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Bell Polynomials and Degenerate Stirling Numbers.

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1. Introduction.

The (exponential) partial Bell polynomials $B_{n,k}(a_1, a_2, \ldots, a_{n-k+1})$ in an infinite number of variables $a_1, a_2, \ldots,$ can be defined by means of

$$k!\sum_{n=k}^{\infty} B_{n,k}(a_1, a_2, \ldots)x^n/n! = \left(\sum_{m=1}^{\infty} a_m x^m/m!\right)^k, \tag{1.1}$$

or, equivalently,

$$B_{n,k}(a_1, a_2, \ldots) = \sum \frac{n!}{c_1!\,c_2!\ldots(1!)^{c_1}(2!)^{c_2}\ldots}\, a_1^{c_1}a_2^{c_2}\ldots \tag{1.2}$$

where the sum takes place over all integers $c_1, c_2, \ldots \geqslant 0$ such that

$$c_1 + 2c_2 + 3c_3 + \ldots = n\,,$$

$$c_1 + c_2 + c_3 + \ldots = k\,.$$

It follows that $B_{n,1} = a_n$ and $B_{n,n} = a_1^n$. Properties of $B_{n,k}$ and a table of values for $k \leqslant n \leqslant 12$ can be found in [5, pp. 133-137, 307]. These polynomials were apparently first introduced by Bell [1].

In this paper we are concerned with certain special cases of the

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Bell polynomials. We shall use the following notation:

$$(1.3) \qquad T_r(n, k) = B_{n,k}(0, \ldots, 0, a_{r+1}, a_{r+2}, \ldots)\,, \quad \text{i.e., } a_i = 0 \text{ for } i \leqslant r\,;$$

$$(1.4) \qquad S_r(n, k|\lambda) = T_r(n, k)\,,$$

where $a_i = (1-\lambda)(1-2\lambda) \ldots \big(1-(i-1)\,\lambda\big)$ for $i > r$;

$$(1.5) \qquad s_r(n, k|\lambda) = T_r(n, k)\,,$$

where $a_i = (1-\lambda)(2-\lambda) \ldots (i-1-\lambda)$ for $i > r$.

We call $S_r(n, k|\lambda)$ a degenerate associated Stirling number of the second kind, and we call $s_r(n, k|\lambda)$ a degenerate associated Stirling number of the first kind. If $r = 0$ in (1.4) and (1.5), we have the degenerate Stirling numbers of Carlitz [3]. If $\lambda = 0$ in (1.4) and (1.5), we have the r-associated Stirling numbers [7]. If $\lambda = 0$ and $r = 0$, we have the ordinary Stirling numbers.

The purpose of this paper is to examine the properties of the degenerate Stirling numbers in the most general possible setting. Thus, in section 2 we develop properties of $T_r(n, k)$ and the polynomials $T_{r,n}(y)$ defined by

$$(1.6) \qquad \sum_{n=0}^{\infty} T_{r,n}(y) x^n/n! = \exp\Big(y \sum_{j=r+1}^{\infty} a_j x^j/j!\Big)\,.$$

When $y = 1$ and $r = 0$, $T_{0,n}(1) = Y_n(a_1, a_2, \ldots, a_n)$, the exponential complete Bell polynomial [5, p. 134]. In section 3 we focus on the degenerate associated Stirling numbers, and we show how some of the properties of the degenerate Stirling numbers can be extended to the associated numbers. In section 4 we generalize some of the Stirling number formulas in [5, chapter 5] and [13] to the degenerate Stirling numbers.

2. Bell polynomials.

The partial Bell polynomials defined by (1.1) or (1.2) have the following interpretation: Let N be the set of integers $1, 2, \ldots, n$. Consider all set partitions of N into exactly k blocks (subsets) such that each block contains at least $r+1$ elements. Assign a « weight » of a_j to any block with exactly j elements. The weight of a partition is the product of the weights of its blocks. Then $T_r(n, k)$, defined by (1.3), is the sum of the weights of all the set partitions of N into exactly k blocks, each block with at least $r+1$ elements. In [4], [7] and [9, chapter 4], this kind of interpretation is discussed for the Stirling numbers and the r-associated Stirling numbers.

The following theorem could be proved by combining several of the properties given in [5, p. 136] for the Bell polynomials. We shall give a different kind of proof, however.

THEOREM 2.1. For $r\geqslant 0$, $k\geqslant 0$,

$$(r!/a_r)^m T_{r-1}(rm+k, m) = (rm+k)(rm+k-1) \dots (m+1) P_{r,k}(m)\,,$$

where $P_{r,k}(m)$ is a polynomial in m of degree k. In fact,

$$P_{r,k}(m) = \sum_{j=1}^{k} m(m-1) \dots (m-j+1)(r!/a_r)^j T_r(rj+k, j)/(rj+k)!\,.$$

PROOF. For convenience, we use the notation $f(i) = a_i/i!$. It follows from (1.1) that

$$\text{(2.1)} \qquad T_{r-1}(rm+k, m) = \\ = (rm+k)(rm+k-1) \dots (m+1) \sum f(u_1) \dots f(u_m)\,,$$

where the sum is taken over all compositions (ordered partitions) $u_1 + \dots + u_m = rm + k$, each $u_i \geqslant r$. We can represent such a composition with a set of $rm+k$ dots arranged in m rows. First place r dots in each of the m rows; then place the remaining k dots in the m rows. Suppose when we do this, there are n_1 rows with h_1 of the k dots, n_2 rows with h_2 of the k dots, etc. Then $k = n_1h_1 + n_2h_2 + \dots$, and there are $m!/((m-j)!\,n_1!\,n_2!\dots)$ ways of arranging these k dots in the m rows, where $j = n_1 + n_2 + \dots$. Thus the sum in (2.1) can be written

$$\text{(2.2)} \qquad \sum_{j=1}^{k} (f(r))^{m-j}(m!/(m-j)!) \sum (f(h_1+r))^{n_1} \cdot \\ \cdot (n_1!)^{-1}(f(h_2+r))^{n_2}(n_2!)^{-1} \dots$$

where the inner sum is over all partitions of k into j parts, i.e.,

$$k = n_1 h_1 + n_2 h_2 + \dots, \qquad j = n_1 + n_2 + \dots .$$

Now, by (1.1),

$$(2.3) \qquad T_r(rj + k, j)/(rj + k)! = (1/j!) \sum f(h_1 + r) \dots f(h_j + r),$$

each $h_i \geqslant 1$, the sum taken over all compositions of k into j parts $k = h_1 + \dots + h_j$. Now if h_i occurs n_i times in the composition, the right side of (2.3) is

$$\sum \big(f(h_1 + r)\big)^{n_1}(n_1!)^{-1}\big(f(h_2 + r)\big)^{n_2}(n_2!)^{-1} \dots ,$$

the sum taken over all partitions of k into j parts, i.e., $k = n_1 h_1 + n_2 h_2 + \dots$, $j = n_1 + n_2 + \dots$. Thus the inner sum in (2.2) is $\big(f(\cdot r)\big)^{-j} \cdot T_r(rj + k, j)/(rj + k)!$.

For small values of k, it is easy to compute $P_{r,k}(m)$ using the method of the proof of Theorem 2.1. The only difficulty is finding all the partition of k. Again letting $f(i) = a_i/i!$, we have

$$P_{r,0}(m) = 1,$$

$$PP_{r,1}(m) = \binom{m}{1} f(1 + r)/f(r),$$

$$P_{r,2}(m) = \binom{m}{1} f(2 + r)/f(r) + \binom{m}{2} \big(f(1 + r)/f(r)\big)^2,$$

$$P_{r,3}(m) = \binom{m}{1} f(3 + r)/f(r) + 2\binom{m}{2} f(2 + r) f(1 + r)/\big(f(r)\big)^2 + \binom{m}{3} \big(f(1 + r)/f(r)\big)^3 .$$

The formulas in the next theorem follow from (1.1), (1.3) and (1.6). The proofs are similar to those of Riordan [9, pp. 76-77] for the Stirling numbers. Most of these formulas can also be deduced by combining properties of the Bell polynomials given in [5, p. 136].

THEOREM 2.2. If $T_r(n, k)$ is defined by (1.3) and $T_{r,n}(y)$ is defined by (1.6), then

$$T_{r-1}(n, k) = \sum_{j=0}^{k} \frac{(a_r/r!)^j n!}{j!(n-rj)!} T_r(n-rj, k-j), \tag{2.4}$$

$$T_r(n, k) = \sum_{j=0}^{k} \frac{(-1)^j (a_r/r!)^j n!}{j!(n-rj)!} T_{r-1}(n-rj, k-j), \tag{2.5}$$

$$T_r(n+1, k) = \sum_{j=0}^{n} \binom{n}{j} a_{n-j+1} T_r(j, k-1), \tag{2.6}$$

$$T_{r,n}(y) = \sum_{j=0}^{n} T_r(n, j) y^j, \tag{2.7}$$

$$T_{r,n+1}(y) = y \sum_{j=0}^{n-r} \binom{n}{j} a_{n-j+1} T_{r,j}(y). \tag{2.8}$$

Many of the formulas in [13] are special cases of formulas involving $T_{r,n}(y)$. We show this in the next three theorems.

THEOREM 2.3. If $T_{r,n}(y)$ is defined by (1.6) or (2.7), then

$$T_{r,0}(y) = 1, \tag{2.9}$$

$$\sum_{k=0}^{n} \binom{n}{k} T_{r,k}(y) T_{r,n-k}(-y) = 0 \qquad \text{if } n > 0, \tag{2.10}$$

$$\sum_{k=0}^{n} \binom{n}{k} T_{r,k}(-y) T_{r,n-k+1}(y) = y a_{n+1}. \tag{2.11}$$

PROOF. Properties (2.9) and (2.10) are obvious from the definition (1.6). Property (2.11) follows after both sides of (1.6) are differentiated with respect to x.

THEOREM 2.4. For all positive $i, j \leqslant n$, let

$$b_{ij} = T_{r,i-j+1}(y)/(i-j+1)!$$

if $i+1 \geqslant j$, and $b_{ij} = 0$ if $i+1 < j$. Then

$$T_{r,n}(-y) = (-1)^n n! \det(b_{ij}) = (-1)^n n! g(n),$$

and

$$(-1)^n T_{r,n}(-y) = n! \sum_{k=0}^{n-1} (-1)^k g(n-k-1) T_{r,k+1}(y)/(k+1)!\,.$$

PROOF. This theorem follows from the fact that the series (1.6) is the reciprocal of the same series with y replaced by $-y$. The determinant expression for the coefficients of the reciprocal of a series is stated in [9, p. 45] and [8], and is well known.

THEOREM 2.5. *For all positive $i, j \leqslant n+1$,*

$$T_{0,n+1}(-y) = (-1)^{n+1} \det(d_{ij}) = (-1)^{n+1} \det(e_{ij})\,,$$

where $d_{ij} = y\binom{i-1}{j-1} a_{i-j+1}$ if $j \leqslant i$, $d_{ij} = 1$ if $j = i+1$ and $d_{ij} = 0$ if $j > i+1$; $e_{ij} = y a_{i-j+1}/(i-j)!$ if $j \leqslant i$, $e_{ij} = i$ if $j = i+1$, $e_{ij} = 0$ if $j > i+1$.

PROOF. We use Cramer's rule on the equations (2.8). The corresponding theorem for Stirling numbers is given in [7], and a special case is given in [13].

3. Degenerate associated Stirling numbers.

We now look at the special cases of the partial Bell polynomials given by (1.4) and (1.5). The cases when $r = 0$ have been investigated by Carlitz [3]. We shall use the notation

$$S_0(n, k|\lambda) = S(n, k|\lambda)\,, \tag{3.1}$$

$$s_0(n, k|\lambda) = s(n, k|\lambda)\,, \tag{3.2}$$

and, following Carlitz, call $S(n, k|\lambda)$ and $s(n, k|\lambda)$ degenerate Stirling numbers of the second and first kinds respectively. Similar numbers have been studied by Toscano [11], [12]; the numbers of Toscano are $(-1)^{n-r} s(n, r|\lambda)$ in our notation. It is easy to see that $S(n, k|0)$ is the ordinary Stirling number of the second kind and $s(n, k|0)$ is the unsigned Stirling number of the first kind.

Definitions (1.4) and (1.5) generalize the degenerate Stirling numbers in the same way that the r-associated Stirling numbers generalize the Stirling numbers [5, pp. 221-222, 256-257), [6], [7], [9, p. 102]. Using the terminology of [7], we see that the r-associated Stirling numbers are the special cases of (1.4) and (1.5) when $\lambda = 0$. It should be noted that, by Theorem 2.1, the numbers $S'(k, j|\lambda)$ and $S_1'(k, j|\lambda)$ defined in [3] are equal to $S_r(2k-j, k-j|\lambda)$ and $s_r(2k-j, k-j|\lambda)$, respectively, with $r = 1$.

In (1.1), if we let $a_m = 0$ for $m \leqslant r$ and $a_m = (1-\lambda)(1-2\lambda) \ldots \ldots (1-(m-1)\lambda)$ for $m > r$, we see that

$$k! \sum_{n=(r+1)k}^{\infty} S_r(n, k|\lambda) x^n/n! = [(1+\lambda x)^{\mu} - 1 - G(r, \lambda)]^k \tag{3.3}$$

where $\lambda\mu = 1$ and $G(r, \lambda) = \sum_{i=1}^{r} (1-\lambda) \ldots (1-(i-1)\lambda) x^i/i!$. If we differentiate both sides of (3.3) with respect to x and then multiply both sides by $1 + \lambda x$, we have, after some simplification,

$$\begin{aligned} k!(1+\lambda x) \sum_{k=0}^{\infty} S_r(n+1, k|\lambda) x^n/n! &= k[(1+\lambda x)^{\mu} - 1 - G(r, \lambda)]^k + \\ &+ k[(1+\lambda x)^{\mu} - 1 - G(r, \lambda)]^{k-1} [(1-\lambda) \ldots (1-r\lambda) x^r/r!] . \end{aligned}$$

Comparison of coefficients gives our next theorem.

THEOREM 3.1. The degenerate associated Stirling numbers of the second kind satisfy the triangular recurrence relation:

$$\begin{aligned} S_r(n+1, k|\lambda) &= (k - n\lambda) S_r(n, k|\lambda) + \\ &+ \binom{n}{r} (1-\lambda) \ldots (1-r\lambda) S_r(n-r, k-1|\lambda) \end{aligned}$$

with

$$S_r(0, 0|\lambda) = 1 ; \qquad S_r(n, 0|\lambda) = 0 \quad \text{if } n > 0 ;$$
$$S_r(n, k|\lambda) = 0 \quad \text{if } n < (r+1)k .$$

The Tate-Goen formula for the r-associated Stirling numbers [10], [7] can be generalized to the degenerate numbers, as the next theorem shows.

THEOREM 3.2.

$$S_r(n, k|\lambda) = (-1)^k n! \sum \frac{(-1)^{k_1} k_1(k_1-\lambda) \dots (k_1-(A-1)\lambda)\, W}{k_1!\, k_2! \dots k_{r+2}!\, A!\, Q}$$

where

$$A = A(k_1, \dots, k_{r+2}) = n - \sum_{i=0}^{r} i k_{i+2}\,,$$

$$Q = Q(k_1, \dots, k_{r+2}) = \prod_{i=0}^{r} (i!)^{k_{i+2}}\,,$$

$$W = W(k_1, \dots, k_{r+1}) = \prod_{i=1}^{r-1} [(1-\lambda) \dots (1-(i-1)\lambda)]^{k_{i+2}}\,,$$

and the sum is taken over all compositions $k_1 + \dots + k_{r+2} = k$, each $k_i \geqslant 0$.

PROOF. The theorem can be proved by induction on r. When $r = 0$, it reduces to the formula given in [3]:

$$k!\, S(n, k|\lambda) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j(j-\lambda) \dots (j-(n-1)\lambda)\,.$$

Assume the theorem is true for $S_{r-1}(n, k|\lambda)$ for all values of n and k. Then from (2.5), after substituting for $S_{r-1}(n-rj, k-j|\lambda)$ and simplifying, we see the theorem is true for $S_r(n, k|\lambda)$.

Theorem 3.2 is not difficult to use for small values of k. For example, if $n > r$,

$$S_r(n, 1|\lambda) = (1-\lambda)(1-2\lambda) \dots (1-(n-1)\lambda)\,,$$

and if $n > 2r+1$,

$$S_r(n, 2|\lambda) = (2-\lambda)(2-2\lambda) \dots (2-(n-1)\lambda) -$$
$$- \sum_{j=0}^{r} \binom{n}{j} (1-\lambda) \dots (1-(n-j-1)\lambda)(1-\lambda) \dots (1-(j-1)\lambda)\,.$$

Thus when $r = 0$, and $n > 1$,

$$S(n, 2|\lambda) = (2-\lambda) \dots (2-(n-1)\lambda) - (1-\lambda) \dots (1-(n-1)\lambda)\,.$$

Theorem 3.2 can also be used to prove the following:

$$k!S_1(n, k|\lambda) = \\ = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{m=0}^{k-j}\binom{k-j}{m}(n)_m j(j-\lambda)\ldots(j-(n-m-1)\lambda),$$

where $(n)_m = n(n-1)\ldots(n+m+1)$.

The numbers $S_1(n, k|\lambda)$ should not be confused with the degenerate Stirling numbers of the first kind in [3]. Unfortunately, the notation in each case is identical.

We now turn our attention to the degenerate associated Stirling numbers of the first kind. In (1.1), if we let $a_m = 0$ for $m \leqslant r$ and $a_m = (1-\lambda)(2-\lambda)\ldots(m-1-\lambda)$ for $m > r$, we have

$$k!\sum_{n=(r+1)k}^{\infty} s_r(n, k|\lambda)x^n/n! = \lambda^{-k}[1 + H(r, \lambda) - (1-x)^\lambda]^k, \tag{3.4}$$

where

$$H(r, \lambda) = \sum_{i=1}^{r}(-1)^{i+1}\lambda(\lambda-1)\ldots(\lambda-i+1)x^i/i!.$$

When $r = 0$, we have the degenerate Stirling numbers of the first kind [3]. Comparison of (3.3) and (3.4) gives us

$$s_r(n, k|\lambda) = (-\lambda)^{n-k}S_r(n, k|\lambda^{-1}), \tag{3.5}$$

so, by Theorem 3.1, we have the following recurrence relation:

THEOREM 3.3. The degenerate associated Stirling numbers of the first kind satisfy the triangular recurrence relation

$$s_r(n+1, k|\lambda) = (n-k\lambda)s_r(n, k|\lambda) + \\ + \binom{n}{r}(1-\lambda)\ldots(r-\lambda)s_r(n-r, k-1|\lambda),$$

with

$$s_r(0, 0|\lambda) = 1\ ;\ s_r(n, 0|\lambda) = 0 \text{ if } n > 0\ ;\ s_r(n, k|\lambda) = 0 \text{ if } n < (r+1)k.$$

We also have

$$s_r(n, 1|\lambda) = (1-\lambda)(2-\lambda)\dots(n-1-\lambda)\,, \qquad \text{if } n > r\,;$$

$$-\lambda s_r(n, 2|\lambda) = (1-2\lambda)(2-2\lambda)\dots$$
$$\dots(n-1-2\lambda) - (1-\lambda)(2-\lambda)\dots(n-1-\lambda) +$$
$$+\lambda\sum_{j=1}^{r}\binom{n}{j}(1-\lambda)\dots(n-j-1-\lambda)(1-\lambda)\dots(j-1-\lambda)\,,$$

if $n > 2r+1$.

Unfortunately, some of the most interesting properties of the degenerate Stirling numbers apparently cannot be extended to the associate numbers. There is the multiplication theorem, for example:

$$S(n, j|\alpha\beta) = \sum_{k=j}^{n}\beta^{n-k}S(n, k|\alpha)S(k, j|\beta)\,, \tag{3.6}$$

where α and β are independent parameters. This theorem can be generalized slightly, however, by considering the polynomials defined by (1.6) with $r = 0$. We show this in the next section.

4. Degenerate Stirling numbers.

Throughout this section we shall use the notations of (3.1) and (3.2). One of our purposes here is to show how formulas for the Stirling numbers given in [5, chapter 5] can be generalized to the degenerate Stirling numbers. We also consider the polynomials defined by (1.6) with $r = 0$, and we use the notation

$$S_n(y|\lambda) = T_{0,n}(y)\,, \qquad \text{where } a_i = (1-\lambda)(1-2\lambda)\dots\big(1-(i-1)\lambda\big)\,; \tag{4.1}$$

$$s_n(y|\lambda) = T_{0,n}(y)\,, \qquad \text{where } a_i = (1-\lambda)(2-\lambda)\dots(i-1-\lambda)\,. \tag{4.2}$$

Thus, by (2.7),

$$S_n(y|\lambda) = \sum_{j=0}^{n} S(n, j|\lambda)y^j\,, \tag{4.3}$$

$$s_n(y|\lambda) = \sum_{j=0}^{n} s(n, j|\lambda)y^j\,. \tag{4.4}$$

We shall show how some of the formulas in [13] can be generalized by means of (4.1) and (4.2). We shall also prove several other properties of the polynomials defined by (4.1) and (4.2).

Carlitz [3] has pointed out that $S(n, k|1) = s(n, k|1) = \delta_{n,k}$. It should also be noted that

$$S(n, k|-1) = s(n, k|-1) = \binom{n-1}{k-1} n!/k! . \tag{4.5}$$

This follows from (3.3) and (3.4), or from the fact that

$$S(n, k|-1) = s(n, k|-1) = B_{n,k}(1!, 2!, 3!, \ldots)$$

[5, p. 135]. A table of values for numbers of the form

$$(-1)^n \binom{n-1}{k-1} n!/k! ,$$

which are called Lah numbers, can be found in [5, p. 156].

It follows from (4.3) and (4.4) that

$$\begin{cases} S_n(y|1) = s_n(y|1) = y^n , \\ S_n(y|-1) = s_n(y|-1) = \sum_{j=0}^{n} y^j \binom{n-1}{j-1} n!/j! . \end{cases} \tag{4.6}$$

In general we have $S_1(y|\lambda) = y$; $S_2(y|\lambda) = y^2 + (1-\lambda)y$; $S_3(y|\lambda) = y^3 + (3-3\lambda)y^2 + (2\lambda^2 - 3\lambda + 1)y$. We can get the corresponding polynomials $s_n(y|\lambda)$ by using

$$s_n(y|\lambda) = (-\lambda)^n S_n(-y\lambda^{-1}|\lambda^{-1}) , \tag{4.7}$$

which follows from (3.5).

Carlitz [3] has proved the multiplication theorem, equation (3.6), and also

$$S(n, k|-\lambda) = \sum_{j=k}^{n} s(n, j) S(j, k) \lambda^{n-j} , \tag{4.8}$$

where $s(n, j)$ is the unsigned Stirling number of the first kind and $S(j, k)$ is the Stirling number of the second kind. If we multiply equations (3.6) and (4.8) by y^j and sum from $j=0$ to $j=n$, we have

$$(4.9)\qquad S_n(y|\alpha\beta) = \sum_{k=0}^{n} \beta^{n-k} S(n, k|\alpha) S_k(y|\beta),$$

$$(4.10)\qquad S_n(y|-\lambda) = \sum_{k=1}^{n} s(n, k) S_k(y|0)\lambda^{n-k}.$$

The formulas in [13] are valid for the special case of (4.3) when $\lambda = 0$ and $y = -1$. All of the formulas can be generalized to the polynomials of (4.3), as we see by (2.7), (2.8), Theorems 2.3, 2.4 and 2.5, and the next theorem.

THEOREM 4.1. The following formulas are valid for $n \geqslant 0$:

$$(4.11)\qquad S_n(y|\lambda) = e^{-y}\sum_{i=0}^{\infty} i(i-\lambda)\dots(i-(n-1)\lambda)y^i/i!,$$

$$(4.12)\qquad yS_n(y|-\lambda) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(1-n\lambda)\dots(1-(j+1)\lambda)S_{j+1}(y|\lambda),$$

$$(4.13)\qquad yS_n(y|\lambda) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(1-\lambda)(1)(1+\lambda)\dots$$
$$\dots(1+(n-j-2)\lambda)S_{j+1}(y|\lambda),$$

$$(4.14)\qquad \sum_{i=0}^{k}\binom{k}{i}S_i(-y|\lambda)S_{n+k-i}(y|\lambda) =$$
$$= \sum_{j=1}^{n}(j-n\lambda)(j-(n+1)\lambda)\dots(j-(n+k-1)\lambda)S(n, j|\lambda)y^j.$$

PROOF. Equation (4.11) follows easily from (1.6) and (4.1). To prove (4.12), let $\lambda\mu = 1$ and differentiate both sides of

$$(4.15)\qquad \exp y[(1-\lambda x)^{-\mu}-1] = \sum_{j=0}^{\infty} S_j(y|-\lambda)x^j/j!$$

with respect to x and then multiply both sides of the resulting equation by $(1-\lambda x)^{n+\mu}$. We now simplify, using (4.6) and (4.8) with $\alpha = -1$, $\beta = \lambda$, and compare coefficients to get equation (4.12). To prove (4.13), we replace λ by $-\lambda$ in (4.15), then differentiate both sides of (4.15) with respect to x, multiply both sides of the resulting equation by $(1+\lambda x)^{1-\mu}$ and compare coefficients. To prove (4.14), we first observe that by (4.15) the left side of (4.14) is equal to the coefficient of $x^n/n!$ in

$$\exp(-y)[(1+\lambda x)^{\mu}-1]D^{(n)}\exp y[(1+\lambda x)^{\mu}-1]\,, \tag{4.16}$$

where $D^{(n)}$ means the n-th derivative with respect to x. Now we can easily prove by induction on n that (4.16) is equal to

$$\sum_{i=1}^{n} S(n, i|\lambda)(1+\lambda x)^{i\mu-n}y^i\,. \tag{4.17}$$

We use the recurrence of Theorem 3.1 with $r=0$ in the induction argument, and we now get (4.14) by comparing coefficients of x in (4.16) and (4.17).

We have already seen in this paper and in [3] how some of the properties of the Stirling numbers carry over to the degenerate Stirling numbers. In the remainder of this paper, we continue to generalize the properties in [5, chapter 5]. In particular, for a fixed n we now look at the numbers $V(j, k|\lambda)$ defined by means of

$$\frac{u^k(1-\lambda u)(1-2\lambda u)\dots(1-(n-1)\lambda u)}{(1-u)(1-2u)\dots(1-ku)}=\sum_{j=0}^{\infty} V(j, k|\lambda)u^j\,. \tag{4.18}$$

THEOREM 4.2. If $V(j, k|\lambda)$ is defined by (4.18) for fixed n, then $V(n, k|\lambda)=S(n, k|\lambda)$. Furthermore

$$S(n, k|\lambda)=\sum_{j=k}^{n} V(j-1, k-1|\lambda)k^{n-j}\,.$$

PROOF. It follows from [5, pp. 207, 214] that

$$V(j, k|\lambda)=\sum_{i=n-j}^{n} s(n, i)S(j-n+i, k)(-\lambda)^{n-i}\,. \tag{4.19}$$

When $j=n$ in (4.19), we have, by (4.8), $V(n, k|\lambda)=S(n, k|\lambda)$. Now call the left side of (4.18) $\varphi(k)$. Then

$$\varphi(k) = u\varphi(k-1)/(1-ku) =$$

$$= \sum_{m=1}^{\infty} k^m u^m \sum_{j=0}^{\infty} V(j, k-1|\lambda) u^j = \sum_{j,\, m\geqslant 0} V(j, k-1|\lambda) k^m u^{j+m} .$$

Comparing coefficients of u, we have

$$V(m, k|\lambda) = \sum_{j=k}^{m} V(j-1, k-1|\lambda) k^{m-j} ,$$

and letting $m = n$, we have the second part of Theorem 4.2.

The numbers $V(j, k|\lambda)$ have properties similar to those of the Stirling numbers. For example, it follows from (4.19) that for fixed n,

$$V(m+1, k|\lambda) = V(m, k-1|\lambda) + kV(m, k|\lambda)$$

with $V(m, k|\lambda) = 0$ if $m < k$; $V(k, k|\lambda) = 1$ for all k; $V(m, k|0) = S(m, k)$; $V(m, 0|\lambda) = (-\lambda)^m s(n, n-m)$. Also from (4.19) we get the formula

$$k!\, V(m, k|\lambda) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^{m-n+1} (j-\lambda) \ldots (j-(n-1)\lambda)$$

for $m \geqslant n$.

To get the corresponding results for $s(n, k|\lambda)$, for fixed n we look at the numbers $V_1(j, k|\lambda)$ defined by means of

$$\text{(4.20)} \qquad \frac{u^k(1+u)(1+2u)\ldots(1+(n-1)u)}{(1+\lambda u)(1+2\lambda u)\ldots(1+k\lambda u)} = \sum_{j=0}^{\infty} V_1(j, k|\lambda) u^j .$$

Theorem 4.3. *If $V_1(j, k|\lambda)$ is defined by (4.20) for fixed n, then $V_1(n, k|\lambda) = s(n, k|\lambda)$. Furthermore,*

$$s(n-1, k|\lambda) = \sum_{j=k}^{n-1} (-1)^{j+k} V_1(n-1-j+k, k|\lambda)(n-1)^{j-k} .$$

Proof. It follows from [5, pp. 207, 214] that

$$\text{(4.21)} \qquad V_1(j, k|\lambda) = \sum_{i=0}^{n} s(n, i) S(j-n+i, k)(-\lambda)^{j+i-n-k} ,$$

and so by (4.8) and (3.5), we have $V_1(n, k|\lambda) = s(n, k|\lambda)$. If we call the left side of (4.20) $\varphi_1(n-1)$, then

$$\varphi_1(n-1) = \varphi_1(n)/(1+nu) =$$

$$= \sum_{m=0}^{\infty}(-nu)^m \sum_{j=0}^{\infty} V_1'(j, k|\lambda)u^j = \sum_{j,m\geqslant 0} V_1'(j, k|\lambda)(-n)^m u^{j+m},$$

where $V_1'(j, k|\lambda)$ is defined by (4.20) with n replaced by $n+1$. Comparing coefficients, we have

$$V_1(m, k|\lambda) = \sum_{j=k}^{m}(-1)^{m-j} V_1'(j, k|\lambda)n^{m-j},$$

and the second part of the theorem follows by replacing n by $n-1$ and letting $m = n-1$.

It follows from (4.21) that for fixed n.

$$V_1(m+1, k|\lambda) = V_1(m, k-1|\lambda) - k\lambda V_1(m, k|\lambda),$$

with $V_1(m, k|\lambda) = 0$ if $m < k$; $V_1(k, k|\lambda) = 1$ for all k;

$$V_1(m, k|0) = s(n, n-m+k); \qquad V_1(m, 0|\lambda) = s(n, n-m).$$

We see from (4.19) and (4.4) that

$$(-\lambda)^{m-k} V_1(m, k|\lambda^{-1}) = V(m, k|\lambda).$$

We close by generalizing two more recurrences to the degenerate Stirling numbers.

THEOREM 4.4. The following recurrences hold:

$$S(n, k|\lambda) = \sum_{j=0}^{n-k}(-1)^j(k+1-n\lambda)\ldots(k+j-n\lambda)S(n+1, k+j+1|\lambda), \tag{4.22}$$

$$s(n+1, k|\lambda) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(\lambda-1)\ldots(\lambda-n+j)s(j, k-1|\lambda). \tag{4.23}$$

PROOF. In the right side of (4.22) replace $S(n+1, k+j+1|\lambda)$ by $S(n, k+j|\lambda) + (k+j+1-n\lambda)S(n, k+j+1|\lambda)$. Then the right side is equal to

$$S(n, k|\lambda) + \sum_{j=0}^{n-k-1}(k+1-n\lambda)\dots(k+j+1-n\lambda)\cdot$$
$$\cdot[(-1)^{j+1} + (-1)^{j}]S(n, k+j+1|\lambda) = S(n, k|\lambda)\,.$$

Equation (4.23) follows from (2.6).

The term « degenerate Stirling number » is probably motivated by the generating functions (3.3) and (3.4). When $G(r, \lambda) = H(r, \lambda) = 0$ in (3.3) and (3.4), the limiting cases $\lambda = 0$ give the generating functions for the Stirling numbers. It should perhaps be noted that this idea has also been applied to some extent to the Bernoulli, Euler and Eulerian numbers [2], [11], [12].

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