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Abelian groups with anti-isomorphic endomorphism rings

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Abelian Groups
with Anti-Isomorphic Endomorphism Rings.

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All groups considered in this paper are abelian. We say that a group $G$ is $E$-dual if there exists a group $H$ such that the endomorphism rings $E(G)$ and $E(H)$ are anti-isomorphic; $G$ is said to be $E$-self-dual if $E(G)$ has an anti-automorphism. In this note we investigate some properties of $E$-dual and $E$-self-dual groups. In section 1, we examine some closure properties of the classes of $E$-dual and $E$-self-dual groups. In fact, we prove that direct summands of $E$-self-dual groups are not necessarily $E$-self-dual, and direct sums of $E$-self-dual groups are not necessarily $E$-dual. In section 2, we show that a torsion group $G$ is $E$-dual if and only if, for every prime $p$, its $p$-component $t_p(G)$ is either a $p$-group of finite rank or a torsion-complete $p$-group with finite Ulm invariants. In section 3, we describe some classes of $E$-dual cotorsion groups. As we shall see, a reduced cotorsion group $G$ is $E$-dual if and only if, for every prime $p$, the $p$-adic component of $G$ is either a $J_p$-module of finite rank or the $p$-adic completion of an $E$-dual reduced $p$-group. We also prove that a divisible group $G$ is $E$-dual if and only if $G$ is either a torsion $E$-dual group or a torsion-free group of finite rank. In section 4, we show that plenty of reduced torsion-free groups are $E$-dual. In fact, every controlled group $G$ such that $E(G)$ is of cardinality $< \aleph_1$, the first strongly inaccessible cardinal, is an $E$-dual group. In the torsion-free case some pathologies of the class of $E$-dual groups appear. For instance, by Corner’s realization

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theorems, completely different reduced torsion-free groups have anti-
isomorphic endomorphism rings. Finally, we remark that there exist
torsion, mixed and torsion-free $E$-dual groups which are not $E$-self-dual.

For all unexplained terminology and notation we refer to ([5]); in
particular $\mathbb{N}$ is the set of natural numbers, $P$ the set of prime
numbers; $\mathbb{Z}, \mathbb{Q}, J_p$ are respectively the groups (or rings) of integers, rational
numbers, $p$-adic integers; $\mathbb{Z}(p)$ is the group (or field) with $p$
elements. If $G$ is a group and $g \in G$, then $o(g)$ is the order of $g$ and,
if $G$ is a $p$-group, $o(g) = p^{e(g)}$, where $e(g)$ is the exponent of $g$. If $G'$
is a pure subgroup of $G$, we write $G' \triangleleft G$. If $G$ is torsion-free and $S$
is a subset of $G$, then $\langle S \rangle_s$ is the pure subgroup of $G$ generated by $S$.
For every set $X$, $G^X$ is the group of all functions from $X$ to $G$ with
finite support. If $R$ is a ring, then $R^a$ is its opposite ring and, for every
$n \in \mathbb{N}$, $M_n(R)$ is the ring of all $n \times n$ matrices with entries in $R$. For
every $p$-group $G$ and every ordinal $\sigma$, $f_\sigma(G)$ is the $\sigma$-th Ulm invariant
of $G$. When we shall say that $B = \bigoplus_{n \in \mathbb{N}} B_n$ is a basic subgroup of the
$p$-group $G$, we always adopt the convention that $B_n$ is a direct sum
of cyclic groups of order $p^n$. If $G$ is a reduced cotorsion group, then
we write $G = \prod_{p \in P} G_p$, where each $G_p$ is the $p$-adic component of $G$.

§ 1. Let $G$ and $H$ be groups and assume there is an anti-
ismorphism between $E(G)$ and $E(H)$. Since idempotents of $E(G)$ are
mapped onto idempotents of $E(H)$, the following lemma is obvious.

**Lemma 1.1.** Direct summands of $E$-dual groups are $E$-dual.

The situation is different in the class of $E$-self-dual groups.

**Lemma 1.2.** Direct summands of $E$-self-dual groups are not neces-
sarily $E$-self-dual.

**Proof.** We shall prove that if $G = \bigoplus_{i=1}^{4} G_i$, where $G_1 = G_2 = \prod_{p \in P} \mathbb{Z}(p)$;
$G_3 = G_4 = \bigoplus_{p \in P} \mathbb{Z}(p)$, then $G$ is $E$-self-dual, but there exists a direct sum-
mand of $G$ which is not $E$-self-dual. In the following $\Pi$ denotes the
group (or ring) $\prod_{p \in P} \mathbb{Z}(p)$ and $\Sigma$ denotes the group $\bigoplus_{p \in P} \mathbb{Z}(p)$. Thus
$\text{Hom}(\Sigma, \Pi) \cong \Pi$; $\text{Hom}(\Pi, \Sigma) \cong \Sigma$; $E(G_i) \cong \Pi$ ($1 < i < 4$). Let $A$ and $A^*$ be the following subrings of $M_4(\Pi)$:

$A = \{a = [a_{ij}] \in M_4(\Pi): a_{ij} \in \Sigma; 3 \leq i \leq 4; 1 \leq j \leq 2\}$

$A^* = \{a = [a_{ij}] \in M_4(\Pi): a_{ij} \in \Sigma; 1 \leq i \leq 2; 3 \leq j \leq 4\}$. 

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Then, by ([5] Theorem 106.1), $A$ is isomorphic to $E(G)$ and, using the decomposition $G = G_1 \oplus G_2 \oplus G_3 \oplus G_4$, the same applies to $A^*$. Since the transposition of $M_4(\Pi)$ induces an anti-isomorphism between $A$ and $A^*$, we conclude that $G$ is $E$-self-dual. To complete the proof, we now show that $G' = \bigoplus_{i=1}^{3} G_i$ is not an $E$-self-dual group. Otherwise, suppose $E(G')$ has an anti-automorphism which takes $e_i$, the projection of $G'$ onto $G_i$, to a suitable $e_i \in E(G')$ $(1 < i < 3)$. Then $G' = \bigoplus_{i=1}^{3} H_i$, where $H_i = e_i(G')$ for every $i$. Evidently $E(H_i) \cong \Pi$ and $\bigcap_{p \in P} H_i = \emptyset$; hence, by ([15] Theorem 2), $\Sigma H_i \cong \Pi$ $(1 < i < 3)$. Also note that $\text{Hom}(H_i, H_i) \cong \text{Hom}(G_i, G_i) \cong \Sigma (1 < i < 2)$. For every prime $p$, let $1_p$ be the unit of $Z(p)$ and let $x = (1_p)_{p \in P} \in G_1$; $y = (1_p)_{p \in P} \in G_2$. On the other hand $x = \sum_{i=1}^{3} x_i$; $y = \sum_{i=1}^{3} y_i$ for some $x_i, y_i \in H_i$ $(1 < i < 3)$. To find a contradiction, we first prove that $x_i, y_i \in \Sigma (1 < i < 2)$. For instance, we show that $x_1 \in \Sigma$. Since $e_1(ax) = ae_1(x) = ax_1 \in H_1 (a \in G_1 = \Pi)$ and $H_3 \cong \Pi$, there exists a homomorphism $p: H_3 \rightarrow H_1$ such that $q(z) = zx_1$ for every $z \in H_3$; therefore $nq = 0$ for some $n \in \mathbb{N}$. Since $nx_1 \Sigma = nq(\Sigma) < nq(H_3) = \emptyset$, we must have $x_1 \in \Sigma$, as claimed. Consequently $P^* = \{ p \in P | t_p(G_1) \oplus t_p(G_2) \leq \ker (e_1 + e_3) \}$ contains all but finitely many primes. Fix $p \in P^*$ and let $j: \bigoplus_{i=1}^{3} G_i \rightarrow \bigoplus_{i=1}^{3} H_i$ denote the identity map of $G'$. Then the choice of $p$ implies $j(t_p(G_1) \oplus t_p(G_2)) \leq H_2$; on the other hand $t_p(G_1), t_p(G_2)$ and $t_p(H_3)$ are all isomorphic to $Z(p)$. This contradiction establishes that $G'$ is not $E$-self-dual, and the lemma follows. Another application of ([5] Theorem 106.1) shows that $E(G')$ is anti-isomorphic to $E(G')$, where $G' = G_2 \oplus G_3 \oplus G_4$. Hence $G'$ is another direct summand of $G$ which is not $E$-self-dual.

**Lemma 1.3.** Finite direct sums of $E$-self-dual groups are not necessarily $E$-dual.

**Proof.** It is enough to observe that $Q, Z(p^\infty), Z$ are clearly $E$-self-dual; however, as we shall see in sections 3 and 4, the groups $Q \oplus Z(p^\infty), Q \oplus Z, Z(p^\infty) \oplus Z$ are not $E$-dual.

Before classifying all $E$-dual and $E$-self-dual torsion groups by means of a suitable realization of their endomorphism ring, we summarize the results previously obtained about this kind of problem.
Liebert has shown ([10] Lemma A) that the endomorphism ring of a finite $p$-group has an anti-automorphism. By a result of Faltings ([4] Lemma 2.10), the same property holds for every torsion-complete $p$-group with finite Ulm invariants. A new theorem of Liebert ([12] Theorem 8.1) states that if $G$ is a torsion-complete $p$-group, then $E(G)$ has an anti-automorphism if and only if $G$ has finite Ulm invariants.

§ 2. In the first part of this section we prove that if $G$ is an $E$-dual reduced $p$-group, then $G$ must be a torsion-complete $E$-self-dual $p$-group. We begin with two lemmas.

**Lemma 2.1.** Let $G$ be a reduced $p$-group. If $G$ is $E$-dual, then $\sigma(G)$ is finite, for every $\sigma < \omega$.

**Proof.** Let $G$ be as in the hypotheses and assume $B = \bigoplus_{n \in \mathbb{N}} B_n$ is basic in $G$. We now prove that $B_1$ is finite. By 1.1, there exists a group $H$ and an anti-isomorphism $f : E(B_1) \to E(H)$. Since $pH = 0$, an application of ([1] General Existence Theorem, p. 193) shows that $B_1$ is finite. An elementary proof of this fact is the following. Assume $|B_1| > \aleph_0$. To find a contradiction, it is enough to prove that $B_1$ cannot be of cardinality $\aleph_0$. Suppose the contrary. Then $H$ is not finite and $E(B_1)$ has only one proper two-sided ideal consisting of all endomorphisms of finite rank ([8], Chapter 4; also see [1], p. 198). Since the endomorphism ring of an uncountable vector space has at least two proper two-sided ideals, i.e. the ideals of all endomorphisms of finite or countable rank, we conclude that $|H| = \aleph_0$. Let $\pi$ be a minimal idempotent of $E(B_1)$ and let $\pi' = f(\pi)$. Then $|E(B_1)\pi'| = |\text{Hom}(\pi(B_1), B_1)| = \aleph_0$, while $|\pi' E(H)| = |\text{Hom}(H, \pi'(H))| = 2^{\aleph_0}$. This contradiction establishes that $B_1$ is finite. To complete the proof, it is enough to check that $B_n$ is finite ($n \in \mathbb{N}$). Fix $n \in \mathbb{N}$, and let $f : E(B_{n+1}) \to E(H)$ be an anti-isomorphism. Remark that $H$ is a direct sum of cyclic groups of order $p^{n+1}$, because $p^{n+1}H = 0$ and $E(H)$ has no idempotent of order $< p^{n+1}$. Let $\sigma_1 : E(B_{n+1}) \to E(B_{n+1})/p^n E(B_{n+1})$ and $\sigma_2 : E(H) \to E(H)/p^n E(H)$ be the natural homomorphisms. Since $\ker \sigma_1 = \text{Ker} \sigma_2 f$ and $f$ is an anti-isomorphism, there exists an anti-isomorphism $\tilde{f} : \sigma_1(E(B_{n+1})) \to \sigma_2(E(H))$ such that $\tilde{f} \sigma_1 = \sigma_2 f$. Evidently $\sigma_1(E(B_{n+1})) \cong E(p^n B_{n+1})$ and $\sigma_2(E(H)) \cong E(p^n H)$. Therefore $B_{n+1}$ is finite, and the lemma follows. □

**Lemma 2.2.** Let $G$ be a reduced $p$-group. If $G$ is $E$-dual, then $p^n G = 0$. 


PROOF. As before, let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be a basic subgroup of $G$. Let $\pi_n$ denote the projection of $G$ onto $B_n$ with $\text{Ker} \, \pi_n = \bigoplus_{m \neq n} B_m + p^n G$ ($n \in \mathbb{N}$). Assume $G$ is $E$-dual; then, there is an anti-isomorphism $f: E(G) \to E(H)$ for some $H$. To see that $G$ is separable, we shall use the following properties of $H$:

1. $t(H) = t_\pi(H)$. Let $q \in P$; $q \neq p$. Since $G$ has no summand whose endomorphism ring is isomorphic to $J_q$ or to $Z(p^n)$ for some $n \in \mathbb{N}$, the same applies to $H$. Therefore $t_q(H) = 0$.

2. $H$ is reduced. By the previous remark, it is enough to observe that $Q$ and $Z(p^\infty)$ cannot be subgroups of $H$.

3. $H$ is a $J_\pi$-module. This follows from the fact that the center of $E(H)$ is isomorphic to the center of $E(G)$, and the center of $E(G)$ is isomorphic to $J_\pi$ or to $Z(p^n)$ for some $n \in \mathbb{N}$ ([5] Theorem 108.3).

4. If $\pi'_n = f(\pi_n)$ and $\pi'_n(H) = B'_n$ ($n \in \mathbb{N}$), then $B' = \bigoplus_{n \in \mathbb{N}} B'_n$ is a basic subgroup of $t_\pi(H)$. Since $B'$ is a direct sum of cyclic groups and $B' < t_\pi(H)$, there is a basic subgroup $B'' = \bigoplus_{n \in \mathbb{N}} B''_n$ such that $B' < B''$.

Our claim is that $B' = B''$. Assume this is not true. Choose $m \in \mathbb{N}$ such that $B'_m < B''_m$. Let $\eta'$ be a projection of $H$ onto $B''_m$, and let $\eta' = f(\eta)$. Therefore $\eta(G)$ is a direct sum of cyclic groups of order $p^m$ and clearly $B_m < \eta(G)$. This contradiction proves that $B' = B''$.

5. $p^\infty t_\pi(H) = 0$. To see this, suppose the contrary. Then, there is $\varphi' \in E(G)$ such that $0 \neq f(\varphi') = \varphi' \in E(H)[p]$ and $\varphi'(H) < p^\infty t_\pi(H)$. Since $\pi'_n \varphi' = 0$, we get $\varphi \pi_n = 0$ ($n \in \mathbb{N}$). It follows that $\varphi = 0$, and this contradicts the hypothesis that $\varphi' \neq 0$; consequently $p^\infty t_\pi(H) = 0$.

The last remark tells us that, if $H$ is a $p$-group, $G$ is separable. To end the proof, assume that $p^\infty G \neq 0$. Then there is a suitable $\varphi \in E(G)[p]$ such that $\varphi 
eq 0$; $\pi_n \varphi = 0$ ($n \in \mathbb{N}$). Let $\varphi' = f(\varphi)$; since $\varphi' \pi'_n = 0$ ($n \in \mathbb{N}$), $\varphi'$ is 0 on $B'$. Using (1), (2) and (4), we conclude that $\varphi'(t(H)) = 0$, and therefore $\text{Hom}(H/t(H), H[p]) \neq 0$. This implies that $H/t(H)$ is not $p$-divisible; hence, by (3), there exists $x \in H$, $x \notin t(H)$ such that $J_\pi x$ is a direct summand of $H$. But this is impossible, because $G$ has no summand isomorphic to $J_\pi$ or $Z(p^\infty)$. This contradiction proves that $p^\infty G = 0$. □

REMARK. In $G$ is an infinite reduced $E$-dual $p$-group, $|G| = 2^{\aleph_0}$. In fact, with the notations of 2.2, $|G| \leq \prod_{n \in \mathbb{N}} B_n = 2^{\aleph_0}$. To prove the
reverse inequality, take \( m \in \mathbb{N} \) such that \( B_m \neq 0 \). Since \( |H/p^mH| > |B'/p_d^mB'| = \mathbb{N}_0 \), we clearly have

\[
|G| > |G[p^m]| = |\text{Hom}(B_m, G)| = |E(G)\tau_m| = |\tau_mE(H)| = \\
= |\text{Hom}(H, B'_m)| = |\text{Hom}(H/p^mH, B'_m)| > 2^{\mathbb{N}_0}.
\]

Following ([12], p. 350), we say that a \( p \)-group \( G \) is torsion-compact if \( G \) is torsion-complete and every Ulm invariant of \( G \) is finite. We now give a realization of the endomorphism ring of a torsion-compact \( p \)-group \( G \). If \( G \) is finite, an application of ([5] Theorem 106.1) shows that there exist \( r, n \in \mathbb{N} \) such that \( E(G) \) is isomorphic to a subring of \( M_r(\mathbb{Z}(p^n)) \) fully invariant under the transposition of \( M_r(\mathbb{Z}(p^n)) \). This is an elementary proof of a result ([10] Lemma A) mentioned in section 1. Suppose now that \( G \) is not finite. Fix a basic subgroup \( B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle \) of \( G \) such that \( o(x_r) < o(x_n) \) if \( r < s \). Regarding \( G \) as embedded in \( \prod_{n \in \mathbb{N}} \langle x_n \rangle \), let \( \pi_n \) denote the projection of \( G \) onto \( \langle x_n \rangle \) \((n \in \mathbb{N})\);

for every \( x \in G \) we may write \( x = (\alpha_n x_n)_{n \in \mathbb{N}} \), where \( \alpha_n x_n = \pi_n(x) \) and \( \alpha_n \) is a suitable \( p \)-adic integer \((n \in \mathbb{N})\). Let \( h \) be the \( p \)-adic valuation of \( J_p \) and let \( A \) be additive group of all \( \mathbb{N}_0 \times \mathbb{N}_0 \) matrices over \( J_p \) of the form \( a = [\alpha_{rs}] \), where \( h(\alpha_{rs}) > \lambda_{rs} = \max (0, e(x_r) - e(x_s)) \) for every \( r, s \in \mathbb{N} \). Then \( A \) is a ring with the usual rows by columns product and the subset \( I \) of all \( a = [\alpha_{rs}] \in A \) such that \( h(\alpha_{rs}) > e(x_r) \) \((r, s \in \mathbb{N})\) is a two-sided ideal of \( A \). Remark that every \( \varphi \in E(G) \) is completely determined by the elements \( \{\varphi(x_n) = (\alpha_{rs} x_{rs})_{r \in \mathbb{N}} \} \). With these notations, let \( \varphi : E(G) \rightarrow A/I \) be the map defined by \( \varphi(\varphi) = [\alpha_{rs}] + I \) for all \( \varphi \in E(G) \). Evidently \( \varphi \) is a group isomorphism. We claim that \( \varphi \) is a ring isomorphism. To see this, choose \( \varphi, \psi \in E(G) \); then there are suitable \( \alpha_{rs}, \beta_{rs}, \delta_{rs} \in J_p \) \((r, s \in \mathbb{N})\) such that \( \varphi(\varphi(\psi)) = [\alpha_{rs}] + I \); \( \varphi(\psi) = [\beta_{rs}] + I \); \( \varphi(\psi \phi) = [\delta_{rs}] + I \). Fix \( r, s \in \mathbb{N} \); then there exists some \( k \in \mathbb{N} \) such that \( e(x_r) > e(x_n) + e(x_s) \). Consequently \( \varphi(x_n) = \sum_{i=1}^{k} \alpha_{is} x_i \mod p^{e(x_r)} \).

Since

\[
\delta_{rs} x_r = \pi_r(\psi \varphi(x_s)) = \pi_r \left( \sum_{i=1}^{k} \alpha_{is} \psi(x_i) \right) = \left( \sum_{i=1}^{k} \beta_{ri} \alpha_{is} \right) x_r,
\]

we obtain \( \delta_{rs} \equiv \sum_{i=1}^{k} \beta_{ri} \alpha_{is} \mod p^{e(x_r)} J_p \). Using the hypothesis that \( h(\alpha_{rs}) > e(x_r) \) for all \( n > k \), we conclude that \( \delta_{rs} \equiv \sum_{n \in \mathbb{N}} \beta_{rn} \alpha_{ns} \mod p^{e(x_r)} J_p \).
This proves that $\varphi$ is a ring isomorphism, because $\varphi, \psi \in E(G)$ and $r, s \in \mathbb{N}$ are arbitrary elements.

The following theorem characterizes all $E$-dual reduced $p$-groups.

**Theorem 2.3.** Let $G$ be a reduced $p$-group. The following are equivalent:

1. $G$ is torsion-compact.
2. $G$ is $E$-self-dual.
3. $G$ is $E$-dual.

**Proof** (1) $\Rightarrow$ (2). As already observed, finite $p$-groups are $E$-self-dual. Assume $G$ is not finite and fix a basic subgroup $B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle$ of $G$ such that $e(x_r) \leq e(x_s)$ if $r < s$ ($r, s \in \mathbb{N}$). Let $A$ be the ring of all matrices $a = [\alpha_{rs}]$, where $\alpha_{rs} = p^{k_{rs}} \gamma_{rs}$ for some $\gamma_{rs} \in J_p$ and $\lambda_{rs} = \max \{0, e(x_r) - e(x_s)\}$ ($r, s \in \mathbb{N}$). Evidently the map $t: A \rightarrow A$ such that $\lambda_a = \{p^{k_{rs}} \gamma_{rs}\}$ ($a \in A$) is a group isomorphism. To show that $t$ is a ring anti-automorphism, pick $a, a' \in A$. Let $a = [p^{k_{rs}} \gamma_{rs}]$; $a' = [p^{k_{rs}} \gamma'_{rs}]$; $a' a = [\beta_{rs}]$ and $t(a') t(a) = [\delta_{rs}]$. For every $r, s \in \mathbb{N}$ we may write $eta_{rs} = \sum_{n \in \mathbb{N}} p^{\sigma_{rn}} \gamma_{rn} \gamma'_{ns}$; $\delta_{rs} = \sum_{n \in \mathbb{N}} p^{\tau_{rn}} \gamma'_{rn} \gamma'_{ns}$ where $\sigma_n = \lambda_{rn} + \lambda_{ns}$ and $\tau_n = \lambda_{rn} + \lambda_{ns}$ ($n \in \mathbb{N}$). Let $r < s$; it is easy to check that

$$
\sigma_n = \begin{cases} 
0 & \text{if } n = r \text{ or } n = s \\
\frac{e(x_r) - e(x_s)}{e(x_r) - e(x_s)} & \text{otherwise}
\end{cases} \quad \text{and} \quad 
\tau_n = \begin{cases} 
\frac{e(x_r) - e(x_s)}{e(x_r) - e(x_s)} & \text{if } n = r \text{ or } n = s \\
0 & \text{otherwise}
\end{cases}
$$

Since $\lambda_{sr} = e(x_r) - e(x_s)$, we clearly have $\tau_n = \sigma_n + \lambda_{sr}$ ($n \in \mathbb{N}$), and therefore $\delta_{sr} = p^{k_{sr}} \beta_{rs}$. Let $r > s$; this hypothesis implies

$$
\sigma_n = \begin{cases} 
\frac{e(x_r) - e(x_s)}{e(x_r) - e(x_s)} & \text{if } n = r \text{ or } n = s \\
\frac{e(x_r) - e(x_s)}{e(x_r) - e(x_s)} & \text{otherwise}
\end{cases} \quad \text{and} \quad 
\tau_n = \begin{cases} 
0 & \text{if } n = r \text{ or } n = s \\
\frac{e(x_r) - e(x_s)}{e(x_r) - e(x_s)} & \text{otherwise}
\end{cases}
$$

Since $\lambda_{sr} = e(x_r) - e(x_s)$, we get $\tau_n = \sigma_n + \lambda_{rs}$ ($n \in \mathbb{N}$); thus $\delta_{sr} = p^{-k_{rs}} \beta_{rs}$. Consequently $t(\beta_{rs}) = [\delta_{rs}]$, and $t$ is a ring anti-automorphism, as required. Let $r, s \in \mathbb{N}$ and $\gamma_{rs} \in J_p$; then $h(p^{k_{rs}} \gamma_{rs}) = e(x_r)$ if and only
if $h(p^\lambda r, \gamma, r_s) > e(x_s)$. Hence $t$ induces an anti-automorphism of $A/I$, that we still call $t$. Since $A/I$ is isomorphic to $E(G)$, $G$ is $E$-self-dual.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Let $G$ be an $E$-dual $p$-group. If $G$ is finite, $G$ is clearly torsion-compact. If $G$ is not finite, Lemmas 2.1 and 2.2 enable us to assume that $G < \tilde{B} = t\left(\prod_{n \in \mathbb{N}} \langle x_n \rangle\right)$, where $B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle$ is a basic subgroup of $G$ and $e(x_r) < e(x_s)$ if $r < s$ ($r, s \in \mathbb{N}$). To prove that $\tilde{B} < G$, we introduce some endomorphisms of $G$ similar to those used in ([9] Theorem 28). Let $\pi_n$ be the projection of $G$ onto $\langle x_n \rangle$ ($n \in \mathbb{N}$). Then, for all $r, s \in \mathbb{N}$ we define $e_{rs} \in E(G)$ as follows: $e_{rs}(1 - \pi_s) = 0$ and $e_{rs}(x_s) = p^{\lambda_{rs}} x_r$, where $\lambda_{rs} = \max\{0, e(x_r) - e(x_s)\}$. Since $G$ is $E$-dual, there exists an anti-isomorphism $f : E(G) \rightarrow E(H)$ for some $H$. Write $f(e_{rs}) = e'_{sr} (r, s \in \mathbb{N})$ and choose $y_n \in H$ such that $e_n(H) = \langle y_n \rangle$ ($n \in \mathbb{N}$). For every $r, s \in \mathbb{N}$, let $e_{rs}$ be the endomorphism of $H$ uniquely determined by the following conditions $e_{rs}(y_n) = p^{\lambda_{rs}} y_r$, $e_{rs}(1 - e'_{ss}) = 0$. Remark that $e_{rs} = e_{sr} e_{ss}$ implies $e_{ss}' = e_{ss}' e_{ss}'$; therefore $e_{rs}' = u_{sr} e_{sr}$ for some $u_{sr} \in J_p / pJ_p$ ($r, s \in \mathbb{N}$). Assume $G < \tilde{B}$ and choose $x \in \tilde{B} \setminus G$. Then there exist $\alpha_n \in J_p$ ($n \in \mathbb{N}$) and $m \in \mathbb{N}$ such that $x = (\alpha_n x_n)_{n \in \mathbb{N}}$ and $o(x_n) > o(x)$ for all $n > m$. Let $\alpha_n = 0$ if $n < m$, and let $\alpha_n = \alpha_m$ if $n > m$. We claim that $x^* = (\alpha_n^* x_n^*)_{n \in \mathbb{N}} \in G$. To see this, let $\varphi$ denote an endomorphism of $H$ with the following properties: $\varphi' = e_{mm} \varphi'$, $\varphi'(y_n) = \alpha_n^* u_{mn} y_m$ ($n \in \mathbb{N}$). Now consider the endomorphism $\varphi$ of $G$ such that $f(\varphi) = \varphi'$. By hypothesis, we clearly have $e_{mm} \varphi' \varphi' = \alpha_n^* u_{mn} e_{mn} = (\alpha_n^* u_{mn}) (u_{mm}^{-1} \varphi'_{mn}) = \alpha_n^* e_{mn}^*$ ($n \in \mathbb{N}$). It follows that $\varphi(x_n) = (\alpha_n^* x_n^*)_{n \in \mathbb{N}} = x^* \in G$. Since $x \in x^* + B$, we obtain $x \in G$. This contradiction shows that $G = \tilde{B}$ and the proof is complete. \hfill \Box

Remark. Let $G$ be an infinite torsion-compact $p$-group. With the previous notations, the anti-automorphism $t$ of $E(G)$ defined in the first part of the proof has the property that $e_{rs}' = e_{rs}$; thus we may assume $u_{rs} = 1$ for every $r, s \in \mathbb{N}$. This follows from the fact that $t$ is induced by the most obvious transposition of the matrix ring $A$.

A remark of ([12], p. 352) states that if $G$ is a $p$-group, then $G$ is $E$-self-dual if and only if $G$ is either a torsion-compact group or a divisible group of finite rank. As we shall see, if a divisible $p$-group is $E$-dual then it is $E$-self-dual, but there exist $E$-dual $p$-groups which are not $E$-self-dual.
THEOREM 2.4. Let $G$ be a divisible $p$-group. Then $G$ is $E$-dual if and only if it is of finite rank.

PROOF. If $G$ is a divisible $p$-group of rank $n$, then $E(G) \cong M_n(J_p)$ clearly has an anti-automorphism. By 1.1, to prove the theorem, it is enough to show that a divisible $p$-group of rank $\aleph_0$ cannot be $E$-dual. Assume this is not true. Write $G = \bigoplus_{n \in \mathbb{N}} G_n$, where $G_n \cong \mathbb{Z}(p^n)$ for every $n \in \mathbb{N}$, and choose an anti-isomorphism $f: E(G) \to E(H)$ for some $H$. Let $\pi_n$ be the projection of $G$ onto $G_n (n \in \mathbb{N})$, and let $\varphi' = f(\varphi) (\varphi \in E(G))$. First note that the groups $\pi_n(H) (n \in \mathbb{N})$ are all isomorphic. Otherwise, there exist $m, n \in \mathbb{N}$ such that $\pi_m(H) \cong \mathbb{Z}(p^\infty)$ and $\pi_n(H) \cong J_p$. But this is impossible, because $\pi_m(E(H)\pi_m = 0$, while $\pi_m E(G)\pi_n \neq 0$ (compare with ([13] Lemma 1.2)). Remark now the following properties of $H$:

(1) $H$ is not a divisible $p$-group. Suppose (1) does not hold. Let $\sigma_1: E(G) \to E(G)/pE(G); \sigma_2: E(H) \to E(H)/pE(H)$ be the natural homomorphisms. Then $\sigma_1(E(G)), \sigma_2(E(H))$ are isomorphic to the endomorphism rings of two infinite vector spaces over $\mathbb{Z}(p)$, and this is a contradiction. In fact, the existence of $f$ implies that $\sigma_1(E(G))$ is anti-isomorphic to $\sigma_2(E(H))$. Hence (1) is true.

(2) $H$ is torsion-free. Since $G$ has no finite summand, the same holds for $H$. Consequently $t_q(H) = 0$ for every prime $q \neq p$, because $G$ cannot have $J_q$ or $\mathbb{Z}(q^\infty)$ for a summand. It remains to check that $t_p(H) = 0$. Assume the contrary. Then $H$ is a mixed group and $t(H)$ is a divisible $p$-group. Thus $t(H)$ is a proper fully invariant direct summand of $H$. On the other hand $G$ has no proper fully invariant direct summand. This contradiction proves that $H$ is torsion-free.

(3) $H$ is a reduced $J_p$-module. Since $H$ is torsion-free and $G$ is a $p$-group, it suffices to repeat the proof of (2) and (3) of Lemma 2.2.

It is now clear that we may assume $\pi_1(H) = J_p$. As an immediate consequence $J_p^N \cong \text{Hom}(G, G_1) \cong \text{Hom}(\pi_1(H), H) \cong \text{Hom}_{J_p}(\pi_1(H), H)$. By (3), the map which takes $\varphi'$ to $\varphi'(1)$ for all $\varphi' \in \text{Hom}(\pi_1(H), H)$ is an isomorphism between $\text{Hom}(\pi_1(H), H)$ and $H$; therefore $H \cong J_p^N$. Since $H$ properly contains the $p$-adic completion of $\bigoplus_{n \in \mathbb{N}} \pi_n(H)$, the group $\overline{H} = H/\bigoplus_{n \in \mathbb{N}} \pi_n(H)$ is not $p$-divisible. This implies that $\overline{H}$ has a pure subgroup, hence a direct summand, isomorphic to $J_p$. Since $\text{Hom}(\overline{H}, H) \neq 0$, there exists a non-zero endomorphism $\varphi'$ of $H$ such...
that \( \varphi' \left( \bigoplus_{n \in \mathbb{N}} \pi_n(H) \right) = 0 \). But this means that if \( \varphi \in E(G) \) and \( f(\varphi) = \varphi' \), then \( \varphi \neq 0 \), while \( \pi_n \varphi = 0 \ (n \in \mathbb{N}) \). This contradiction establishes that divisible \( p \)-groups of infinite rank are not \( E \)-dual. \( \square \)

**Remark.** Let \( G \) be a divisible \( p \)-group of infinite rank \( m \). Then, as observed in ([5], vol. II, p. 220) \( E(G) \) is isomorphic to the ring of all column-convergent \( m \times m \) matrices with entries in \( J_p \) (i.e. in every column almost all entries are divisible by \( p^n \) for any \( n \in \mathbb{N} \)). In this case the asymmetry between rows and columns cannot be removed by means of a suitable transposition, as in the case of infinite torsion-compact \( p \)-groups.

**Theorem 2.5.** Let \( G \) be a \( p \)-group. Then \( G \) is \( E \)-dual if and only if either \( G \) is torsion-compact or \( G \) is of finite rank.

**Proof.** By the previous results, we assume \( G \) is neither reduced not divisible. Suppose first that \( G \) has finite rank. Then \( G \) has a decomposition \( G = \bigoplus_{i=1}^{r} G_i \) with the following properties: \( G_i \) is a cyclic group, if \( 1 \leq i < n \), and \( G_i \cong \mathbb{Z}(p^\infty) \), if \( n + 1 \leq i < r \). Using ([5], Theorem 106.1), we identify \( E(G) \) with the ring of all \( r \times r \) matrices \( [\alpha_{ij}] \), where \( \alpha_{ij} \in \text{Hom}(G_i, G_i) \). Define \( H \) to be the group \( H = \bigoplus_{i=1}^{r} H_i \), where \( H_i = G_i \) if \( 1 \leq i < n \), and \( H_i = J_p \) if \( n + 1 \leq i < r \). Another application of ([5], Theorem 106.1) shows that \( E(H) \) is isomorphic to the ring of all \( r \times r \) matrices \( [\alpha_{ij}] \), where \( \alpha_{ij} \in \text{Hom}(H_i, H_i) \). Identifying the groups \( \text{Hom}(G_i, G_i) \) and \( \text{Hom}(H_i, H_i) \) \( (1 \leq i, j < n) \), let \( t: E(G) \rightarrow E(H) \) be the map that sends \( a = [\alpha_{ij}] \) to \( t'a = [\alpha_{ij}] \) for all \( a \in E(G) \). Since \( t' \) is a ring anti-isomorphism, \( G \) is \( E \)-dual. Conversely, let \( G = D \oplus R \) be an \( E \)-dual \( p \)-group, where \( D \cong (\mathbb{Z}(p^\infty))^r \) for some \( r \in \mathbb{N} \) and \( R \) is reduced. Our claim is that \( R \) is finite. Suppose \( R \) is not finite and fix an anti-isomorphism \( f: E(G) \rightarrow E(H) \). Let \( \pi_1, \pi_2 \) be the projections of \( G \) onto \( D \) and \( R \) respectively, and let \( \pi'_i = f(\pi_i), \ H_i = \pi_i(H) \) \( (1 < i < 2) \). Then \( H = H_1 \oplus H_2 \). Since \( \pi'_1 E(H) \pi'_2 = 0 \), \( H_1 \) is isomorphic to \( J_p^r \). The proof of Lemma 2.2 enables us to regard \( H_2 \) as embedded in a group of the form \( \prod_{n \in \mathbb{N}} B_n \), where \( B' = \bigoplus_{n \in \mathbb{N}} B_n \) is isomorphic to a basic subgroup \( B = \bigoplus_{n \in \mathbb{N}} B_n \) of \( R \). Since \( |R/B| = 2^{\aleph_0} \), we get \( |\text{Hom}(R, D)| = |\text{Hom}(B, \mathbb{Z}(p^\infty)) \bigoplus J_p^{2\aleph_0}| > 2^{\aleph_0} \) ([5], Theorem 47.1). This contradicts the fact that \( |\text{Hom}(H_1, H_2)| = \prod_{n \in \mathbb{N}} |\text{Hom}(J_p^r, B_n)| < 2^{\aleph_0} \). Therefore \( R \) is finite, and the proof is complete. \( \square \)
COROLLARY 2.6. Let G be a torsion group. Then G is E-dual if and only if \( t_p(G) \) is E-dual for every prime \( p \).

Proof. Necessity follows from 1.1. Assume that \( t_p(G) \) is E-dual \((p \in P)\). Let \( H_p = t_p(G) \) if \( t_p(G) \) is reduced, and let \( H_p = J_p \oplus R \) if \( t_p(G) \cong (\mathbb{Z}(p^\infty))^r \oplus R \), where \( r \in \mathbb{N} \) and \( R \) is reduced. If \( H = \bigoplus_{p \in P} H_p \), then \( E(H) \cong \prod_{p \in P} E(H_p) \). Since \( E(t_p(G)) \) is anti-isomorphic to \( E(H_p) \) for all \( p \in P \), \( E(G) \) is anti-isomorphic to \( E(H) \). Thus \( G \) is E-dual. \( \Box \)

COROLLARY 2.7. If \( G \) and \( H \) are torsion groups with anti-isomorphic endomorphism rings, the following conditions hold for every prime \( p \):

(i) \( t_p(G) \) is either reduced or divisible.

(ii) \( t_p(G) \) is isomorphic to \( t_p(H) \).

Proof. (i) Assume \( t_p(G) \) is neither reduced nor divisible. Then our hypotheses imply that \( t_p(H) \) is a group of the form \( t_p(H) = D \oplus R \), where \( D \) is divisible, \( R \) is reduced and \( \text{Hom}(D, R) \neq 0 \). Since this is clearly impossible, (i) holds.

(ii) If \( t_p(G) \) is reduced, then the proof of 2.2 indicates that \( t_p(G) \) and \( t_p(H) \) must be torsion-compact \( p \)-groups with the same Ulm invariants. Consequently \( t_p(G) \) is isomorphic to \( t_p(H) \). If \( t_p(G) \) is divisible, the statement is obvious, because \( t_p(G) \) and \( t_p(H) \) must have the same rank. \( \Box \)

Remark. By 2.6 and 2.7, if \( G \) and \( H \) are torsion groups and there exists an anti-isomorphism \( f: E(G) \rightarrow E(H) \), then \( G \) belongs to a restricted class of torsion groups and \( H \) is isomorphic to \( G \). In particular, let \( G, H, f \) be as above; then the following conditions are equivalent:

1. \( f \) is induced by a group isomorphism \( \tau: G \rightarrow H \) (i.e. \( f(\varphi) = \tau \varphi \tau^{-1} \) for \( \varphi \in E(G) \)).

2. \( E(G) \) is commutative.

In fact, assume first that (1) is true. Since \( \tau \varphi \varphi \tau^{-1} = \tau \varphi \varphi \tau^{-1} \) \((\varphi, \varphi \in E(G))\), (2) clearly holds. This completes the proof, because the implication \( (2) \Rightarrow (1) \) follows from the Baer-Kaplansky theorem. Hence, by ([15] Theorem 1), condition (1) is not generally satisfied.
§ 3. The characterization of all E-dual torsion groups enables us to prove the following

**Theorem 3.1.** Let $G = \prod_{p \in P} G_p$ be a reduced cotorsion group. Then $G$ is E-dual if and only if, for every $p \in P$, its $p$-adic component $G_p$ is either a $J_p$-module of finite rank or the $p$-adic completion of a torsion-compact $p$-group.

**Proof.** Necessity. Suppose $G$ is E-dual and fix a prime $p$. To prove that $G_p$ has the required properties, we distinguish three cases.

(i) $G_p$ adjusted. Since $E(G_p)$ is isomorphic to $E(t_p(G_p))$ ([13] Theorem 3.3), from Theorem 2.3 we deduce that $G_p$ is the $p$-adic completion of a torsion compact $p$-group.

(ii) $G_p$ torsion-free. An application of ([11] Theorem 5.5) shows that $E(G_p) \cong E(Q_p/J_p \otimes G_p)$, where $Q_p$ is the field of $p$-adic numbers. Using 2.4, we conclude that $G_p$ is isomorphic to $J'_p$ for some $r \in \mathbb{N}$.

(iii) $G_p$ neither adjusted nor torsion-free. It is not restrictive to assume $G_p = J'_p \oplus G''_p$, where $r \in \mathbb{N}$ and $G''_p$ is adjusted. We claim that $G''_p$ is finite. Suppose the contrary. Let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be a basic subgroup of $t_p(G''_p)$ and assume $G''_p \ll \prod_{n \in \mathbb{N}} B_n$. Fix a group $H$ and an anti-isomorphism between $E(G_p)$ and $E(H)$, which takes $\varphi$ to $\varphi'$ ($\varphi \in E(G_p)$). Let $\pi_1, \pi_2$ be the projections of $G_p$ onto $J'_p$ and $G''_p$ respectively, and let $H_i = \pi_i(H)$ ($1 \leq i \leq 2$). Evidently $H = H_1 \oplus H_2$ and $H_1 \cong (Z(p^n))^r$ because $H_1$ is fully invariant in $H$. Since $E(G''_p)$ is isomorphic to $E(t_p(G''_p))$, the proof of Lemma 2.2 shows that $H_2$ is a reduced $J_p$-module. The hypothesis that $G''_p$ is not finite and the fact that $H_1 \cong (Z(p^n))^r$ guarantee that $H_2$ is not a $p$-group. Let $x$ be a torsion-free element of $H_2$, and let $y = ux$, where $u$ is a $p$-adic integer algebraically independent over $Z_p$. Choose $\bar{x}, \bar{y} \in H_1$ such that $\bar{y} \neq u\bar{x}$. Then $H$ has an endomorphism $\varphi'$ that maps $x$ and $y$ onto $\bar{x}$ and $\bar{y}$ respectively. Since $\text{Hom}(J'_p, G''_p) \cong \text{Hom}_{J_p}(J'_p, \prod_{n \in \mathbb{N}} B_n)$, the center of $E(G)$ is isomorphic to $J'_p$ and the same applies to $E(H)$. But this is a contradiction, because $u\varphi' \neq \varphi' u$. Consequently $G''_p$ is finite.

Sufficiency. Assume $G = \prod_{p \in P} G_p$, where each $G_p$ is as in (i), (ii) or (iii). Then 2.5, 2.6 and the result used in (i) tell us that, for every $p \in P$, there exists a group $H_p$ such that $E(G_p)$ is anti-isomorphic to $E(H_p)$.
and $H_p$ is fully invariant in $H = \bigoplus_{p \in P} H_p$. Since $E(H) \cong \prod_{p \in P} E(H_p)$,
$G$ is $E$-dual. \qed

**Corollary 3.2.** Let $G = \prod_{p \in P} G_p$ be a reduced cotorsion group. The following are equivalent:

1. For every prime $p$, $G_p$ is either a torsion-free $J_p$-module of finite rank or the $p$-adic completion of a torsion-compact $p$-group.

2. $G$ is $E$-self-dual.

**Proof.** Since $G$ is $E$-self-dual if and only if the $p$-adic component $G_p$ of $G$ is $E$-self-dual ($p \in P$), the result is an immediate consequence of the previous theorem. In fact, the first part of the proof of 3.1 shows that if $G_p$ is $E$-dual, then $G_p$ is $E$-self-dual if and only if it is either adjusted or torsion-free. \qed

**Remark.** In ([9], p. 73) Kaplansky asserts that there are reasons for believing that two modules with isomorphic (or anti-isomorphic) endomorphism rings are isomorphic or «dual». This suggests that we translate 3.1 and 3.2 as follows: The correspondence given by Harrison ([7]) between torsion groups and reduced cotorsion groups induces a correspondence between $E$-dual ($E$-self-dual) torsion groups and $E$-dual ($E$-self-dual) reduced cotorsion groups. It is natural to compare this statement with a result of May and Tubassi ([13] Main Theorem) about groups with isomorphic endomorphism rings, i.e. the characterization of all groups $G$ and $H$ such that $E(G) \cong E(H)$ and $t(G) \preceq t(H)$. Even in this case, a theory of duality, more precisely Harrison’s duality, clarifies the situation.

**Theorem 3.3.** Let $G$ be a divisible group. Then $G$ is $E$-dual if and only if either $G = \bigoplus_{p \in P} D_p$ with $D_p$ a divisible $p$-group of finite rank or $G$ is a torsion-free group of finite rank.

**Proof.** Sufficiency immediately follows from 2.5, because if $G$ is torsion-free of finite rank $r$, then $E(G) \cong M_r(\mathbb{Q})$. Since arguments very similar to those used in the first part of 2.1 show that infinite dimensional vector spaces over $\mathbb{Q}$ cannot be $E$-dual, it remains to prove that the group $G = \mathbb{Q} \oplus \mathbb{Z}(p^\infty)$ is not $E$-dual ($p \in P$). Suppose this does not hold. Let $\pi_1, \pi_2$ be the projections of $G$ onto $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ respectively, and let $\pi_1, \pi_2$ be the corresponding elements under an anti-isomorphism between $E(G)$ and $E(H)$ for some $H$. Write
\(H = H_1 \oplus H_2\), where \(H_i = \pi'_i(H)\) (1 ≤ \(i\) ≤ 2). Since \(H_1 \cong \mathbb{Q}\) and \(\pi'_i E(H) \pi'_i = 0\), we must have \(H_2 \cong \mathbb{Z}_p\). Therefore \(|E(H)| > 2^{\aleph_0} = |E(G)|\), and this contradiction proves that divisible mixed groups are not \(E\)-dual. \(\square\)

The following result is an obvious consequence of Theorem 3.3.

**Corollary 3.4.** Let \(G\) be a divisible group. Then \(G\) is \(E\)-dual if and only if \(G\) is \(E\)-self-dual.

**Corollary 3.5.** Let \(G = D \oplus R\) and let \(D\), the divisible part of \(G\), be non-zero and torsion-free. Then \(G\) is \(E\)-dual if and only if \(D\) and \(R\) are \(E\)-dual and \(R\) is a torsion group.

**Proof.** Let \(G\) be an \(E\)-dual group as in the hypotheses. Fix a group \(H\) such that \(E(G)\) and \(E(H)\) are anti-isomorphic. Write \(H = H_1 \oplus H_2\), where \(\text{Hom}(H_1, H_2) \cong \text{Hom}(R, D); \text{Hom}(H_2, H_1) = 0\) and \(E(H_1), E(H_2)\) are anti-isomorphic to \(E(D)\) and \(E(R)\) respectively. Then \(H_1\) is isomorphic to \(D\), while \(H_2\) is a reduced torsion group. By symmetry, we conclude that \(R\) is a torsion group. The other assertions follow from Lemma 1.1 and the fact that \(D\) and \(R\) are fully invariant in \(G\). \(\square\)

**Corollary 3.6.** Let \(G = D \oplus R\); let \(D\) be a non-zero divisible torsion group and \(R = \prod_{p \in \mathbb{P}} R_p\) an adjusted cotorsion group. Then \(G\) is \(E\)-dual if and only if \(D\) is \(E\)-dual and \(R\) is finite.

**Proof.** By 1.1 and 2.6, we need only prove that if \(G\) is an \(E\)-dual group as in the hypotheses, then \(R\) is finite. To see this, fix a group \(H\) and an anti-isomorphism between \(E(G)\) and \(E(H)\) mapping \(\varphi\) onto \(\varphi'\) for every \(\varphi \in E(G)\). Let \(\pi_1, \pi_2\) be the projections of \(G\) onto \(D\) and \(R\) respectively, and let \(H = H_1 \oplus H_2\), where \(H_i = \pi'_i(H)\) (1 ≤ \(i\) ≤ 2). Assume first that \(R = R_p\) for some prime \(p\). Our claim is that \(R\) is finite. Suppose the contrary. Then there exists \(\varphi \in E(G)\) such that \(\varphi \neq 0, \varphi(t(G)) = 0\). Since \(G/t(G)\) is divisible and torsion-free, \(\varphi \in p^\infty E(G)\) ([15] vol. I, p. 182) and, obviously, \(\varphi = q\pi_2\). Hence \(\varphi'(H)\) is a non-zero subgroup of \(p^\infty H_2\). On the other hand, by 3.1, \(t(R)\) is a torsion-compact \(p\)-group. Since \(E(R)\) is isomorphic to \(E(t(R))\), the proof of Lemma 2.2 assures us that \(p^\infty H_2 = 0\). This contradiction establishes that \(R\) is finite. To complete the proof, it remains to show that the hypothesis that \(G\) is \(E\)-dual always implies that \(R\) is finite. Assume this is not
true. Then, as before, there exists \( q \in E(G) \) such that \( q \neq 0 \) and \( \varphi(t(G)) = 0 \). For every prime \( p \), let \( e_p \) denote the projection of \( G \) onto \( R_p \). Remark that \( \varphi'(H) \cap t_p(H_2) = 0 \), because \( t_p(H_2) = e_p'(H) \) and \( \varphi'(H) \cap e_p'(H) = 0 \) (\( p \in P \)). Since \( G/t(G) \) is divisible and torsion-free, it follows that \( \varphi \in \bigcap_{p \in P} p^\omega E(G) \). Therefore \( \varphi'(H) \) must be a torsion-free divisible subgroup of \( H_2 \), and this is clearly impossible. In fact, \( R \) has no subgroup isomorphic to \( \mathbb{Q} \) and the same applies to \( H_2 \). This contradiction proves that \( R \) is finite, and the proof is complete. \( \Box \)

**Remark 1.** Let \( G \) be as in 3.6. Then a necessary and sufficient condition for \( G \) to be \( E \)-dual is that \( D \) and \( R \) are \( E \)-dual with \( H_2 \) a torsion group. In fact, by ([5] Corollary 54.4), reduced cotorsion torsion groups are bounded. The result now follows from 2.6.

**Remark 2.** The hypotheses of 3.6 cannot be weakened, because there exist reduced \( E \)-dual groups \( G \) of the form \( G = T \oplus R \), where \( T \) is a non-zero torsion group and \( R \) is an infinite adjusted cotorsion group. For instance, Lemma 1.2 tells us that the group \( G = \bigoplus_{p \in P} \mathbb{Z}(p) \oplus \bigoplus_{p \in P} \mathbb{Z}(p) \) is \( E \)-dual.

**Proposition 3.7.** If \( G \) is a mixed \( E \)-dual group, the following facts hold:

(i) \( G/t(G) \) is not necessarily \( E \)-dual.

(ii) \( t_p(G)/p^\omega t_p(G) \) is \( E \)-dual, for every prime \( p \).

**Proof.** (i) Since \( G = \bigoplus_{p \in P} \mathbb{Z}(p) \) is \( E \)-dual and \( G/t(G) \) is a divisible torsion-free group of rank \( 2^\aleph_0 \), (i) follows from 3.3.

(ii) Let \( B = \bigoplus_{n \in \mathbb{N}} B_n \) be a basic subgroup of \( t_p(G) \). Since \( B_n \) is a summand of \( G \), \( B_n \) is finite \( (n \in \mathbb{N}) \). If \( B \) is finite, then the statement clearly holds. Assume \( B \) is not finite. Then there exist suitable \( x_n \in B \) \( (n \in \mathbb{N}) \) such that \( B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle \) and \( o(x_r) < o(x_s) \) \( (r, s \in \mathbb{N}; r < s) \).

Fix pairwise orthogonal projections \( \pi_n : G \to \langle x_n \rangle \) \( (n \in \mathbb{N}) \) so that if \( \eta : t_p(G) \to \prod_{n \in \mathbb{N}} \langle x_n \rangle \) is the product map, i.e. \( \eta(x) = (\pi_n(x))_{n \in \mathbb{N}} \) \( (x \in t_p(G)) \), then \( \ker \eta = p^\omega t_p(G) \). It remains to show that \( t_p(G) \cap \mathbb{N} \).

Let \( e_{rs} \) denote the endomorphism of \( G \) uniquely defined by the following conditions: \( e_{rs}(1 - \pi_s) = 0 \) and \( e_{rs}(x_s) = p^{\lambda_{rs}} x_r \), where \( \lambda_{rs} = \)
Remark. Condition (ii) indicates that only very particular torsion groups may be the torsion part of an $E$-dual group. We don't know examples of $E$-dual groups $G$ such that, for some prime $p$, $p^\omega t_p(G)$ is not divisible. However, we can give a sufficient condition in order that $p^\omega t_p(G)$ is divisible. In fact, if $G$ and $H$ have anti-isomorphic endomorphism rings and $H/t(H)$ is $p$-divisible, then $t_p(G)$ is $E$-dual. To see this, assume the anti-isomorphism between $E(G)$ and $E(H)$ takes $\varphi$ to $\varphi'$ ($\varphi \in E(G)$). Write $t_p(G) = D \oplus R$; $t_p(H) = D' \oplus R'$ where $D$, $D'$ are divisible and $R$, $R'$ are reduced. We claim that $p^\omega R = 0$. Suppose this does not holds. Then there is an endomorphism $\varphi$ of $G$ such that $0 \neq \varphi(G) < p^\omega R[p]$. Let $\pi_n (n \in \mathbb{N})$ be as before. An argument similar to that used in 2.2 shows that $\bigoplus_{n \in \mathbb{N}} \pi_n(H)$ is a basic subgroup of $H$.

Since $\pi_n \varphi = 0$ ($n \in \mathbb{N}$), $\varphi'$ is 0 on $t(H)$ and clearly $0 \neq \varphi'(H) < H[p]$. But this is impossible, because $H/t(H)$ is $p$-divisible. This contradiction establishes that $p^\omega R = 0$. Consequently $p^\omega t_p(G)$ is divisible.

§ 4. In this section we investigate some properties of torsion-free $E$-dual groups. Since Corollary 3.5 gives the structure of an $E$-dual group containing $\mathbb{Q}$, we can confine ourselves to the reduced case. First we recall some definitions.

If $G$ is any group, the finite topology of $E(G)$ has the family of all $U_X = \{\varphi \in E(G) : \varphi(X) = 0\}$, with $X$ a finite subset of $G$, as a basis of neighborhoods of 0. It is well known ([5] Theorem 107.1) that $E(G)$, with respect to the finite topology, is a complete Hausdorff topological ring. According to ([2], p. 63), reduced torsion-free groups of cardinality $< 2^{\mathfrak{S}}$ are called control groups. If $G$ is a group and, for some control group $C$, every subgroup of $G$ of finite rank is isomorphic to a subgroup of $C$, then $G$ is a controlled group. In the following, $\mathfrak{S}$ denotes the first strongly inaccessible cardinal ([5] vol. II, p. 129).

Theorem 4.1. If $G$ is a controlled group and $E(G)$ is of cardinality $< \mathfrak{S}$, then $G$ is $E$-dual.

Proof. It is enough to show that the ring $A = (E(G))^\omega$, equipped with the discrete topology, satisfies the hypotheses of ([2] Theorem 2.2). This clearly holds, if we only show that the group $E(G)$ is controlled. To this purpose, regard $E(G)$ as embedded in $\prod_X E(G)/U_X$, the product
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being extended over all finite subsets $X$ of $G$, and let $K$ be a subgroup of $E(G)$ of finite rank. Take linearly independent elements $\varphi_1, \ldots, \varphi_r \in E(G)$ such that $K \lhd \langle \varphi_1, \ldots, \varphi_r \rangle \lhd E(G)$. Then there exist a finite subset $X'$ of $G$ such that the natural projection $\pi: \prod_{X} E(G)/U_X \to E(G)/U_{X'}$ maps $\varphi_1, \ldots, \varphi_r$ onto linearly independent elements. Since $K \cap \ker \pi = 0$, $K$ is isomorphic to a subgroup of $E(G)/U_{X'}$. The choice of $K$ assures us that every subgroup of $E(G)$ of finite rank is isomorphic to a subgroup of $\bigoplus_{X} E(G)/U_X$. Using ([2] Proposition 2.1), we conclude that $E(G)$ is controlled. This completes the proof. 

\textbf{Corollary 4.2.} If $G$ is a reduced torsion-free separable group and $E(G)$ is of cardinality $< \aleph_1$, then $G$ is $E$-dual.

\textbf{Proof.} Let $C = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}^{(N)}_p$. Since $C$ is a control group and every subgroup of $G$ of finite rank may be embedded in $C$, the result follows from the previous theorem. 

\textbf{Remark.} There exists an $E$-dual group $G$ such that $G^{(N)}$ is $E$-dual. In fact, the group $\mathbb{Z}^{(N)}$ satisfies the hypotheses of 4.1. Observe that, by 2.5 and 3.3, this possibility cannot occur if $G$ either a torsion or a divisible group.

Comparing 3.1 and 3.3 with 4.1, we see that the behaviour of torsion-free cotorsion groups is completely different from that of torsion-free non cotorsion groups. Also note that, by Corner’s theorems, very complicated torsion-free groups have uncomplicated, even commutative, endomorphism rings ([14], p. 180; [15], p. 62). On the other hand, if $G$ and $H$ are arbitrary reduced torsion-free groups with anti-isomorphic endomorphism rings, then $H$ does not generally inherit many properties of $G$. For instance, it has been proved ([6] Theorem 1.2) that if $G = \mathbb{Z}^{(N)}$, then there is no reduced torsion-free group $H$ of the same type as $\mathbb{Z}$ such that $E(G)$ and $E(H)$ are anti-isomorphic. More generally, we have the following

\textbf{Proposition 4.3.} There exist a free group $G$ and a non controlled group $H$ such that $E(G)$ and $E(H)$ are anti-isomorphic.

\textbf{Proof.} Let $G = \mathbb{Z}^{(N)}$ and $A = (E(G))^\circ$. We shall show first that $A$, endowed with the discrete topology, satisfies the hypotheses of ([3] Theorem 1). Fix a prime $p$. Since $|A| = 2^{\aleph_0}$ and $p^\alpha A = 0$, it suffices to prove that $J_p$ is linearly disjoint from the group $A$, that
is from the group $E(G)$, over $\mathbb{Z}_p$ \(i.e.\) if \(\sum_{i=1}^{n} \alpha_i \varphi_i = 0\) in $\widehat{E(G)}$, the $p$-adic completion of $E(G)$, with $\varphi_1, \ldots, \varphi_n \in \widehat{E(G)}$; $\alpha_1, \ldots, \alpha_n \in J_\mathbb{p}$ and linearly independent over $\mathbb{Z}_p$, then $\varphi_1 = \ldots = \varphi_n = 0$. Assume this is not true. Then we may write $\sum_{i=1}^{n} \alpha_i \varphi_i = 0$, where $\varphi_i \in E(G)$ \((1 < i < n)\), $\varphi_1 \neq 0$ and the $\alpha_i$'s are as before. Let $G = \bigoplus_{n \in \mathbb{N}} G_n$, where $G_n = \mathbb{Z}$, $x_n = 1 \in G_n$ and $\pi_n$ is the projection of $G$ onto $G_n \ (n \in \mathbb{N})$. By hypothesis, there exist $r, s \in \mathbb{N}$ such that $\pi_r \left( \sum_{i=1}^{n} \alpha_i \varphi_i (x_s) \right) = 0$ is a linear combination of the $\alpha_i$'s with coefficients in $\mathbb{Z}$ not all equal to 0. This contradiction establishes that $A$ has the required property. We claim that there exists a non controlled group $H$ whose endomorphism ring, with the finite topology, is the discrete ring $A$. In fact, for every $a \in A$ we can choose a $p$-adic integer $\alpha(a)$ with the following properties:

(i) The set \(\{ \alpha(a) \colon a \in A \}\) is algebraically independent over $\mathbb{Z}_p$.

(ii) $J_\mathbb{p}$ has transcendent degree $2^{\aleph_0}$ over the subring generated by the $\alpha(a)$'s. Let $H$ be the following pure subgroup of the $p$-adic completion $\hat{A}$ of $A$

\[ H = \langle A, A\alpha(a) \colon a \in A \rangle_{*} < \hat{A} \ . \]

Since ([3] Theorem 1) assures that $E(H)$ is isomorphic to $A$, it remains to check that $H$ is not controlled. To see this, let $S$ denote the subset of all $\varphi \in E(G)$ such that $\pi_r \varphi \pi_s = \pi_1$; $\pi_r \varphi \pi_s = 0 \ (r, s \in \mathbb{N}; r \neq s)$. Now consider the pure subgroups $S'$ and $S''$, where

\[ S' = \langle 1, \alpha(\varphi) \colon \varphi \in S \rangle_{*} < J_\mathbb{p}; \quad S'' = \langle \pi_1, \pi_1 \varphi \alpha(\varphi) \colon \varphi \in S \rangle_{*} < \widehat{E(G)} \ . \]

By ([3] Proposition 1), $S'$ is not controlled. Since $S''$ is isomorphic to $S'$ and $S'' < H$, we conclude that $H$ is not controlled. □

**Proposition 4.4.** There exists a countable reduced torsion-free group $G$ such that $E(G)$ is not anti-isomorphic to the endomorphism ring of a countable reduced torsion-free group.

**Proof.** Let $G = \mathbb{Z}^{\mathbb{N}}$. With the same notations of 4.3, let $e_r$, be the endomorphism of $G$ defined by $e_r(x_s) = x_r$; $e_r(1 - \pi_s) = 0 \ (r, s \in \mathbb{N})$. Let $f \colon E(G) \to E(H)$ be any anti-isomorphism. To end the proof, it is enough to show that $H$ is not countable. Assume the contrary.
Then $H$ is a countable reduced torsion-free group and $f$ is continuous with respect to the finite topologies of $E(G)$ and $E(H)$. This is an immediate consequence of ([14] Lemma 4.3), because if $U$ is a subgroup of $E(H)$, then

$$U \text{ open } \iff E(H)/U \cong E(G)/f^{-1}(U) \text{ countable}$$

reduced torsion-free $\iff f^{-1}(U)$ open.

Since $E(G)$ is not discrete, the same applies to $E(H)$. Therefore $E(H)$ has a proper open left ideal $U$ and $f^{-1}(U)$ is a proper open right ideal of $E(G)$. It is now clear that there exists $m \in \mathbb{N}$ such that \{ \(q \in E(G)\colon q_r \pi_r = 0 \ (1 < r < m)\} \subset V$, where $V$ is an open two-sided ideal of $E(G)$ and $V < f^{-1}(U)$. Choose $q \in E(G) \setminus V$ and define $q', q'' \in E(G)$ as follows: $q' \pi_r = q \pi_r$, $q'' \pi_r = 0$ $(1 < r < m)$; $q' \pi_r = 0$, $q'' \pi_r = q \pi_r$ $(r > m)$.

Evidently $q = q' + q''$ and $q'' \in V$. Write $q' = \sum_{r=1}^{m} q_r$, where $q_r \pi_r = q \pi_r$, $\pi_r (1 - \pi_r) = 0$ $(1 < r < m)$. Then there are suitable $n_{k,r} \in \mathbb{Z}$ $(k \in \mathbb{N}; 1 < r < m)$ almost all 0 such that $q_r = \sum_{k \in \mathbb{N}} n_{k,r} e_{k,r}$. Since

$$e_{k,r} = e_{k,m+1} e_{m+1,r} \in V \quad (k \in \mathbb{N}; 1 < r < m),$$

we conclude that $q \in V$. This contradiction proves that $H$ is not countable, and the proof is complete. \(\square\)

**Remark 1.** There exists a non-commutative topological ring $A$ such that $A$ is the endomorphism ring of a countable reduced torsion-free group and the same applies to its opposite ring. In fact, let $G = \mathbb{Z}^{(\mathbb{N})}$ and, using the notations of 4.4, let $A$ be the subring of $E(G)$ consisting of all $q$ such that $\pi_r q \pi_s = 0$ $(r, s \in \mathbb{N}; r > s)$, i.e. $A$ is isomorphic to the subring of all upper triangular $\mathbb{N} \times \mathbb{N}$ matrices with entries in $\mathbb{Z}$ ([5] Theorem 106.1). It is easy to see that $A$, with the topology induced by the finite topology of $E(G)$, has a family of two-sided ideals as a basis of neighborhoods of 0 and satisfies the hypotheses of ([2] Theorem 1.1).

**Remark 2.** The direct sum of two reduced torsion-free $E$-self-dual groups is not necessarily $E$-dual. To prove this, fix a prime $p$ and let $G = \mathbb{Z} \oplus J_p$. We claim that $G$ is not $E$-dual. Otherwise, $E(G)$ is anti-isomorphic to $E(H)$ for some group $H$ of the form $H = H' \oplus H''$, where $E(H') \cong \mathbb{Z}$; $E(H'') \cong J_p$; $\text{Hom}(H', H'') = 0$ and
Hom \( (H', H') \cong J_p \). Since \( H' \) is reduced and torsion-free, we may assume \( H'' = J_p \). Consequently \( \text{Hom}(H'', H') = \text{Hom}_{J_p}(H'', H') \). Choose a non-zero homomorphism \( \varphi: H' \rightarrow H' \) and regard \( H' \) as a pure subgroup of its \( \mathbb{Z} \)-adic completion \( \hat{H}' = \prod_{q \in \mathbb{P}} \hat{H}'_q \). Since \( \varphi(H') = J_p \varphi(1) \) is torsion-free, there exists \( x \in H' \setminus pH' \) such that \( \varphi(1) = p^nx \) for some \( n \in \mathbb{N} \). These conditions imply that \( J_p x \) is a pure subgroup, hence a direct summand, of \( H' \). But this is clearly impossible, because \( E(H') \) is isomorphic to \( \mathbb{Z} \). This contradiction establishes that \( G' \) is not \( E \)-dual.

The previous example suggests that we determine some properties of all \( E \)-dual groups admitting a free summand.

**Proposition 4.5.** If \( G = \mathbb{Z} \oplus G' \) is \( E \)-dual, then the following conditions hold:

(i) \( G' \) is reduced and torsion-free.

(ii) \( G' \) is not cotorsion.

(iii) \( G' \) is not necessarily a controlled group.

**Proof.** (i) We first prove that \( G' \) is torsion-free. Suppose \( E(G) \) is anti-isomorphic to \( E(H) \). Then \( H \) has a decomposition \( H = H' \oplus H'' \), where \( E(H') \cong \mathbb{Z} \) and \( \text{Hom}(H', H') \cong \text{Hom}(\mathbb{Z}, G') \cong G' \). Since \( H' \) is torsion-free, the same applies to \( G' \). Using Corollary 3.5, we conclude that \( G' \) is reduced.

(ii) This immediately follows from (i) and Remark 2.

(iii) Fix a prime \( p \). Let \( G' \) denote a pure subgroup of \( J_p \) with the following properties: \( 1 \in G' ; |G'| = 2^{\aleph_0} \) and the transcendence degree of \( J_p \), over the subring generated by \( G' \) is \( 2^{\aleph_0} \). Let \( G = \mathbb{Z} \oplus G' \); then, as in 4.3, one can show that the ring \( (E(G))_0 \), with the discrete topology, satisfies the hypotheses of ([3] Theorem 1). Thus \( G \) is \( E \)-dual and, by ([3] Proposition 1), \( G' \) is not controlled. □

**Remark.** More generally, if \( R \) is a rational group, \( p \) is a prime and \( G = R \oplus G' \) is \( E \)-dual, then \( pR \neq R \) implies \( t_p(G') = 0 \). In fact, we can find a group \( H = H' \oplus H'' \) such that \( E(G) \) is anti-isomorphic to \( E(H) \), \( E(H') \cong E(R) \) and \( \text{Hom}(H', H') \cong \text{Hom}(R, G') \). Since \( H' \) is torsion-free and \( R \neq pR \), \( G' \) has no element of order \( p \). Finally note that if \( R = \mathbb{Q} \), then the structure of \( G' \) is completely determined by 3.5.
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