak{m})}\big\}\circ\varphi_1\simeq\sum_{\substack{\mu=\sum_{i=1}^{n} b_i\bar{\varepsilon}_i\in D_{Sp(1)\times Sp(n-1)}\\ \sum_{i=1}^{n}(a_i+b_i)\in 2\mathbf{Z}\\ a_i\geqslant b_{i+1}\quad i=1,\dots,n-1\\ b_i\geqslant a_{i+1}\quad i=2,\dots,n-1}}\big(\tilde{m}_\lambda(\mu)(\pi_\mu)_*\otimes\pi_{a_{n+1}}\big)|_{\varphi_1(\mathfrak{m})}\circ\varphi_1\simeq$$

$$\simeq\big\{\big(\sum\tilde{m}_\lambda(\mu)(\pi_\mu)_*\otimes\pi_{a_{n+1}}\circ\varphi^{-1}|_{\varphi\varphi_1(\mathfrak{m})}\big)\circ\varphi\big\}\circ\varphi_1\simeq$$

$$\simeq\big(\sum\tilde{m}_\lambda(\mu)(\pi_{\mu_1})_*\otimes\pi_{b_1}\otimes\pi_{a_{n+1}}|_{\varphi\varphi_1(\mathfrak{m})}\big)\circ\varphi\varphi_1\simeq$$

$$\simeq\Big(\sum\tilde{m}_\lambda(\mu)(\pi_{\mu_1})_*\otimes\sum_{j=0}^{\min(b_1,a_{n+1})}\pi_{b_1+a_{n+1}-2j)}\Big)\circ\varphi\varphi_1\simeq$$

$$\simeq\sum\tilde{m}_\lambda(\mu)(\pi_{\mu_j})_*\,.$$

The theorem follows.

6. Branching theorem for F_4.

6.1. Let $G_{\mathbf{C}} = (F_4)_{\mathbf{C}}$ and F_4 be the analytic subgroup of $G_{\mathbf{C}}$, whose Lie algebra is $\mathfrak{g} = f_{4(-20)}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$, then $\mathfrak{k} = so(9)$ and $K = Spin(9)$.

Let $\mathfrak{h} \subset \mathfrak{k}$ be a Cartan subalgebra for both $\mathfrak{g}$ and $\mathfrak{k}$. Let:

$$\underset{\alpha_1}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_4}{\circ}$$

be a choice of simple roots for $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Define $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ in terms of the dual basis of the α_i's (cf. [3]).

Then $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$, $\alpha_3 = \varepsilon_4$, $\alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$. Let $\Delta(\Delta_k)$ be the roots for $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$, $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$. Then

$$\Delta = \{\pm \varepsilon_i,\ 1 \leqslant i \leqslant 4,\ \pm \varepsilon_i \pm \varepsilon_j,\ 1 \leqslant i < j \leqslant 4,\ \tfrac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$$

and $\Delta_k = \{\pm \varepsilon_i,\ 1 \leqslant i \leqslant 4,\ \pm \varepsilon_i \pm \varepsilon_j,\ 1 \leqslant i < j \leqslant 4\}$.

Let $\mathfrak{k}_{\mathbf{C}}$: $\underset{\alpha_2+2\alpha_3+2\alpha_4}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_1}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ}$ be the simple roots for Δ_k. Let $\Delta^+(\Delta_k^+)$

be positive for $\Delta(\Delta_k)$ determined by this choice. We choose the root vectors X_{α_4}, $X_{-\alpha_4}$ satisfying $[X_{\alpha_4}, X_{-\alpha_4}] = H_{\alpha_4}$ and $X_{\alpha_4} + X_{-\alpha_4} \in \mathfrak{p}$. Then $\mathfrak{a} = \mathbb{R}(X_{\alpha_4} + X_{-\alpha_4})$ is a maximal abelian subalgebra of $\mathfrak{p}$, and $\mathfrak{m} = so(7)$.

Define $\mathfrak{h}^- = \{H \in \mathfrak{h}: [H, \mathfrak{a}] = 0\} = \{H \in \mathfrak{h}: \alpha_4(H) = 0\}$, then $\mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{m}$. As usual let $\mathfrak{h}_0 = \mathfrak{h}^- + \mathfrak{a}$.

Let $u_{\alpha_4} = \exp \pi/4(X_{\alpha_4} - X_{-\alpha_4})$ and consider the Cayley transform $\operatorname{Ad} u_{\alpha_4}$, with respect to the noncompact root α_4. Then $\operatorname{Ad} u_{\alpha_4}$ carries $\mathfrak{h}_{\mathbf{C}}$ to $(\mathfrak{h}^- + \mathfrak{a})_{\mathbf{C}}$.

Let Φ be the roots of $\mathfrak{g}_{\mathbf{C}}$ relative to $(\mathfrak{h}_0)_{\mathbf{C}}$; then $\Phi = \Delta \circ \operatorname{Ad}(u_{\alpha_4})^{-1}$ and $\Phi^+ = \Delta^+ \circ \operatorname{Ad}(u_{\alpha_4})^{-1}$ is positive for Φ. Let Φ_m be the roots of $(\mathfrak{m}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}^-)$ and $\Phi_m^+ = \Phi^+ \cap \Phi_m$.

LEMMA 6.2. $\mathfrak{m}_{\mathbf{C}}$: $\underset{\alpha_2}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_1}{\circ} \Rightarrow \underset{\frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)}{\circ}$ *relative to* Φ_m^+.

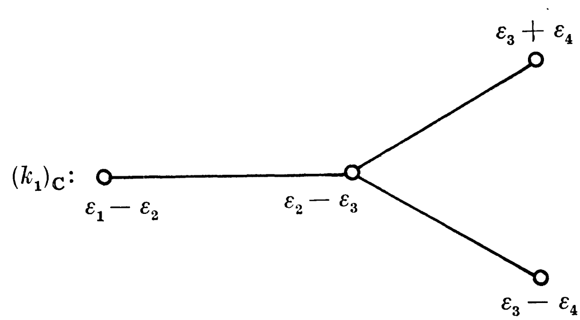
PROOF. The roots of $\mathfrak{h}_{\mathbf{C}}^-$ are the roots in Φ which are zero on $\mathfrak{a}$, therefore are of the form $\alpha = \beta \circ \operatorname{Ad}(u_{\alpha_4})^{-1}$, with $\beta \in \Delta$, and $(\beta, \alpha_4) = 0$.

$$\{\beta \in \Delta^+;\ (\beta, \alpha_4) = 0\} = \{\varepsilon_2 - \varepsilon_3,\ \varepsilon_2 - \varepsilon_4,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_1 + \varepsilon_2,\ \varepsilon_1 + \varepsilon_3,\ \varepsilon_1 + \varepsilon_4,$$
$$\tfrac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4),\ \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4),\ \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)\} = A\ .$$

Thus the positive roots for $\mathfrak{h}_{\mathrm{C}}^{-}$ are of the form $\alpha\circ\mathrm{Ad}\,(u_{\alpha_4})^{-1}$, for $\alpha\in A$.

On the other hand $\mathrm{Ad}\,(u_{\alpha_4})^{-1}|_{\mathfrak{h}^-}=I$, hence $\alpha\circ\mathrm{Ad}\,(u_{\alpha_4})^{-1}|_{\mathfrak{h}^-}=\alpha|_{\mathfrak{h}^-}$. It is now clear that the simple roots for $(\mathfrak{m}_{\mathrm{C}},\mathfrak{h}_{\mathrm{C}}^{-})$ are the ones described.

Let K_1 be the subgroup of K isomorphic to $Spin(8)$, so that a system of positive roots for the Lie algebra of K_1, $\mathfrak{k}_1=so(8)$, is given in the following way:



Then $\mathfrak{k}_1$ is contained in $\mathfrak{k}$ in the standard way. Relative to this choice the branching from $(\mathfrak{k}_1)_{\mathrm{C}}$ to $\mathfrak{m}_{\mathrm{C}}$ is not standard, while the one from $\mathfrak{k}_{\mathrm{C}}$ to $(\mathfrak{k}_1)_{\mathrm{C}}$ is. So we want, as we did for all the other cases, to define an automorphism φ of $(\mathfrak{k}_1)_{\mathrm{C}}$ which preserves the roots, and such that $\mathfrak{m}'=\varphi(\mathfrak{m}_{\mathrm{C}})$ is standard in $\varphi((\mathfrak{k}_1)_{\mathrm{C}})$, i.e.

$$\varphi:\mathfrak{m}_{\mathrm{C}}:\ \underset{\varepsilon_3-\varepsilon_4}{\circ}\!\!-\!\!-\!\!\underset{\varepsilon_2-\varepsilon_3}{\circ}\Rightarrow\underset{\frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)}{\circ}\ \rightarrow\mathfrak{m}':\ \underset{\varepsilon_1-\varepsilon_2}{\circ}\!\!-\!\!-\!\!\underset{\varepsilon_2-\varepsilon_3}{\circ}\Rightarrow\underset{\varepsilon_3}{\circ}\,.$$

It is now clear that φ must be defined in the following way:

$$\varphi(\varepsilon_1-\varepsilon_2)=\varepsilon_3-\varepsilon_4\,,$$
$$\varphi(\varepsilon_3-\varepsilon_4)=\varepsilon_1-\varepsilon_2\,,$$
$$\varphi(\varepsilon_2-\varepsilon_3)=\varepsilon_2-\varepsilon_3\,,$$
$$\varphi(\varepsilon_3+\varepsilon_4)=\varepsilon_3+\varepsilon_4\,,$$

that is

$$\varphi(\varepsilon_1)=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)\,,$$
$$\varphi(\varepsilon_2)=\tfrac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)\,,$$
$$\varphi(\varepsilon_3)=\tfrac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)\,,$$
$$\varphi(\varepsilon_4)=\tfrac{1}{2}(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)\,.$$

We recall that since $K_1 = Spin(8)$ and $M = Spin(7)$,

$$D_K \quad = \left\{\lambda = \sum_{i=1}^{4} a_i \varepsilon_i \colon a_1 \geqslant \ldots \geqslant a_4 \geqslant 0,\ 2a_i \in \mathbf{Z},\ a_i - a_j \in \mathbf{Z},\ i, j = 1, \ldots, 4\right\}.$$

$$D_{K_1} \quad = \left\{\mu = \sum_{i=1}^{4} b_i \varepsilon_i \colon b_1 \geqslant b_2 \geqslant b_3 \geqslant |b_4|,\ b_i - b_j \in \mathbf{Z},\ 2b_i \in \mathbf{Z},\ i = 1, 2, 3, 4\right\}.$$

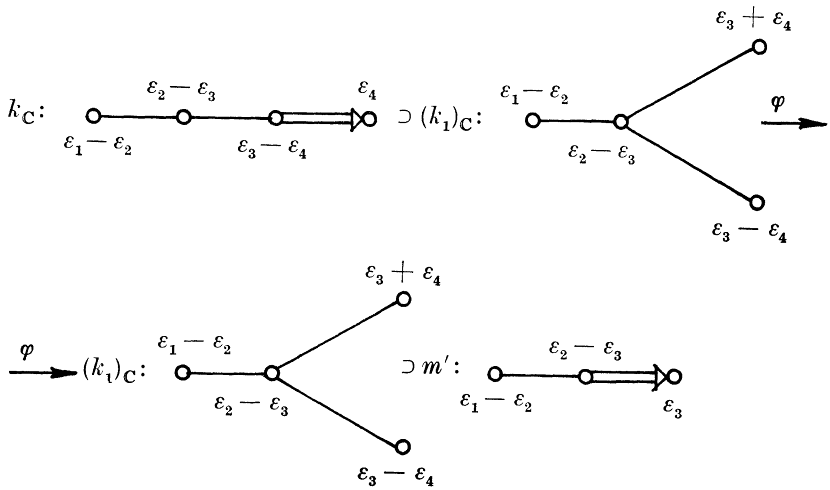
$$D_{\varphi(M)} = \left\{\gamma = \sum_{i=1}^{3} q_i \varepsilon_i \colon q_1 \geqslant q_2 \geqslant q_3 \geqslant 0,\ 2q_i \in \mathbf{Z},\ q_i - q_j \in \mathbf{Z}\right\}.$$

We use the notation $(a_1, \ldots, a_4) = \sum_{1}^{3} a_i \varepsilon_i$ for $a_i \in \mathbf{C}$.

THEOREM 6.3. *Let* $\lambda = \sum_{i=1}^{4} a_i \varepsilon_i \in D_K$, *then*:

$$\pi_\lambda|_M = \sum_{\substack{(q_1', \ldots, q_4') = \varphi(q_1, \ldots, q_4) \\ (q_1, \ldots, q_4) \in D_{K_1} \\ a_1 \geqslant q_1 \geqslant \ldots \geqslant a_4 \geqslant |q_4| \\ a_i - q_j \in \mathbf{Z}}} \ \sum_{\substack{\gamma = (b_1, b_2, b_3) \in D_{\varphi(M)} \\ q_1' \geqslant b_1 \geqslant \ldots \geqslant q_3' \geqslant b_3 \geqslant |q_4'| \\ q_i' - b_j \in Z}} \pi_\gamma \circ \varphi .$$

PROOF. We make use in the proof of the branching laws suggested by the following diagram:



$$(1)\quad (\pi_\lambda)_*|_{(\mathfrak{k}_1)_{\mathbf{C}}} = \sum_{\substack{\mu = (q_1, \ldots, q_4) \in D_{K_1} \\ a_1 \geqslant q_1 \geqslant \ldots \geqslant a_4 \geqslant |q_4| \\ a_i - q_j \in \mathbf{Z}}} (\pi_\mu)_* .$$

This is the standard branching theorem from $Spin(9)$ to $Spin(8)$.

(2) For each π_μ appearing in the above sum, let V_μ be the representation space. Then $((\pi_\mu)_* \circ \varphi^{-1}, V_\mu) \simeq (\pi_{\varphi(\mu)}, V_{\varphi(\mu)})$ as $(\mathfrak{k}_1)_{\mathbf{C}}$ representation, where $\pi_{\varphi(\mu)}$ is the irreducible $(\mathfrak{k}_1)_{\mathbf{C}}$ module of highest weight $\varphi(\mu) = (q'_1, \ldots, q'_4)$.

$$(3) \quad \pi_{\varphi(\mu)}|_{m'} \simeq \sum_{\substack{\gamma=(b_1,b_2,b_3)\in D_{\varphi(M)} \\ a'_1 \geqslant b_1 \geqslant a'_2 \geqslant b_2 \geqslant a'_3 \geqslant b_3 \geqslant |a'_4| \\ a'_i - b_j \in \mathbf{Z}}} (\pi_\gamma)_* .$$

This is the classical branching from *Spin*(8) to *Spin*(7).

(4) By Lemma 2.1 and (1)-(3) we thus have:

$$(\pi_\lambda)_*|_{\mathfrak{m}_{\mathbf{C}}} = ((\pi_\lambda)_*|_{(\mathfrak{k}_1)_{\mathbf{C}}})|_{\mathfrak{m}_{\mathbf{C}}} \simeq \sum_{\substack{\mu=(a_1,\ldots,a_4) \\ a_1 \geqslant \ldots \geqslant |a_4| \\ a_i - a_j \in \mathbf{Z}}} (\pi_\mu)_*|_{\mathfrak{m}_{\mathbf{C}}} \simeq \left(\sum (\pi_\mu)_* \circ \varphi^{-1}|_{\mathfrak{m}'}\right) \circ \varphi \simeq$$

$$\simeq \Bigg(\sum_{\substack{\varphi(\mu) = \Phi(a_1,\ldots,a_4) = (a'_1,\ldots,a'_4) \\ (a_1,\ldots,a_4) \in D_{K_1} \\ a_1 \geqslant \ldots \geqslant |a_4| \\ a_i - a_j \in \mathbf{Z}}} (\pi_{\varphi(\mu)})_*|_{\mathfrak{m}'}\Bigg) \circ \varphi \simeq \sum \sum_{\substack{\gamma=(b_1,b_2,b_3)\in D_{\varphi(M)} \\ a'_1 \geqslant b_1 \geqslant \ldots \geqslant b_3 \geqslant |a'_4| \\ a' - b_j \in \mathbf{Z}}} (\pi_\gamma)_* \circ \varphi$$

where the first sum of the right-hand side is on the same set as on the left-hand side.

The theorem follows.

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