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Construction of Positive Definite Functions

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1. Introduction.

The purpose of this paper is to improve and extend a method for constructing positive definite functions on unimodular groups, which was first introduced in [7]. The improved method allows us to *prove the same results proved in* [7] *without the hypothesis of separability of the group.*

Let G be a locally compact unimodular group. We shall assume throughout the paper that the *regular representation* λ *of* G *is not completely reducible.* This assumption is clearly equivalent to hypothesis H) of [7]. We denote by $A(G)$, $B_\lambda(G)$, $B(G)$, respectively, the algebras of coefficients of the regular representation, of coefficients of the representations weakly contained in the sense of Fell in the regular representation, and of coefficients of all unitary representations. It is clear that $A(G) \subseteq B_\lambda(G) \subseteq B(G)$. We refer to [6] for the properties of $A(G)$, $B_\lambda(G)$ and $B(G)$ (the algebra $B_\lambda(G)$ is denoted by $B_\varrho(G)$ in [6]). We only recall here that if $C^*_\lambda(G)$ is the C^*-algebra generated by L^1 acting by convolution on $L^2(G)$, if $VN(G)$ is the von Neumann algebra generated by $C^*_\lambda(G)$, and if $C^*(G)$ is the enveloping C^*-algebra of $L^1(G)$, then the dual space of $A(G)$ is $VN(G)$, the dual space of $C^*_\lambda(G)$ is $B_\lambda(G)$ and the dual space of $C^*(G)$ is $B(G)$. The norms that the elements of A, B_λ and B have as linear functionals on VN, C^*_λ and C^*, respectively, are Banach algebra norms, and the inclusions $A \subseteq B_\lambda \subseteq B$ are all isometric; furthermore A and B_λ are closed ideals in B. Finally we recall that the maximal ideal space of $A(G)$ is exactly G.

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We will also use the notation $B_0(G)$ to indicate the closed subalgebra of $B(G)$ consisting of the elements vanishing at infinity: thus $B_0(G) = B(G) \cap C_0(G)$.

Under the additional hypothesis of separability of G, we proved in [7] that $B_0(G)$ properly contains $A(G)$, we constructed in fact an element of $B(G) \cap C_0(G)$ which is not in $A(G)$. In the present paper, without assuming that G is separable, we show that if D is the Cantor group, $D = \prod_{j=1}^{\infty} \{-1, 1\}_j$, there exists a map which assigns to every positive measure μ on D a positive definite continuous function φ on G, which belongs to $B_\lambda(G)$. The value at the identity of φ is the same as the mass of μ; if μ singular, with respect to the Haar measure on D, then for almost all $t \in D$, the translates μ^t of $(\mu^t(E) = \mu(E+t))$, yield positive definite functions which are not in $A(G)$. Finally if μ is a « Riesz product », with Fourier-Stieltjes coefficients vanishing at infinity, that is μ is the weak* limit of products of the type

$$\prod_{j=1}^{n} (1 + \beta_j r_j)\,, \qquad \text{with } -1 \leqslant \beta_j \leqslant 1\,,\ \lim \beta_j = 0$$

(the $r_j(t)$ are the Rademacher functions on D), then the positive definite function corresponding to D vanishes at infinity on G. We remark that the map $\mu \to \varphi$ is not injective in general, as one can easily ascertain considering the simplest examples ($G = \mathbb{Z}$, or $G = \mathbb{R}$). Therefore it cannot be extended to an isometry between $M(D)$ and a subspace of $B(G)$. In order to define the map from positive measures to positive definite functions, we construct a conditional expectation $\mathcal{E}'$ mapping $VN(G)$ onto a subalgebra isomorphic to $L^\infty(D)$, with the additional property that *if* $T \in C^*_\lambda(G)$, *then* $\mathcal{E}'(T)$ *corresponds to a continuous function on* D. This last property of $\mathcal{E}'$ is what makes the present construction different from the construction of [7] (where a conditional expectation $\mathcal{E}$ was also used) and allows us to do without the separability of G.

We remark that the results of [7] were extended by L. Baggett and K. Taylor to separable nonunimodular groups [2] (that is assuming only that the regular representation is not completely reducible and that the group is separable). Presumably the condition of separability may also be removed in the context of nonunimodular groups, as the authors of [2] seem to indicate in the introduction to their paper.

Properties of noncompact unimodular groups with completely reducible regular representations are studied in [11] and [13]; many examples are exhibited in [12] and in [1], where nonunimodular examples are also studied. All known examples of unimodular groups with completely reducible regular representation have the property that all continuous positive definite functions vanishing at infinity are in $A(G)$ [11], [15], [12]. On the other hand L. Baggett and K. Taylor exhibit [1] an example of a nonunimodular group with completely reducible regular representation and such that $B_0(G) \neq A(G)$.

The relationship between $B_0(G)$ and $A(G)$ is very close in many important noncommutative groups for which $A(G) \neq B_0(G)$. For instance $B_0(G)/A(G)$ is a nilpotent algebra when G is the group of rigid motions of $\mathbb{R}^n$ (or a similar compact extension of an Abelian group [10]). We also have that B_0/A is a radical algebra if $G = SL(2, R)$ [4] and even when G is any semisimple Lie group with compact center [5]. This contrast sharply with the situation for commutative noncompact groups. When G is commutative and noncompact only entire functions operate in $B(G)$ [14] which implies that B_0/A is not a radical algebra. The same properties may be proved for separable groups with noncompact center, but outside this case (or strictly related cases) they might never hold for connected Lie groups [8].

2. The construction.

We start with the same method used in [7]. We will summarily review the contents of Section 2 of [7] where the reader can find further details. Let $\Gamma = (L^2(G), VN(G), \mathrm{tr})$ be the canonical of gage space of G. We may assume that there exists a projection $P \in VN(G)$ with $\mathrm{tr}\,(P) = 1$, and such that P contains no minimal projections. Starting from P we construct a sequence of Rademacher operators $\{R_n\}_{n=1}^{\infty}$, with $R_n = \sum_{k=1}^{2^n} (-1)^{k+1} P_{nk}$ where $\pi_n = \{P_{n1}, \ldots, P_{n2n}\}$ is a partition of P into projections of equal trace, and $P_{n-1,k} = P_{n,2k-1} + P_{n,2k}$, for $k = 1, \ldots, 2^{n-1}$. We construct the Walsh operators W_n as products of the Rademacher operators, in such a way that $W_0 = P$, and $W_n = R_{j_1} \ldots R_{j_s}$ if $n = 2^{j_1-1} + \ldots + 2^{j_s-1}$. We define then, for $T \in L^1(\Gamma)$

$$\mathcal{E}_n(T) = \sum_{k=0}^{2^n-1} \mathrm{tr}\,(W_k T)\, W_k \,.$$

The operator $\mathcal{E}$ mapping $L^1(\Gamma)$ into $L^1(\Gamma)$ is defined as the limit, in the strong operator topology of the sequence $\mathcal{E}_n$. We also have that $\mathcal{E}$ extends to a projection of norm one mapping $VN(G)$ onto the subalgebra $\mathcal{E}(VN)$, generated (as a von Neumann algebra) by the Walsh operators. We have that

$$\mathcal{E}_n(\mathcal{E}(T)) = \mathcal{E}(\mathcal{E}_n(T)) = \mathcal{E}_n(T)$$

and that, for $T \in L^1(\Gamma)$ $\lim_n \|\mathcal{E}_n(T) - \mathcal{E}(T)\|_{L^1(\Gamma)} = 0$.

As mentioned in the introduction we will now define conditional expectations $\mathcal{E}_n'$ and $\mathcal{E}'$ which will have the property that

$$\lim_n \|\mathcal{E}_n'(T) - \mathcal{E}'(T)\|_{VN(G)} = 0\,, \qquad \text{for } T \in C(G)\,. \tag{1}$$

In order to do this we will define inductively a sequence R_{j_n} of Rademacher functions, set $R_n' = R_{j_n'}$, define the operators W_n' as the products of the R_n', and define $\mathcal{E}_n'$ and $\mathcal{E}'$ the same way we defined $\mathcal{E}_n$ and $\mathcal{E}$, but with reference to the sequence $\{R_n'\}$.

First of all we remark that, without loss of generality, we may assume that G is a countable union of compact sets. In fact the Fourier transform $\operatorname{tr}(L_x^* W_n)$ of each W_n vanishes outside an open set which is the countable union of compact sets. Since the operators W_n are countable, there exists an open set S which is a countable union of compact sets and such that, for all n, $\operatorname{tr}(L_n^* W_n) = 0$ if $x \notin S$. The set S is contained in an open ubgroup G_0 which is also a countable union of compact sets. Every element of $A(G_0)$, $B_\lambda(G_0)$ or $B(G_0)$ may be extended isometrically to an element of the corresponding spaces of G, by defining it to be zero off G_0. We can therefore make our entire construction on G_0, or, equivalently, we can assume that G is a countable union of compact sets. Assume now that V_n is an increasing sequence of compact sets and that every compact subset of G is contained in some V_n. Let α_n be a sequence of real numbers decreasing to zero. (The sequence α_n is used in the construction only to allow to prove Theorem 2, below. For the proof of Theorem 1 it is not necessary to introduce such a sequence). Recall that if $D = \prod_{j=1}^{\infty} \{-1, 1\}_j$, and $t = \{\varepsilon_j\} \in D$, then $r_j(t) = \varepsilon_j$, in other words $t = \{r_j(t)\}$. For each $t \in D$ we shall construct now a sequence ψ_n^t of positive definite elements of $A(G)$, and show that this sequence converges uniformly on compact sets to positive definite functions $\psi^t \in B_\lambda(G)$. We construct ψ_n^t as the Fourier transform

$$\psi_n^t(x) = \mathrm{tr}\,(L_x^* \Psi_n^t)\,,$$

of operators $\Psi_n^t \in \mathcal{E}(VN) \subseteq L^1(\Gamma)$. We let, first of all $\Psi_1^t = W_0 = P$, for each $t \in D$. The other operators Ψ_n^t will have the form

$$(2) \qquad \Psi_n^t = \prod_{k=1}^{n-1} (P + r_k(t) R_{j_k})\,, \qquad \text{for } t = \{r_j(t)\} \in D\,,$$

where the sequence j_k is to be determined inductively. Together with the sequence of operators Ψ_n^t, we shall construct an increasing sequence of compact sets, $K_n \supseteq V_n$, which will be used in the induction procedure. We let

$$K_1 = \{x \in G\colon |\mathrm{tr}\,(L_x^* W_0)| \geqslant \alpha_1\} \cup V_1\,.$$

Suppose that the operators $\Psi_1^t, \ldots, \Psi_n^t$ (of the form (2) if $n > 1$) and the compact sets $K_1, \ldots, K_{n-1}$, have been determined. Define

$$K_n = \bigcup_{k=0}^{2^{j_{n-1}}} \left\{x \in G\colon |\mathrm{tr}\,(L_x^* W_k)| \geqslant \frac{\alpha_n}{2^{n-1}}\right\} \cup K_{n-1} \cup V_n\,.$$

(What we really need is that $K_n \supseteq K_{n-1}$, $K_n \supseteq V_n$, and, for the purpose of proving Theorem 2, that $|\mathrm{tr}\, L_x^* W)| < \alpha_n/2^{n-1}$, for every $x \notin K_n$ and every W appearing in the expansion of Ψ_n^t. The definition just given fulfils these requirements). Let $\nu(x) = \sum_{j=0}^{\infty} |\mathrm{tr}\,(L_x^* W_j)|^2$. Then, as shown in [3], $\nu(x)$ is a continuous function of x. Therefore, by Dini's theorem, for some positive integer h_n,

$$\sum_{j \geqslant h_n} |\mathrm{tr}\,(L_x^* W_j)|^2 < 2^{4-n} \qquad \text{for } x \in K_n\,.$$

Let j_n be such that $j_n > j_{n-1}$ and $2^{j_n - 1} > h_n$. Define

$$\Psi_{n+1}^t = \Psi_n^t + r_n(t) R_{j_n} \Psi_n^t = \prod_{k=1}^{n} (P + r_k(t\mathrm{P} R_{j_k})\,.$$

All the Walsh operators appearing in the expansion of $R_{j_n}\Psi_n^t$ have indices greater than h_n. Therefore if W is any such operator

$$|\operatorname{tr}(L_x^* W)| < 2^{-2n}, \qquad \text{for } x \in K_n .$$

Since the number of terms appearing in the expansion of $R_{j_n}\Psi_n^t$ is exactly 2^{n-1}, we deduce that

$$|\operatorname{tr}(L_x^* R_{j_n}\Psi_n^t)| < (\tfrac{1}{2})^{n+1}, \qquad \text{for } x \in K_n .$$

We now prove that for each $t \in D$, the sequence $\psi_n^t(x) = \operatorname{tr}(L_x^*\Psi_n^t)$, which consists of positive definite elements of A, converges uniformly, on compact sets, to an element $\psi^t \in B_\lambda$. Let K be a compact subset of G and let $\varepsilon > 0$. Choose n so that $K \subseteq K_n$ and $2^{-n} < \varepsilon$. Then if $x \in K$, we also have $x \in K_{n+i}$, with $i \geqslant 0$, and hence

$$|\psi_{n+i}^t(x) - \psi_{n+i+1}^t(x)| = |\operatorname{tr}(L_x^* R_{j_{n+i}}\Psi_{n+1}^t)| < \frac{1}{2^{n+i+1}} .$$

Therefore

$$(3) \qquad |\psi_n^t(x) - \psi_{n+p}^t(x)| \leqslant \sum_{i=0}^{m-1} 2^{-n-i-1} \leqslant 2^{-n} < \varepsilon .$$

We choose now a new set of Rademacher operators, namely $R_n' = R_{j_n}$, and we form a new system of Walsh operators W_n', by taking finite products of the R_n'.

We can write then

$$\Psi_n^t = \prod_{j=1}^{n-1} (P + r_j(t) R_j') .$$

We define then $\mathcal{E}_n'$ and $\mathcal{E}'$ with respect to the new system of Walsh operators. Let θ be the weak* continuous isometric linear map which maps the subalgebra $\mathcal{E}'(VN)$ of VN onto $L^\infty(D)$, and such that $\theta W_n' = w_n(t)$. We shall prove that θ maps $\mathcal{E}'(C_\lambda^*(G))$ into the space $C(D)$ of *continuous* functions on D. It is enough to prove one of the two equivalent statements:

$$\lim_n \|\mathcal{E}_n'(T) - \mathcal{E}'(T)\|_{VN(G)} = 0 \qquad \text{for } T \in C_\lambda^*(G) ,$$

or

$$(4)\qquad \lim_n \|\theta\mathcal{E}'_n(T)-\theta\mathcal{E}'(T)\|_{L^\infty(D)}=0\,, \qquad \text{for } T\in C^*_\lambda(G)\,.$$

Since the projections $\mathcal{E}'_n$ are bounded in norm as operators on VN, it suffices to prove one of the above statements for a dense set of operators in $C^*_\lambda(G)$. We shall prove that (4) holds if $T=L_f$ is the left convolution operator by $f\in L^1(G)$ and $\operatorname{supp} f=K$ is compact. In this case

$$|\operatorname{tr}(\Psi^t_n T)-\operatorname{tr}(\Psi^t_{n+p}T)|\leqslant\int_K|\psi^t_n(x)-\psi^t_{n+p}(x)|\,|f(x)|\,dx\leqslant 2^{-n}\int_K|f(x)|\,dx\,,$$

provided that n is chosen so large that $K\subseteq K_n$. We notice now that

$$\operatorname{tr}(\psi^t_n T)=\sum_{k=0}^{2^n-1}\operatorname{tr}(W'_k T)\,w_k(t)=\theta\mathcal{E}'_n(t)\,.$$

Therefore the sequence $\theta\mathcal{E}'_n(T)$ of continuous functions on D is uniformly Cauchy and (4) holds. Incidentally we have proved that if $\langle\ ,\ \rangle$ denotes the pairing between $C^*_\lambda(G)$ and $B_\lambda(G)$, and if $T\in C^*_\lambda(G)$, then

$$\theta\mathcal{E}'(T)=\lim_n\operatorname{tr}(\Psi^t_n T)=\langle T,\psi^t\rangle\,.$$

In particular if $T=L_f$, with $f\in L^1(G)$ $\theta\mathcal{E}'(L_f)=\int_G f(x)\,\psi^t(x)\,dx$.

THEOREM 1. *Let μ be a positive Borel measure on the Cantor group D. Let*

$$\varphi^t_n(x)=\sum_{k=0}^{2^n-1}\hat{\mu}(w_k)\,w_k(t)\operatorname{tr}(L^*_x W'_k)\,.$$

Then, for each t, φ^t_n is a positive definite element of $A(G)$ and $\lim_n\varphi^t_n(x)=\varphi^t(x)$, uniformly on compact set. If μ is singular then, for almost all $t\in D$, $\varphi^t\notin A(G)$.

PROOF. Recalling that $\psi^t_n(x)=\sum_{k=0}^{2^n-1}w_k(t)(L^*_x W'_k)$, one verifies that

$$\varphi^t_n(x)=\int_D\psi^{t+s}_n(x)\,d\mu(s)\,.$$

It follows that φ_n^t is positive definite (because φ_n^{t+s} is positive definite for all s and μ is a positive measure), and that

$$|\varphi_{n+p}^t(x) - \varphi_n^t(x)| \leqslant \mu(D) \sup_{s \in D} |\psi_{n+p}^{t+s}(x) - \psi_n^{t+s}(x)| .$$

Since ψ_n^t converges uniformly on each compact set, uniformly in t, so does the sequence $\varphi_n^t(x)$. We let $\varphi^t(x) = \lim\limits_n \varphi_n^t(x)$. We will show now that for almost all $t \in D$, $\varphi^t \notin A$. Notice that if $T \in C_\lambda^*(G)$

$$\langle T, \varphi^t \rangle = \lim_n \langle T, \varphi_n^t \rangle = \lim_n \sum_{k=0}^{2^n-1} \hat{\mu}(w_k) \operatorname{tr}(W_k' T) w_k(t) .$$

Let $u(t) = \theta\mathcal{E}'(T)$, for $T \in C_\lambda^*(G)$. Then, as we saw, $u(t)$ is a continuous function on D and by definition $\hat{u}(w_k) = \operatorname{tr}\big(W_k' T)\big)$. Therefore, for every $t \in D$

$$\langle T, \varphi^t \rangle = \lim_n \sum_{k=0}^{2^m-1} \hat{\mu}(w_k) \hat{u}(w_k) w_k(t) = \mu * u(t) .$$

Now let T_α be a bounded net of elements of C_λ^* converging in the weak* topology of $VN(G)$ to a Walsh operator W'. Let $u_\alpha(t) = \theta\mathcal{E}'(T_\alpha)$, then $\mu * u_\alpha \in C(D)$ and $\lim \mu * u_\alpha = \mu * w = \hat{\mu}(w) w(t)$, in the weak* topology of $L^\infty(D)$. (We use here the fact that $\theta\mathcal{E}'$ is weak* continuous, and the fact that convolution by a bounded measure is a weak* continuous linear transformation on $L^\infty(D)$). Suppose now that there is a nonnegligible set $E \subseteq D$ such that, for $t \in E$, $\varphi^t \in A$. Then for all $t \in E$ there exists $\Phi^t \in L^1(\Gamma)$, such that $\varphi^t(x) = \operatorname{tr}(L_x^* \Phi^t)$. If T_α is the net defined above and $t \in E$

$$\lim_\alpha \langle T_\alpha, \varphi^t \rangle = \lim_\alpha \mu * u_\alpha(t) = \lim_\alpha \operatorname{tr}(T_\alpha \Phi^t) = \operatorname{tr}(W \Phi^t) .$$

But, on E, $\mu * u_\alpha$ converges weak* to $\hat{\mu}(w) w(t)$, we conclude that for almost all $t \in E$, $\operatorname{tr}(W' \Phi^t) = \hat{\mu}(w) w(t)$. Since the Walsh system is denumerable we can also conclude that for all W_n', $\operatorname{tr}(W_n' \Phi^t) = \hat{\mu}(w_n) w_n(t)$, for almost all $t \in E$. Let t be such that $\operatorname{tr}(\Phi^t W_n') = \hat{\mu}(w_n) w_n(t)$, for all n. Then $\mathcal{E}_n'(\Phi^t) = \sum_{k=0}^{2^n-1} \hat{\mu}(w_k) w_k(t) W_k'$, and $\mathcal{E}_n'(\Phi^t)$ converges in the norm of $L^1(\Gamma)$ to $\mathcal{E}'(\Phi^t)$. It follows that

$$\theta\mathcal{E}_n(\Phi^t) = \sum_{k=0}^{2^n-1} \hat{\mu}(w_k) w_k(t) w_k(s) \tag{5}$$

must converge in the norm of $L^1(D)$. But this is impossible because μ is singular and so is its translate μ^t, while (5) is a partial sum of the Fourier series of μ^t. We conclude that for almost all t, $\varphi^t \notin A(G)$.

THEOREM 2. *Let β_j be a sequence of real numqers, $-1 \leqslant \beta_j \leqslant 1$, such that* $\lim \beta_j = 0$. *Let μ be the positive measure on D which is the weak* limit of $\prod_{j=1}^{n}(1+\beta_j r_j)$. Let φ^t, for $t \in D$, be the positive definite function obtained, as in Theorem 1. Then $\varphi^t \in B_0$. In particular if $\sum \beta_j^2 = +\infty$, then for almost all $t \in D$, $\varphi^t \notin A$ and $\varphi^t \in B_0$.*

PROOF. Recall that $\varphi^t(x) = \lim_n \varphi_n^t(x)$ on compact sets, where $\varphi_n^t(x) = \operatorname{tr}(L_x^* \Phi_n^t)$ and (because of the special form of μ)

$$\Phi_n^t = \prod_{k=1}^{n} (P + \beta_k r_k(t) R_k'),$$

$$\Phi_{n+1}^t = \Phi_n^t + \beta_n r_n(t) R_n' \Phi_n^t .$$

Let K_n and α_n be the increasing sequence of compact sets and the decreasing sequence of real numbers used in the construction of ψ_n^t. Recall that because of the way we defined the operators W_m' and the sets, K_m,

$$K_m \supseteq \bigcup_{k=0}^{2^m-1} \left\{x \in G\colon |\operatorname{tr}(L_x^* W_k)| \geqslant \frac{\alpha_m}{2^{m-1}}\right\}, \tag{6}$$

and that for all Walsh operators W' appearing in the expansion of $R_m' \Phi_m^t$, one has

$$|\operatorname{tr}(L_x^* W)| < 2^{-2m}, \qquad \text{for } x \in K_m ,$$

which implies

$$|\operatorname{tr}(R_m' \Phi_m^t)| < 2^{-m-1} \qquad \text{for } x \in K_m . \tag{7}$$

We shall prove that, for all positive integers p,

$$|\varphi_{n+p}^t(x)| \leqslant \alpha_{n-1} + 2 \max_{n-1 \leqslant j < n+p} |\beta_j| , \qquad \text{for } x \notin K_n . \tag{8}$$

This clearly implies that $|\varphi^t(x)|$ is arbitrarily small outside a sufficiently large compact set. If $x \notin K_{n+p}$, then, by (5)

$$|\varphi^t_{n+p}(x)| \leq \sum_{k=0}^{2^{n+p}-1} |\mathrm{tr}\,(L_x W'_k)| \leq \alpha_{n+p} \leq \alpha_n .$$

Suppose now that $x \in K_{n+p} \setminus K_n$. Then for some $m > n$, $x \in K_m \setminus K_{m-1}$ and hence $x \in K_{m+i}$, with $i \geqslant 0$.

Therefore, by (7)

$$\begin{aligned}
&\varphi^t_{n+p}(x \leqslant | = |\mathrm{tr}\,(L^*_x \Phi^t_{n+p})| \leqslant \\
&\leqslant |\beta_{n+p-1} r_{n+p-1}(t)\,\mathrm{tr}\,(L^*_x R'_{n+p-1} \Phi^t_{n+p-1}) + \ldots + \beta_m r_m(t)\,\mathrm{tr}\,(L^*_x R'_m \Phi^t_m)| + \\
&+ |\beta_{m-1} r_{m-1}(t)\,\mathrm{tr}\,(L^*_x R'_{m-1} \Phi^t_{m-1})| + |\mathrm{tr}\,(L^*_x \Phi^t_{m-1})| \leqslant \\
&\leqslant |\beta_{n+p-1}| \frac{1}{2^{n+p}} + \ldots + |\beta_m| \frac{1}{2^{m+1}} + |\beta_{m-1}| + \alpha_{m-1} \leqslant 2 \max_{n-1 \leqslant j < n+p} |\beta_j| + \alpha_{n-1} .
\end{aligned}$$

This shows that (8) is true for $x \notin K_n$, and the proof is complete. We should only remark that the condition $\sum_{j=1}^{\infty} \beta_j^2 = +\infty$ implies that the measure is singular [9], therefore, under this hypothesis, by Theorem 1, $\varphi^t \in B_0$ and $\varphi^t \notin A$ for almost all t.

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