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Extremal Units in an Archimedean Riesz Space.

Anthony W. Hager (*) (**) - Lewis C. Robertson (*)

Let A be an archimedean Riesz space (= vector lattice) with distinguished weak unit e_A, and for any $e \in A$, let $X(e)$ be the compact space of e-maximal ideals. A *natural map* $\sigma^e: X(e) \to X(e_A)$ is a continuous extension of the inclusion $X(e) \cap X(e_A) \hookrightarrow X(e_A)$; natural $\sigma_e: X(e_A) \to X(e)$ is defined dually, only for weak units e.

This paper concerns when natural σ^e (or σ_e) exists, and those (A, e_A) such that for every $e \in A$, σ^e (or σ_e) exists. We then call e_A X-strong (or X-costrong). These conditions are treated in terms of the Yosida representation $\hat{A}$ in $D(X(e_A))$.

Some of the results: (2.5 and 3.1) σ^e exists iff $p \neq q$ in $X(e_A)$ implies $a \in A$ with $a \in O_p$ and $e - a \varepsilon O_q$. (§ 6) σ_e exists iff whenever U_1 and U_2 are $\hat{A}$-cozeros in $X(e_A)$ for which there is $\hat{a} \in \hat{A}$ which is $\hat{e}$ on U_1 and 0 on U_2, then $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. (§ 4) e_A is X-strong iff each prime ideal of A contains a unique O_p $\big(p \in X(e_A)$ iff to each open cover of $X(e_A)$ are subordinate finite $\hat{A}$-partitions of every $e \in A$. (§ 5) e_A will be X-strong if e_A is a strong unit, or if A is an l-algebra with identity e_A, or if A has the principal projection property. e_A will be X-costrong if A is Cantor complete or has the principal projection property.

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1. Representation.

We sketch those aspects of the Yoshida representation [Y] which are needed for the sequel. More detail appears in [HR]; see also [LZ].

Let A be an archimedean Riesz space ($=$ vector lattice over the reals R), and let $0 < e \in A$. A Riesz ideal M which is maximal with respect to the property of not containing e will be called e-maximal, and the set of these will be denoted $X(e)$. Any such M is prime, hence A/M is totally ordered (see also I 3, here). Regarding the following, see also 4.5, below.

1.1. Let P be a prime ideal in A, let $A \xrightarrow{q} A/P$ denote the quotient, and let $e \in A^+$. These are equivalent: $P \in X(e)$; there is no nonzero $q(e)$-infinitesimal in A/P; the principal ideal $I(q(e))$ is the smallest nonzero ideal in A/P.

When this occurs, $I(q(e)) = \{tq(e) | t \in R\}$.

Consider the extended reals $\bar{R} = R \cup \{\pm\infty\}$, with the obvious order and topology and partly defined addition and scaler multiplication extending these operations from R.

1.2. (a) Let $M \in X(e)$, with $A \xrightarrow{q} A/M$ the quotient.
Define $\gamma^e_M\colon A \to \bar{R}$ by:

$$\gamma^e_M(a) = t \quad \text{if } q(a) = tq(e) \quad (t \in R);$$

$$\gamma^e_M(a) = +\infty \quad \text{if } 0 < q(a) \notin I(q(e)), \text{ and}$$

$$= -\infty \quad \text{if } 0 > q(a) \notin I(q(e)).$$

(b) Define $\gamma^e\colon A \to \bar{R}^{X(e)}$ by:

$$\gamma^e(a)(M) \equiv \gamma^e_M(a).$$

Now, when X is a topological space, let $D(X)$ denote those continuous $f\colon X \to \bar{R}$ with $\mathfrak{R}(f)$ dense, where $\mathfrak{R}(f) = f^{-1}(R)$. $D(X)$ is a lattice admitting scaler multiplication. For $f, g, h \in D(X)$, $f + g = h$ means that $f(x) + g(x) = h(x)$ for $x \in \mathfrak{R}(f) \cap \mathfrak{R}(g)$. A « Riesz space in $D(X)$ » is a sublattice A with $ra \in A$ when $a \in A$ and $r \in R$, and « closed under addition ».

Let $0 < e \in A$.

1.3. THEOREM. (*a*) In the hull-kernel topology, $X(e)$ is a nonvoid compact Hausdorff space.

(*b*) γ^e is a homomorphism of A onto a Riesz space in $D(X(e))$, with kernel $\gamma^e = e^{\perp}$ and with $\gamma^e(e)$ the constant function 1. So γ^e is an isomorphism iff $e^{\perp} = (0)$, i.e., e is a weak unit.

(*c*) If K_1 and K_2 are disjoint closed sets in $X(e)$, then there is $a \in A$ with $0 \leqslant a \leqslant e$, hence $0 \leqslant \gamma^e(a) \leqslant 1$, and $\gamma^e(a) = 1$ on K_1 and 0 on K_2.

(*d*) Let e be a weak unit and let $\gamma\colon A \to D(X)$ be an isomorphism, with X compact, $\gamma(e) = 1$, and with $\gamma(A)$ separating the points of X. Then there is a homeomorphism $h\colon X(e) \to X$ with $\gamma(a) = \gamma^e(a) \circ h$ for each $a \in A$.

1.4. NOTATION. Throughout the paper, we use the following abbreviations: $A \in \mathfrak{L}$ means that A has a distinguished positive weak unit e_A. For $A \in \mathfrak{L}$ the isomorphic representation $\gamma^{e_A}\colon A \to D(X(e_A))$ is denoted $\hat{A}$. For another $e \in A^+$, we always write γ^e.

2. Natural mappings: topology and functions.

We begin the comparision of representations and maximal ideal spaces. Throughout the section, $A \in \mathfrak{L}$ (which presumes e_A), and $0 < e \in A^+$. We state the results and sketch the development, then proceed to the proofs.

2.1. DEFINITION. A *natural mapping* $\sigma^e\colon X(e) \to X(e_A)$ is a continuous extension of the inclusion $X(e) \cap X(e_A) \hookrightarrow X(e_A)$. (Such a mapping is unique.)

2.2. MAIN LEMMA. Let $Y_e \equiv \operatorname{coz} \hat{e} \cap \mathfrak{R}(e) \subset X(e_A)$, and for $p \in Y_e$, let $\tau(p) = M_p \equiv \{a | \hat{a}(p) = 0\}$. Then

(*a*) Each $M_p \in X(e)$.

(*b*) τ is a homeomorphism of Y_e onto $\operatorname{coz} \gamma^e(e_A) \cap \mathfrak{R}(\gamma^e(e_A))$.

(*c*) $\gamma^e(a) \cdot \tau = (1/e)\hat{a} | Y_e$.

2.3. COROLLARY. A natural map σ^e is exactly a continuous function $\sigma^e\colon X(e) \to X(e_A)$ with $\sigma^e \circ \tau$ the identity on Y_e.

2.4. THEOREM. σ^e exists iff whenever K_1 and K_2 are disjoint closed sets in $X(e_A)$, then there is $a \in A$ with $\hat{a} = \hat{e}$ on K_1, and $\hat{a} = 0$ on K_2.

The condition in 2.4 is reminiscent of the theory of normal topological spaces. The analogy goes quite far:

2.5. PROPOSITION. The following are equivalent (to σ^e exists):

(a) If K_1 and K_2 are disjoint closed sets in $X(e_A)$, then there is $a \in A$ with $\hat{a} = \hat{e}$ on K_1 and $\hat{a} = 0$ on K_2.

(b) If $G_1, G_2 \in \operatorname{coz} \hat{A}$ and $\bar{G}_1 \cap \bar{G}_2 = \emptyset$, then there is $a \in A$ with $\hat{a} = \hat{e}$ on G_1 and $\hat{a} = 0$ on G_2.

(c) If $p \neq q$ in $X(e_A)$, then there is $a \in A$ with $\hat{a} = \hat{e}$ an a neighborhood of p and $\hat{a} = 0$ on a neighborhood of q.

(d) If G an H are open in $X(e_A)$ with $\bar{G} \subset H$, then there is $a \in A$ with $\hat{a} = \hat{e}$ on G and $a = 0$ off H.

In each case, we may take $0 \leqslant a \leqslant e$.

2.6. DEFINITION. A *partition of* e is a family $\psi \subset A$ such that $\bigvee \psi = e$ (supremum in A).

Let $\mathcal{G}$ be an open cover of $X(e_A)$. A family $\psi \subset A$ is *subordinate* to $\mathcal{G}$ if $\operatorname{coz} \hat{\psi}$ refines $\mathcal{G}$. (I.e., for each $f \in \psi$, $\operatorname{coz} \hat{f}$ is contained in some $g \in \mathcal{G}$; we do not assume $\operatorname{coz} \hat{\psi}$ to be a cover).

2.7. PROPOSITION. σ^e exists iff to each (finite) open cover of $X(e_A)$ is subordinate a finite partition of e.

(2.6 and 2.7 are suggested by the proof of 5.5 (a)).

We turn to the proofs.

PROOF OF 2.2. (a) Obviously, M_p is prime. To show e-maximality consider the quotient $q: A \to A/M_p$. Using § 1, we show that $Z(q(e)) = (0)$.

Suppose $a \in A^+$, and $tq(a) \leqslant q(e)$. Then $0 \leqslant q(e - ta)$, and by definition of the order in A/M_p, there is $m \in M_p$ with $m \leqslant e - ta$.

Then $0 = \hat{m}(p) \leqslant \hat{e}(p) - t\hat{a}(p)$. If this holds for every $t \in R$, then $a(p) = 0$ since $\hat{e}(p) \in R$. Thus, $a \in M_p$ and $q(a) = 0$ as desired.

(b) We use the facts that $\operatorname{coz} \hat{A}$ is an open base in $X(e_A)$ and $\operatorname{coz} \gamma^e(A)$ is an base in $X(e)$: τ is continuous from the equations $\tau^{-1} \operatorname{coz} \gamma^e(a) = \operatorname{coz} \hat{a} \cap Y_e$ $(a \in A)$. τ has dense image and is open onto its range from the equations $\tau(\operatorname{coz} \hat{a} \cap Y_e) = \operatorname{coz} \gamma^e(a) \cap \tau(Y_e)$ $(a \in A)$.

τ is one-to-one (and thus a homeomorphism) because $\hat{A}$ separates the points in $X(e_A)$.

We finish the proof of (*b*) below.

(*c*) Let $a \in A$, and $p \in Y_e$. Consider the definition of $\gamma^e(a)(\tau(p)) = \gamma^e(a)(M_p)$ from $A \xrightarrow{q} A/M_p \to \bar{R}$ described in § 1.

Suppose first that $\hat{a}(p) = t \in R$. Then $\hat{e}(p)\hat{a}(p) = t\hat{e}(p)$, or $a - (t/\hat{e}(p))^e \in M_p$: Thus,

$$0 = q\left(a - \frac{t}{\hat{e}(p)}\, e\right),$$

whence

$$0 = \gamma^e(a)(M_p) - \frac{t}{\hat{e}(p)}\,\gamma^e(e)(M_p)\,, \qquad \text{or} \qquad \gamma^e(a)(M_p) = \frac{1}{\hat{e}(p)}\,\hat{a}(p)\,,$$

as desired.

Thus the equation in (*c*) holds on the dense set $\mathfrak{R}(\hat{a}) \cap Y_e$, and by continuity, it holds on Y_e.

(*b*) *continued.* $\tau(Y_e) \subset \operatorname{coz} \gamma^e(e_A) \cap \mathfrak{R}(\gamma^e(e_A))$ follows from the equation in (*c*), with $a = e_A$. For the reverse inclusions let $x \in \operatorname{coz} \gamma^e(e_A) \cap \mathfrak{R}(\gamma^e(e_A))$ and consider $M \equiv \{a | \gamma^e(a)(x) = 0\}$. This is a prime ideal $e_A \notin M$ because $x \in \operatorname{coz} \gamma^e(e_A)$ and an argument as in (*a*) shows M is e_A-maximal, because $x \in \mathfrak{R}(\gamma^e(e_A))$. Thus $M = M_p$ for unique $p \in X(e_A)$, by § 1. It follows that $x = \tau(p)$.

Proof of 2.1 (uniqueness) and 2.3. By 2.2, we may identify Y_e and $\tau(Y_e)$ as subspaces of, say, the prime ideal space with the hull kernel topology and this space is $X(e) \cap X(e_A)$, which is dense in $X(e)$ by 2.2 (*b*).

Thus σ^e is unique when it exists. Upon the identification, the inclusion $X(e) \cap X(e_A) \hookrightarrow x(e_A)$ restricted to $\tau(Y_e)$ is τ^{-1}. Thus 2.3.

2.8. Lemma ([E], p. 110). Let Y be dense in X' and let $i\colon Y \to X$ be continuous, with X compact. There is a continuous extension $\sigma\colon X' \to X$ iff whenever K_1 and K_2 are disjoint closed sets in X, then $i^{-1}(K_1)$ and $i^{-1}(K_2)$ have disjoint closures in X'.

2.9. Lemma. Let $A \in \mathfrak{L}$, $0 < e \in A^+$, and let $K_1, K_2 \subset X(e_A)$. These are equivalent:

(*a*) The closures in $X(e)$ of $\tau(K_1)$ and $\tau(K_2)$ are disjoint.

(*b*) There is $a \in A$ with $\gamma^e(a) = 1$ on $\tau(K_1)$ and $\gamma^e(a) = 0$ on $\tau(K_2)$.

(*c*) There is $a \in A$ with $\hat{a} = \hat{e}$ on K_1 and $\hat{a} = 0$ on K_2.

PROOF. Any Yosida representation separates closed sets (§ 1), so $(a) \Leftrightarrow (b)$. The equation (c) in 2.1 shows $(b) \Leftrightarrow (c)$.

PROOF OF 2.4. By 2.3, 2.8, and 2.9 (using τ^{-1} as the i of 2.8).

PROOF OF (2.5). $(a) \Rightarrow (b)$. Obvious.

$(b) \Rightarrow (c)$. Choose $\operatorname{coz}\hat{A}$-neighborhoods with disjoint closures.

$(c) \Rightarrow (a)$. Let K_1 and K_2 be given. For each $p \in K_1$ and $q \in K_2$, choose $a_q^p \in A$ with

$$a_q^p = \begin{cases} \hat{e} & \text{on a neighborhood } G_q^p \text{ of } p\,, \\ 0 & \text{on a neighborhood } H_q^p \text{ of } q\,. \end{cases}$$

Fix p. $\{H_q^p | q \in K_2\}$ covers K_2, and so does a finite subset $\{H_{q_i}^p\}$. Let $a^p \equiv \bigwedge_i a_{q_i}^p$. Then

$$a^p = \begin{cases} \hat{e} & \text{on } G^p \equiv \bigcap_i G_{q_i}^p\,, \\ 0 & \text{on } K_2\,. \end{cases}$$

Then $\{G^p | p \in K_1\}$ covers K_1, and so does a finite subset $\{G^{p_i}\}$. Let $a \equiv \bigvee_i a^{p_i}$, so that

$$a = \begin{cases} \hat{e} & \text{on } \bigcup_i G^{p_i} \supset K_1\,, \\ 0 & \text{on } K_2\,. \end{cases}$$

$(d) \Leftrightarrow (a)$ is routine.

Finally, in any of the conditions we may replace a by $a_1 = |a| \wedge e$. Then a_1 also works and $0 \leqslant a_1 \leqslant e$.

PROOF OF 2.7. Suppose σ^e exists, let $\mathcal{G}$ be an open cover and $\{G_i\}$ a finite subcover. Let $\{W_i\}$ be a « shrinkage » of $\{G_i\}$: an open cover with $\overline{W}_i \subset G_i$ for each i. (Such exists by [E], p. 266). For each i, choose $a_i \in A$ with $a_i = e$ on W_i and $a_i = 0$ off of G_i and $0 \leqslant a_i \leqslant e$ (by 2.5 (d)). Then $\{a_i\}$ is a finite partition of a subordinate to $\{G_i\}$ and to $\mathcal{G}$. To show $\bigvee_i \hat{a}_i = \hat{e}$: $\hat{e} \geqslant \bigvee_i \hat{a}_i$ because $\hat{e} \geqslant$ each $\hat{a}_i$. And, if $\varepsilon X(e_A)$, then $x \in$ some $W_i \subset G_i$, and $\hat{e}(x) = \hat{a}_i(x)$, hence $e(x) \leqslant \bigvee_i a_i(x)$.

Conversely, let G, H be open with $\overline{G} \subset H$. Then $\{X - \overline{G}, H\}$ is an open cover, and we assume there is a finite partition ψ of e subordinate. Let $a = \bigvee \{f \in \psi | \operatorname{coz} \hat{f} \subset H\}$. Then a satisfies 2.5 (d).

3. Natural mappings: ideals.

We find some what more algebraic conditions that σ^e exists in terms of certain ideals of A. We state the results, then proceed to the proofs.

3.1. Let $A \in \mathfrak{L}$, let $p \in X(e_A)$, and define

$$O_p \equiv \{a \in A \mid \hat{a} = 0 \text{ on a neighborhood of } p\} .$$

Then O_p is an ideal, $e_A \notin O_p$, and M_p is the unique e_A-maximal ideal containing O_p.

3.2. THEOREM. Let $A \in \mathfrak{L}$, and let $0 < e \in A^+$.

(*a*) There is natural σ^e: $X(e) \to X(e_A)$ iff each $M \in X(e)$ contains a unique O_p $(p \in X(e_A))$.

(*b*) Assuming this, $\sigma^e(M) = p$ iff $M \supseteq O_p$.

The point in (*a*) is the *uniqueness* of O_p as the following shows.

3.3. PROPOSITION. Let $A \in \mathfrak{L}$. Each prime ideal of A contains on O_p.

3.2 (*b*) says that the condition in (*a*) provides a canonical description of σ^e. Another such comes from the following, applied to $M \in X(e)$.

(3.4 was obtained jointly with Giuseppe De Marco).

3.4. THEOREM. Let $A \in \mathfrak{L}$, let $p \in X(e_A)$, and let P be a prime ideal of A. Then $P \supseteq O_p$ iff P is comparable with M_p.

3.5. COROLLARY. These are equivalent.

(*a*) σ^e exists.

(*b*) Each prime P with $e \notin P$ contains unique O_p (or is comparable with unique M_p) for $p \in X(e_A)$.

(*c*) Each $M \in X(e)$ is comparable with unique $M' \in X(e_A)$.

(*d*) Each $M \in X(e)$ with $e_A \in M$ contains unique $M' \in X(e_A)$;

3.4 and 3.5 can be used to derive conditions on mappings of prime ideal spaces. We postpone such a discussion to a later paper, as it would curry us too far afield.

We turn to the proofs.

PROOF OF 3.1. $O_p \subseteq M_q \Rightarrow p = q$.

3.6. Let $p \in X(e_A)$. Define

$$X^p(e) \equiv \{M \in X(e) | M \supseteq O_p\} .$$

Then, clearly,

$$p \in Y_e \Rightarrow e \notin O_p \Leftrightarrow X^p(e) \neq \emptyset.$$

3.7. LEMMA. Let $p, q \in Y_e$:

(a) $X^p(e) = \bigcap \{\overline{\tau(G)} | G$ a neighborhood of $p\}$.

(b) $X^p(e) \cap X^q(e) = \emptyset$ iff there is $a \in O_p$ with $e - a \in O_q$.

PROOF. Recall from 2.1 that

$$(*) \qquad \gamma^e(a) \circ \tau = \frac{1}{e} \hat{a} | Y_e .$$

We shall use this several times.

(a) Let $M \in X(e)$, and let G always be a neighborhood of p.

If $M \in X^p(e) - \overline{\tau(G)}$, then there is $a \in A$ with $0 \leqslant a \leqslant e$, $\gamma^e(a) = 0$ on $\tau(G)$ and $\gamma^e(a)(M) = 1$. ($*$) shows that $\hat{a} = 0$ on $G \cap Y_e$. Since $0 \leqslant a \leqslant e$, we have $\hat{a} = 0$ on G. Thus, $a \in O_p \subseteq M$, contradiction.

Suppose $M \supseteq O_p$, so there is $a \; \varepsilon O_p - M$. Then there is G with $\hat{a} = 0$ on G, and ($*$) shows that $\gamma^e(a) = 0$ on $\tau(G \cap Y_e)$, hence on $\overline{\tau(G)}$. Since $a \notin M$, $\gamma^e(a)(M) \neq 0$ and $M \notin \overline{\tau(G)}$.

(b) Given such a, there are neighborhoods G, H of p, q with $\hat{a} = 0$ on G, $\hat{a} = \hat{e}$ on H. Using ($*$) as before, it follows that $\gamma^e(a) = 0$ on $\overline{\tau(G)}$ and $\gamma^e(a) = 0$ on $\overline{\tau(H)}$. So $\overline{\tau(G)}$ and $\overline{\tau(H)}$ are disjoint, hence so are $X^p(e)$ and $X^q(e)$, by (a).

Let $X^p(e) \cap X^q(e) = \emptyset$. We claim there are neighborhoods G, H of p, q respectively, with $\overline{\tau(G)} \cap \overline{\tau(H)} = \emptyset$. Then choose $a \in A$ with $0 \leqslant a \leqslant e$, $\gamma^e(a) = 0$ on $\overline{\tau(G)}$ and $\gamma^e(a) = 1$ on $\overline{\tau(H)}$. Using ($*$) as usual, we get $a \in O_p$, $e - a \in O_q$. To obtain such G, H: If for all such G, H, $\overline{\tau(G)} \cap \overline{\tau(H)} \neq \emptyset$, then $\mathfrak{I} \equiv \{\overline{\tau(G)} \cap \overline{\tau(H)} | G, H\}$ has the finite intersection property and there is $M \in \bigcap \mathfrak{I}$ by compactness. Such $M \in X^p(e) \cap X^q(e)$ by (a), a contradiction.

This completes the proof of 3.3.

PROOF OF 3.2. (a) follows immediately from 3.7 and 2.6.

(b) Let $\sigma^e(M) = p$. If G is a neighborhood of p, then $M \in (\sigma^e)^{-1}(G)$. Now, $(\sigma^e)^{-1}(G) \cap Y_e = \tau(G)$, and this set is dense in $(\sigma^e)^{-1}(G)$. Thus $M \in \tau(G)$. Since this is true for every G, $M \in X^p(e)$ by 3.7.

If $M \in X^p(e)$, let $\sigma(M) = q$. The preceding shows that $M \in X^p(e)$. By 3.2 (a), $p = q$.

3.8. LEMMA [LZ]. Let A be a Riesz space and P an ideal. These are equivalent.

(a) $a \wedge b \in P \Rightarrow a \in P$ or $b \in P$ (P is prime).

(b) $|a| \wedge |b| = 0 \Rightarrow a \in P$ or $b \in P$:

(c) A/P is totally ordered.

(d) The set of ideals containing P is totally ordered by set-inclusion.

PROOF OF 3.3. Suppose $O_q \not\subset P$ for each q. Then, for each q, there is $a_q \notin P$ with $\hat{a}_q = 0$ on a neighborhood G_q of q. From $\{G_q | q \in X(e_A)\}$, we extract the finite subcover $\{G_{q_i}\}$. Then $\bigwedge_i |a_{q_i}| = 0 \in P$, and by 3.8, P is not prime.

The following interesting lemma was contributed by Giuseppe De Marco, considerally simplifying our proofs and essentially extending part of 3.5 to 3.4.

3.9. LEMMA (De Marco). Let $q \in X(e_A)$. If P is a prime ideal of A with $O_q \subseteq P$, then there is a prime ideal Q with $Q \subseteq P$ and $Q \subseteq M_q$.

PROOF. First, let S be any subset of positive elements (of any Riesz space) such that $0 \notin S$ and $u, v \in S \Rightarrow u \wedge v \in S$. Then (with an argument by Zorn's lemma), there is an ideal Q which is maximal with respect to the property $Q \cap S = \emptyset$. And Q is prime: if $u \wedge v \in Q$, then one of u, v is not in S; say $u \notin S$. Then $u \in Q$, for if not, the ideal generated by Q and u still misses S and contradicts maximality of Q.

Now let P be prime, $O_q \subseteq P$. Let $S_1 = (A - M_q)^+$, $S_2 = (A - P)^+$. These are «meet-closed» because M_q and P are prime ideals. Then $S = S_1 \cup S_2 \cup \{u \wedge v | u \in S_1, v \in S_2\}$ is meet-closed too. Also $0 \notin S$: Certainly $0 \notin S_1 \cup S_2$. Suppose $0 = u \wedge v$ for $u \in S_1$. Then $0 = \hat{u} \wedge \hat{v}$ (identically in $D(X(e_A))$). Since $u \notin M_q$, $\hat{u}(q) \neq 0$, and it follows that $\hat{v}$ is 0 on a neighborhood of q, i.e., $v \in O_q$, so $v \notin S_2$.

Applying the first paragraph to this S products the desired prime Q.

PROOF OF 3.4. Let $P \supseteq O_p$. By 3.9, choose prime Q with $Q \subseteq P$, M_p. By 3.8 then, P and M_p are comparable.

Conversely, let P and M_p be comparable. If $P \supseteq M_p$, certainly $P \subseteq O_p$. When $P \subseteq M_p$, choose q with $P \subseteq O_q$ by 3.3. Then $O_q \subseteq M_p$, and $q = p$ follows.

3.10. For any subset M of A: M is comparable with M_p iff either $e_A \notin M$ and $M \subseteq M_p$, or $e_A \in M$ and $M \supset M_p$ properly.

Proof of 3.5. Each $M \in X(e)$ is prime, of course: $(a) \Leftrightarrow (c)$ by 3.2 and 3.4. $(c) \Leftrightarrow (d)$ by 3.10. The two parts of (b) are equivalent by 3.4. $(b) \Rightarrow (c)$, clearly.

$(c) \Rightarrow (b)$. Let $e \notin P$. By 3.3, $P \supseteq$ some O_p. Suppose also that $O_q \subseteq P$. Now, P is contained in unique $M \in X(e)$ (by Zorn's lemma and 3.8). So $O_p, O_q \subseteq M$. By (c) (and 3.4 and 3.2), $p = q$.

4. X-strong units.

This section is essentially a summary of conditions on A and e_A such that σ^e's always exists.

4.1. Definition. Let $A \in \mathfrak{L}$. If for each $e \in A^+$, natural $\sigma^e\colon X(e) \to X(e_A)$ exists, we call e_A an *X-strong unit.*

We are not convinced that the terminology is the best. The motivation is that such an e_A behaves like a strong unit with respect to the spaces $X(e)$:

4.2. Proposition. (a) If $te_A \geqslant e$ for some $t \in R^+$, then σ^e exists.

(b) A strong unit is X-strong.

Proof. (a) Given $\bar{G} \subset H$, choose $u \in A$ with $0 \leqslant \hat{u} \leqslant 1$, $\hat{u} = 1$ on G and $\hat{u} = 0$ off H (from § 1). Then $a = tu \wedge e$ (when $te_A \geqslant e$) satisfies $\hat{a} = \hat{e}$ on G, $\hat{a} = 0$ off H.

(b) follows from (a).

4.3. Proposition. These conditions on $A \in \mathfrak{L}$ are equivalent.

(a) e_A is X-strong.

(b) σ^e exists $\forall$ weak unit $e \in A^+$.

(c) σ^e exists $\forall e \geqslant e_A$.

(d) $\forall e \geqslant e_A$, the natural map $\sigma_e^{e_A}\colon X(e_A) \to X(e)$ (existing by 4.2) is a homeomorphism.

PROOF. $(a) \Rightarrow (b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (d)$. Assuming (c), $\sigma^e \circ \sigma_e^{e_A}$ and $\sigma_e^{e_A} \circ \sigma^e$ are identities on dense sets, hence identities because the spaces are Hausdorff. So each is a homeomorphism.

$(d) \Rightarrow (c)$. $\sigma^e = (\sigma_e^{e_A})^{-1}$.

$(c) \Rightarrow (a)$. Let $0 < e \in A^+$, and let $\bar{G} \subset H$. Since $e_1 = e \vee e_A \geqslant e_A$, there is σ^{e_1} and hence there is $a_1 \in A$ with $\hat{a}_1 = \hat{e}_1$ on G and $\hat{a}_1 = 0$ off H. Since $e_1 \geqslant e$, we have $e_1 \wedge e = e$ and $a_1 \wedge e$ works.

4.3 will be useful later. The following just restates part of § 3.

4.4. THEOREM. These conditions on $A \in \mathfrak{L}$ are equivalent.

(a) e_A is X-strong.

(b) Whenever $e \in A$ and $G_1, G_2 \subset X(e_A)$ (which may be assumed arbitrary, open, or in $\operatorname{coz} \hat{A}$, with $G_1 \cap G_2 = \emptyset$, then there is $a \in A$ with $\hat{a} = \hat{e}$ on G_1 and $\hat{a} = 0$ on G_2.

(c) Whenever $e \in A$ and G, H are open (or in $\operatorname{coz} \hat{A}$) with $\bar{G} \subset H$, then there is $a \in A$ with $\hat{a} = \hat{e}$ on G and $\hat{a} = 0$ off H.

(d) Whenever $a \in A$ and $\mathcal{G}$ is an open cover of $X(e_A)$, then there is a finite partition of e subordinate to $\mathcal{G}$.

(e) Whenever $e \in A$ and $p \neq q$ in $X(e_A)$, then there is $a \in A$ with $a \in O_p$ and $e - a \in O_q$.

Finally, there are the more algebraic conditions from § 3. For a better statement of the results, we insert a preliminary

4.5. LEMMA (cf. 1.1). Let A be a Riesz space and M an ideal. These are equivalent.

(a) There is $e \in A$ with $M \in X(e)$.

(b) A/M is totally ordered, and there is $x \in A/M$ such that 0 is the only x-infinitesimal.

(c) A/M is a subdirectly irreducible Riesz space.

(d) In A/M there is a smallest nonzero ideal.

(e) If $\mathcal{G}$ is a family of ideals in A each properly containing M, then $\cap\, \mathcal{G}$ properly contains M.

We call such an M completely meet-irreducible (cmm).

PROOF (sketch). $(a) \Leftrightarrow (b)$. Use $x = e + M$ (and 3.8; see § 1).

$(b) \Rightarrow (d)$. The principal ideal generated by x.

$(c) \Leftrightarrow (c)$. See the treatment in [B] of subdirectly irreducible abstract algebras.

$(d) \Leftrightarrow (e)$. From the correspondence between ideals in A/M and ideals in A containing M.

From 4.5 and § 3, we have immediately.

4.6. THEOREM. These conditions on A are equivalent.

(a) e_A is X-strong.

(b) Each proper prime ideal contains unique O_p (or, is comparable with unique M_p) for $p \in X(e_A)$.

(c) Each cmm ideal contains unique O_p (or, is comparable with unique M_p) for $p \in X(e_A)$.

5. X-strong units versus other properties.

We recall some definitions and relevant facts:

5.1. Let $A \in \mathfrak{L}$. A is called a *Φ-algebra* [HJ] if e_A is the identity for an f-ring multiplication on A. It is shown in [HR] that when A is a Φ-algebra, the Riesz isomorphism $A \to \hat{A} \subseteq D(X(e_A))$ preserves the multiplication.

Let $A \in \mathfrak{L}$. A is called *convex* [AH] if $\hat{A}$ is a convex subset of $D(X(e_A))$, that is, if $f \in D(X(e_A))$ and $|f| \leqslant \hat{a}$ for some $a \in A$ imply $f \in A$.

If A is convex and $a \in A$, then $\mathfrak{R}(\hat{a})$ is C^*-embedded in $X(e_A)$ and A is e_A-uniformly complete, whence [HR] $\hat{A}^* = C(X(e_A))$.

Let A be a Riesz space. A has the *principal projection property*, or *ppp*, if for each $a \in A$, $A = a^{\perp\perp} \oplus a^{\perp}$. Then, given $f \in A$, $f = p_a f + b$ with $p_a f \in a^{\perp\perp}$ and $b \in a^{\perp}$, uniquely. See Chapt. 4 of [LZ]. Such A is archimedean. For $A \in \mathfrak{L}$, if *follows easily from* 24.9 of [LZ] that A has the *ppp* iff for each $a \in A$, $\overline{\operatorname{coz} \hat{a}}$ is open. Then, $(p_a f)^\wedge = \hat{f}$ on $\overline{\operatorname{coz} \hat{a}}$ and 0 off $\overline{\operatorname{coz} \hat{a}}$.

5.2. THEOREM. Let $A \in \mathfrak{L}$. Any of the following imply that e_A is X-strong: A is a Φ-algebra; A is convex; e_A is a strong unit; A has the *ppp*.

PROOF. Let $a \in A^+$, and using 2.6, let G, H be open with $\bar{G} \subset H$. Choose $u \in A$ with $\hat{u} = 1$ on G, 0 off H, and $0 \leqslant u \leqslant e_A$.

If A is a Φ-algebra, then $ua \in A$, and $(ua)^\wedge = \hat{u}\hat{a}$ is the desired function.

If A is convex, $\hat{u}\hat{a}$ is continuous on $\mathfrak{R}(\hat{a})$, hence extends to $f \in D(X(e_A))$. Clearly, $0 \leqslant f \leqslant \hat{a}$, so by convexity $f \in \hat{A}$. And f is the desired function.

If e_A is strong, then $a \leqslant te_A$ for some real t. Then $t\hat{u} \wedge \hat{a}$ is the desired function.

If A has the *ppp*, then we resort to assuming that $G \varepsilon \operatorname{coz} A$ (*per* 4.4 (*c*)). Thus $\bar{G}$ is open. Then the function $f = \hat{a}$ on G and $f = 0$ off G is in $\hat{A}$ (*per* 5.1; f is a certain $(p_b a)^\wedge$), and serves the purpose.

REMARK. If A is either a Φ-algebra or convex, then $\hat{A}^* \cdot \hat{A} \subseteq \hat{A}$ (see [AH]) and as the proof above shows, this property implies that e_A is X-strong.

5.3. DEFINITION. Let $A \in \mathfrak{L}$. Let $\operatorname{loc} A$ be the set of functions which are locally in A, that is, $f \in \operatorname{loc} A$ iff $f\colon X(e_A) \to \bar{R}$ is a function such that for each $p \in X(e_A)$ there are a neighborhood G of p and $a \in A$ such that $f = \hat{a}$ on G.

If $\operatorname{loc} A = A$, we call A *local.*

5.4. REMARKS. Note that $\operatorname{loc} A$ is a Riesz space in $D(X(e_A))$. So A is local iff $(\operatorname{loc} A)^+ \subseteq \hat{A}$.

By compactness, if $f \in \operatorname{loc} A$, then there is a *finite* open cover $\{G_i\}$ of $X(e_A)$ and $\{a_i\} \subseteq A$, with $f = \hat{a}_i$ on G_i for each i.

Each $\mathfrak{L}$-morphism (see § 1) $\varphi\colon A \to L$ with L local extends to an $\mathfrak{L}$-morphism $\bar{\varphi}\colon \operatorname{loc} A \to L$ (using [HR]). Thus $A \hookrightarrow \operatorname{loc} A$ is what is called a *reflection* in category theory.

5.5. THEOREM. Let $A \in \mathfrak{L}$.

(*a*) If e_A is X-strong, then A is local.

(*b*) If A is local and $X(e_A)$ is totally disconnected, then e_A is X-strong.

PROOF. (*a*) Let $f \in (\operatorname{loc} A)^+$. For each $p \in X(e_A)$, choose a neighborhood G_p and $a_p \in A$ with $f = \hat{a}_p$ on G_p. Let $\{G_i\}$ be a finite subcover of $\{G_p | p \in X(e_A)\}$, with $\{a_i\}$ the associated elements of A. We may take $\{a_i\} \subseteq A^+$.

The finite cover $\{G_i\}$ has a « shrinkage » by [E], p. 266: A finite open cover $\{W_i\}$ with $\bar{W}_i \subset G_i$ for each i.

For each i, choose $b_i \in A$ with $0 \leqslant b_i \leqslant a_i$, $\hat{b}_i = \hat{a}_i$ on W_i and $\hat{b}_i = 0$ off of G_i. This is possible because e_A is X-strong. Note that $0 \leqslant \hat{b}_i \leqslant f$: Off G_i, $\hat{b}_i(x) = 0 \leqslant f(x)$, and on G_i, $\hat{b}_i(x) \leqslant \hat{a}_i(x) = f(x)$.

Then $f = \bigvee\limits_i \hat{b}_i$ as in the proof of 2.7.

(*b*) Let $a \in A$ and let G and H be open with $\bar{G} \subset H$. A compactness argument products clopen C with $\bar{G} \subset C \subset H$. Let $f = \hat{a}$ on C, O off of C. Since C is clopen, $f \in \operatorname{loc} A = A$.

For X a Hausdorff uniform space, let $U(X)$ be the Riesz space of all uniformly continuous functions to the reals R (R having the usual uniformity), with weak unit 1. (See [I]).

5.6. PROPOSITION. (*a*) Any $U(X)$ is local.

(*b*) In $U(R)$, 1 is not X-strong.

5.7. LEMMA. The Yosida representation of $U(X)$ is extension over the Samuel compactification sX.

PROOF. Essentially by definition, sX is the « compact reflection » of X in Hausdorff uniform spaces: there is a uniformly continues dense homeomorphism $s_x\colon X \to sX$ such that whenever $f\colon X \to K$ is uniform with K compact, there is unique $sf\colon sX \to K$ with $(sf) \circ s_x = f$. See [I].

Let $\hat{A} = \{sf \mid f \in U(X)\}$. It follows that $\hat{A}^* = C(sX)$, hence A separates the points of sX. Since $1 \in \hat{A}$, from 1.3 we see that $sX = X(1)$ and $\hat{A}$ is the Yosida representation.

PROOF OF 5.6. (*a*) Let $f \in \operatorname{loc} U(X)$. We are to show that given $\varepsilon > 0$, $f^{-1}\mathcal{S}(\varepsilon)$ is a uniform cover, where $\mathcal{S}(\varepsilon)$ is the cover of R consisting of ε-balls. (We are using the covering description of uniform spaces *per* [I]).

There is a finite cover $\{G_i\}$ of sX and $\{a_i\} \subseteq U(X)$ such that $f = \hat{a}_i$ on G_i. Thus each $f|G_i \cap X$ is uniformly continuous, and so there is a uniform cover $\mathcal{U}_i$ such that $f^{-1}\mathcal{S}(\varepsilon)|G_i > \mathcal{U}_i|G_i$ (the notation meaning the cover traced on the subset; $>$ means « is refined by ».) Thus, $f^{-1}\mathcal{S}(\varepsilon) > \{G_i\} \wedge \bigwedge\limits_i \mathcal{U}_i$ (where « $\wedge$ » means « least common refinement).

Now $\{G_i\}$ is an open cover of compact sX, hence uniform, and its trace on X is uniform. Since $\bigwedge\limits_i \mathcal{U}_i$ is uniform, so is $f^{-1}\mathcal{S}(\varepsilon)$.

(*b*) We use 2.5. Ler $\hat{f}$ be the extension of $|\sin x|$ over $sR = X(1)$. Let $K_1 = \{x | \hat{f}(x) = 1\}$, $K_2 = \{x | \hat{f}(x) = 0\}$. Let $\hat{a}$ be the extension of

$a(x) = x$. There is no $g \in A$ with $\hat{g} = \hat{a}$ on K_1 and $\hat{g} = 0$ on K_2. Because, for such g, $g = \hat{g}|X$ would be « x » on $\{(2n+1)\pi/2 | n$ integral$\}$ and 0 on $\{n\pi | n$ integral$\}$, and therefore not uniformly continuous.

6. X-costrong units.

We discuss those $A \in \mathfrak{L}$ for which e_A has the property « dual » to being X-strong.

6.1. Definition. e_A is X-costrong if there is a natural mapping $\sigma_e\colon X(e_A) \to X(e)$ whenever e is a positive weak unit.

The condition is « dual » to 4.3 (b). It doesn't make sense to postulate such σ_e when e is not a weak unit: the existence of σ_e implies $e^\perp = e_A^\perp$ (using 2.2).

6.2. A natural $\sigma_e\colon X(e_A) \to X(e)$ is exactly an extension of the $\tau\colon Y_e \to X(e)$ of 2.2.

Proof. Such σ_e is (by 2.3) a function with $\sigma_e \circ \tau'$ the identity on a certain subset of $X(e)$, where τ' is as in 2.2 with e and e_A interchanged. By 2.2, τ itself is $(\tau')^{-1}$.

One can get a lot of properties equivalent to the existence of σ_e by interchanging e and e_A in the results of §'s 2 and 3. This interchanging can get confusing, and we shall be content with essentially one condition anyhow (the converse of 2.5 (b)); so we proceed directly from 2.8 and 2.9.

6.2. Proposition. Let $A \in \mathfrak{L}$ and let $e \in A^+$ be a weak unit. Then these are equivalent.

(a) Natural $\sigma_e\colon X(e_A) \to X(e)$ exists.

(b) If $U_1, U_2 \in \operatorname{coz} \hat{A}$ and there is $a \in A$ with $\hat{a} = \hat{e}$ on U_1 and $\hat{a} = 0$ on U_2, then $\overline{U}_1 \cap \overline{U}_2 = \emptyset$.

Proof. By 2.4, there is $a \in A$ with $\hat{a} = \hat{e}$ on U_1 and $\hat{a} = 0$ on U_2 iff $\tau(U_1)$ and $\tau(U_2)$ have disjoint closures in $X(e)$.

(a) $\Rightarrow$ (b). Let U_1, U_2, and a be as in (b). If σ_e exists, then by 2.8, $\tau^{-1}\tau(U_1)$ and $\tau^{-1}\tau(U_2)$ have disjoint closures in $X(e_A)$. But $\overline{U}_i = \overline{\tau^{-1}\tau(U_i)}$ since $U_i \cap Y_e = \tau^{-1}\tau(U_i)$ and Y_e is dense,

$(b) \Rightarrow (a)$. Applying 2.8, let K_1 and K_2 be closed and disjoint in $X(e)$. Choose $a_i \in A$ with $K_i \subset \operatorname{coz} \gamma^e(a_i)$ and with the closures in $X(e)$ of $\operatorname{coz} \gamma^e(a_i)$ disjoint. Let $U_i = \operatorname{coz} \hat{a}_i$. Using (c) 2.2, $\tau(U_i) = \tau(Y_e) \cap \cap \operatorname{coz} \gamma^e(a_i)$; and $\overline{\tau(U_i)} = \overline{\operatorname{coz} \gamma^e(a_i)}$. Thus, $\overline{\tau(U_1)} \cap \overline{\tau(U_2)} = \emptyset$ and by 2.9 there is $a \in A$ with $\hat{a} = \hat{e}$ on U_1 and $\hat{a} = 0$ on U_2. By (b), $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Now $\overline{U}_1 \subset \overline{\tau(K_1)}$; 2.8 yields σ_e.

6.2 immediately gives a workable condition that e_A be X-costrong. The following makes the statement of the result more concise.

6.3. Terminology. Let $U_1, U_2 \in \operatorname{coz} \hat{A}$. U_1 and U_2 are *adjacent* if $\overline{U}_1 \cap \overline{U}_2$ is nonempty with empty interior.

Let $a_1, a_2 \in A$. We say a_1 *is adjacent to* a_2 if there are adjacent U_1, U_2, and $a \in A$, with $\hat{a} = a_1$ on U_1 and $\hat{a} = \hat{a}_2$ on U_2.

Thus, immediately from 6.2:

6.4. Theorem. Let $A \in \mathfrak{L}$. These conditions are equivalent.

(a) e_A is X-costrong.

(b) No weak unit is adjacent to 0.

(c) If a is adjacent to 0, then $\hat{a} = 0$ on some nonvoid open set in $X(e_A)$.

6.5. Corollary. Let $A \in \mathfrak{L}$. The following are equivalent, and implied by « e_A is X-costrong ».

(a) If e is a weak unit, then $\overline{\operatorname{pos} \hat{e}} \cap \overline{\operatorname{neg} \hat{e}} = \emptyset$.

(b) If $U_1, U_2 \in \operatorname{coz} \hat{A}$, $U_1 \cap U_2 = \emptyset$, and $U_1 \cap U_2$ is dense, then $\overline{U}_1 \cap \overline{U}_2 = \emptyset$.

(c) If $U_1 \in \operatorname{coz} A$ is complemented (meaning: there is $U_2 \in \operatorname{coz} \hat{A}$ with $U_1 \cap U_2 = \emptyset$, $U_1 \cup U_2$ dense), then $\overline{U}_1$ is open.

(d) If $a, b \in A$, then either $\{x | \hat{a}(x) = b(x)\}$ has interior, or $\{x | \hat{a}(x) \, \overline{G} \, \hat{b}(x)\}$ has open closure.

Proof. Let e_A be X-costrong, and let e be a weak unit. Then $|e|$ is a positive weak unit. Let $a = e^+$. Then clearly, $\hat{a} = |\hat{e}|$ on $\operatorname{pos} \hat{e}$ and $\hat{a} = 0$ on $\operatorname{neg} \hat{e}$. Of course, $\overline{\operatorname{pos} \hat{e}} \cap \overline{\operatorname{neg} \hat{e}} = \emptyset$ by 6.2 (b) (or 6.4 (b)).

$(a) \Rightarrow (b)$. $U_i = \operatorname{coz} \hat{a}_i$ for $a_i \geqslant 0$. Then apply (a) to $e = a_1 - a_2$.

$(b) \Rightarrow (c)$. By (b), $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Since $\overline{U}_1 = X(e_A) - \overline{U}_2$, $\overline{U}_1$ is open.

$(c) \Rightarrow (d)$. If $\operatorname{int}\{x|\hat{a}(x) = \hat{b}(x)\} = \emptyset$, then $\{x|\hat{a}(x) > \hat{b}(x)\}$ « is complemented by » $\{x|\hat{a}(x) < \hat{b}(x)\}$.

$(d) \Rightarrow (a)$. Apply (d) to $a = e \vee 0$ and $b = (-e) \vee 0$. Then $Z(\hat{e}) = \{x|\hat{a}(x) = \hat{b}(x)\}$ has no interior, so $\operatorname{pos} \hat{e} = \{x|a(x) > \hat{b}(x)\}$ has open closure. Since $\overline{\operatorname{pos} \hat{e}} \subset \operatorname{pos} \hat{e} \cup Z(e)$, we have $\overline{\operatorname{pos} \hat{e}} \cap \operatorname{neg} e = \emptyset$. Since $\overline{\operatorname{pos} \hat{e}}$ is open, (a) follows.

The converse to 6.5 fails. We postpone the examples to a later paper treating the ideas of this section with more care.

6.6. A topological space is called *quasi-F* [DHH] if each dense cozero set is C^*-embedded. This is equivalent to each of: [HJ] $D(X)$ is a Riesz space; $C(X)$ (or $D(X)$) is Cantor complete (Dashiell), where a Riesz space A is called *Cantor complete* (Everett, Papangelou) if each order-Cauchy sequence order-converges, where $\{a_n\}$ is called order-Cauchy if there is $\{u_n\}$ with $u_1 \geqslant u_2 \geqslant \ldots \geqslant 0$ with $\bigwedge_n u_n = 0$ such that for each n, $|a_n - a_{n+p}| \leqslant u_n$ for all $p \geqslant 0$; and order-convergence is similarly defined.

Every Riesz space (archimedean or not) has a Cantor completion. It was shown in [AH], and independently by Dashiell, that for $A \in \mathfrak{L}$, A is Cantor complete iff $X(e_A)$ is quasi-F and A is convex $\big(=$ an ideal in the Riesz space $D(X(e_A))\big)$.

6.7. Corollary. Let $A \in \mathfrak{L}$. If A is Cantor complete, or if only $X(e_A)$ is quasi-F, then e_A is X-costrong.

Proof. If e is a weak unit, then Y_e is a dense cozero set, hence C^*-embedded and hence $\tau\colon Y_e \to X(e)$ has the extension $\sigma_e\colon X(e_A) \to (e)$.

6.8. There is $A \in \mathfrak{L}$ with e_A X-costrong, but $X(e_A)$ not quasi-F.

A class of examples is as follows: Let Y be a compact totally disconnected space, and let A consist of all locally constant functions on Y, with $e_A = 1$. (Otherwise put, A is the linear span of the continuous characteristic functions). Since Y is totally disconnected, A separates the points. Hence the given presentation of A is the Yosida representation by § 1. By compactness, each $\operatorname{coz} a$ is clopen. Thus 6.4 (b) holds vacuously.

These examples can be « classified » in two ways: First, whenever $\operatorname{coz} A$ has no proper dense member, then whenever e is a weak unit, e is never 0 and $e_A \leqslant te$ for some $t \in R$; thus $\sigma_e\colon X(e_A) \to X(e)$ exists by 4.2. Second, the examples above have the *ppp* (because any $\operatorname{coz} a$ is clopen; see 5.1), and

6.9. Corollary. If A has the *ppp*, then any weak unit is X-costrong.

Proof. Given weak units e and e', e' is X-strong by § 5, so $\sigma: X(e) \to X(e')$ exists and thus e is X-costrong.

6.10. Remarks. As in the proof of 4.11, it follows that these conditions on A are equivalent:

(*a*) Each weak unit is X-strong.

(*b*) Each weak unit is X-costrong.

(*c*) Each weak unit is both X-strong and -costrong.

(*d*) There is a weak unit which is X-strong and -costrong (if there is any weak unit).

(*e*) All spaces $X(e)$ (e a weak unit) are naturally homeomorphic.

(For (*e*), the existence of natural maps $X(e) \rightleftarrows X(e')$ implies the maps are mutually inverse, hence each is a « natural » homeomorphism).

Hence, if A has the *ppp* or is Cantor complete, then the above conditions hold.

We shall return to the general subject of « X-equivalence » and « X-uniqueness » in another paper.

Added in proof. The topic of this paper is explored further in « Retracting the prime spectrum of a Riesz space », by Giuseppe De Marco and the first author (to appear).

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