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Remarks on the Asymptotical Behaviour of Solutions to Some Nonlinear Parabolic Equations.

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RIASSUNTO - Si dimostra un risultato di attrattività delle soluzioni di equilibrio non banali per una classe di equazioni paraboliche non lineari.

1. Introduction.

We are interested in the asymptotical behaviour for $t \to +\infty$ of the solutions of the problem:

$$
(1) \qquad \begin{cases} \partial_t u = Au + \lambda u - |u|^\alpha u & \text{in } \Omega \times (0, +\infty)\,, \\ u \;\; = 0 & \text{in } \partial\Omega \times (0, +\infty)\,, \\ u \;\; = \xi & \text{in } \Omega \times \{0\}\,, \end{cases}
$$

where $(Au)(x) = \sum_{i,j=1}^{n} \partial_i\big(a_{ij}(x)\partial_j u(x)\big) + c(x)u(x)$ defines a linear second-order elliptic operator, λ and α are positive parameters; Ω is an open bounded subset of $\mathbf{R}^n$ with boundary $\partial\Omega$, ξ is a given function.

The above equation is suggested by concrete diffusion problems occurring in different domains [1]; moreover, its investigation gives

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useful hints for studying a wider class of reaction-diffusion systems (e.g., see [4]).

It is known that, when λ is greater than the principal eigenvalue of A with homogeneous Dirichlet boundary conditions, there exist two nontrivial equilibrium solutions $\pm\varphi$ of (1), namely

$$(1') \qquad A\varphi + \lambda\varphi - |\varphi|^\alpha\varphi = 0 \qquad \text{in } \Omega\,,$$

φ denoting a positive function in the Banach space $H_0^1(\Omega)\cap L^{2+\alpha}(\Omega)$. Such equilibrium solutions can be proved to be: *a*) stable in $H_0^1(\Omega)\cap \cap L^{2+\alpha}(\Omega)$; *b*) asymptotically stable in $L^2(\Omega)$ (1), under some restrictions on the coefficient α depending on the space dimension [3].

In the present note we shall prove a satisfactory refinement of the above properties, namely an attractivity result for $\pm\varphi$ in the space $H_0^1(\Omega)\cap L^{2+\alpha}(\Omega)$, under the same restrictions on α. The argument of the proof is suggested by a linearized stability procedure: in fact, the main tool to be used is a convergence result as $t\to+\infty$ of the derivative of the mapping $u\to Au+\lambda u-|u|^\alpha u$ when evaluated along the solutions of (1), as well as a monotonicity property of the same derivative at the equilibrium solutions.

2. The main result.

Let Ω be an open bounded subset of $\mathbf{R}^n$ with boundary $\partial\Omega$. We shall denote by $(u, v)=\int_\Omega u(x)\,v(x)\,dx$ the scalar product in the space $L^2(\Omega)$, with norm $|u|_2=(u, u)^{\frac12}$; for $\alpha>0$ we shall consider the norm

$$\|u\| = |Du|_2 + |u|_{2+\alpha} = \left(\sum_{i=1}^n(\partial_i u, \partial_i u)\right)^{\frac12} + \left(\int_\Omega |u(x)|^{2+\alpha}\,dx\right)^{1/(2+\alpha)},$$

under which $X=H_0^1(\Omega)\cap L^{2+\alpha}(\Omega)$ is a Banach space.

For $a_{ij}=a_{ji}$, $c\in L^\infty(\Omega)$ $(i, j=1, \ldots, n)$, let us assume a real con-

(1) We recall that the asymptotical stability of an equilibrium solution φ in a Banach space amounts to both its stability (i.e., for any sufficiently small $\varepsilon>0$ there is $\delta_\varepsilon>0$ such that $\|\xi-\varphi\|<\delta_\varepsilon$ implies $\|u(t;\xi)-\varphi\|<\varepsilon$ for all $t\geq 0$) and its attractivity (i.e., for any sufficiently small $\eta>0$, $\|\xi-\varphi\|<\eta$ implies $\|u(t;\xi)-\varphi\|\to 0$ as $t\to+\infty$).

stant $\eta > 0$ to exist, such that

$$\eta|\zeta|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \qquad (\forall \zeta \in \mathbb{R}^n, \text{ a.e. in } \Omega)\,.$$

Then the uniformly elliptic operator A is defined as follows:

$$\begin{cases} D(A) &= \{u \in H_0^1(\Omega) \,|\, a(u,\cdot) \text{ is } L^2(\Omega)\text{-continuous on } H_0^1(\Omega)\}\,, \\ (Au, v) &= a(u, v)\,, \qquad (\forall v \in H_0^1(\Omega))\,, \end{cases}$$

where $a(u, v)$ denotes the following bilinear form on $H_0^1(\Omega)$:

$$a(u, v) = -\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\,\partial_i u(x)\,\partial_j v(x)\,dx + \int_\Omega c(x)\,u(x)\,v(x)\,dx\,.$$

By regularity results [5] it follows that $D(A) \subseteq L^\infty(\Omega)$ if $n < 4$, or $D(A) \subseteq L^{2n/(n-4)}(\Omega)$ if $n > 4$. We shall denote by λ_0 the principal eigenvalue of A:

$$A\varphi_0 + \lambda_0\varphi_0 = 0\,, \qquad \varphi_0 \in H_0^1(\Omega)\,, \qquad \varphi_0 \geqslant 0\,, \qquad |\varphi_0|_2 = 1\,.$$

We shall need in the following several results concerning existence, uniqueness and regularity of the solution of problem (1). For this purpose, it is convenient to consider the map f defined as follows:

$$\begin{cases} D(f) &= D(A) \cap L^{2\alpha+2}(\Omega)\,, \\ f(u) &= -Au - \lambda u + |u|^\alpha u\,. \end{cases}$$

It can be proved that, if $\xi \in D(f)$, there exists a unique strict solution $u = u(t;\xi)$ of (1) belonging to $L^2_{\text{loc}}([0, +\infty); L^2(\Omega))$, namely:

(i) $u \in H^1_{\text{loc}}([0, +\infty); L^2(\Omega)) \cap L^2_{\text{loc}}([0, +\infty); D(f))$;

(ii) $u_t = Au + \lambda u - |u|^\alpha u$, t-a.e. in $[0, +\infty)$;

(iii) $u(0) = \xi$.

Moreover, such solution belongs to $\text{Lip}(0, T; L^2(\Omega))$ for any $T > 0$ [2] In particular, $u(t) \in L^\infty(\Omega)$ if $n < 4$, or $u(t) \in L^{2n/(n-4)}(\Omega)$ if $n > 4$, almost for any $t \in [0, +\infty)$.

The main result to be proved is as follows:

THEOREM 1. *Assume* $\lambda > \lambda_0$; *let* $\alpha < 4/n$ *if* $n > 2$ (*and* $\alpha < 2$ *if* $n \leqslant 2$). *Then there exists a neighbourhood* N *of* φ (*resp.* $-\varphi$) *in* $X = H_0^1(\Omega) \cap L^{2+\alpha}(\Omega)$ *such that, for any* $\xi \in N$, $u(t;\xi)$ *converges to* φ (*resp.* $-\varphi$) *in* X.

3. A regularity result.

Let us first prove a regularity property of the solutions of (1), which will be use in the following.

LEMMA. *Let* $\xi \in D(f)$, *and suppose* $\alpha < 4/n$ *for any* n. *Then*:

(i′) $u_t \in H^1_{\mathrm{loc}}([0,+\infty); L^2(\Omega)) \cap L^2_{\mathrm{loc}}([0,+\infty); D(A))$;

(ii′) $u_{tt} = Z(t)u_t \equiv Au_t + \lambda u_t - (1+\alpha)|u|^\alpha u_t$, *t-a.e. in* $[0,+\infty)$;

(iii′) $u_t(0;\xi) = -f(\xi)$.

As for the proof, consider the family $\{Z(t)\}_{t\in[0,+\infty)}$ of closed operators in $L^2(\Omega)$ defined as follows (2):

$$\begin{cases} D(Z(t)) = D(A)\,, \\ Z(t) \quad = A + \lambda - (1+\alpha)|u(t;\xi)|^\alpha, \qquad (t \in [0,+\infty))\,, \end{cases}$$

$u(t;\xi)$ denoting the unique strict solution of (1) in $L^2_{\mathrm{loc}}([0,+\infty); L^2(\Omega))$, which is Lipschitz continuous on the compact subsets of $[0,+\infty)$. Assume that the temporally inhomogeneous problem

$$\begin{cases} w_t(t) = Z(t)w(t)\,, \\ w(0) = -f(\xi)\,, \end{cases} \tag{2}$$

admits a unique strict solution $w \in L^2_{\mathrm{loc}}([0,+\infty); L^2(\Omega))$; then a standard approximation procedure [2] shows that

$$u_t(t;\xi) = w(t; -f(\xi))$$

t-almost everywhere in $[0,T]$ for any $T > 0$, whence the conclusion follows.

(2) It can be remarked that the definition of $\{Z(t)\}_{t\in[0,+\infty)}$ makes sense for arbitrary $\alpha > 0$ if $n < 4$, and for $\alpha < 4/(n-4)$ if $n > 4$.

To prove that a unique solution of (2) in the above sense does exist, suffice it [6] to exhibit $k > 0$, $\gamma \in (0, 1]$ such that, for any $T > 0$ and $r, s, t \in [0, T]$, the following inequality holds:

$$|[Z(t) - Z(r)]Z(s)^{-1}|_B \leqslant k|t-r|^\gamma$$

(where $|\cdot|_B$ denotes the norm of bounded operators on $L^2(\Omega)$). In the case $n > 4$, $Z(s)^{-1}\psi \in L^{2n/(n-4)}(\Omega)$ for any $\psi \in L^2(\Omega)$, $s \in [0, T]$; as $\alpha < 4/n < 1$, it follows:

$$\begin{aligned}|[Z(t) - Z(r)]Z(s)^{-1}|_B &= (1+\alpha)\sup_{|\psi|_2=1}\left(\int_\Omega \{|u(t)|^\alpha - |u(r)|^\alpha\}^2 (Z(s)^{-1}\psi)^2\, dx\right)^{1/2} \leqslant \\ &\leqslant (1+\alpha)\sup_{|\psi|_2=1} |Z(s)^{-1}\psi|_{2n/(n-4)} \left(\int_\Omega |u(t) - u(r)|^{\alpha n/2}\, dx\right)^{2/n} \leqslant \\ &\leqslant (1+\alpha)k'|t-r|^\alpha,\end{aligned}$$

where use has been made of the inequality

$$(\mathrm{d}_1) \qquad |b^\alpha - a^\alpha| \leqslant |b-a|^\alpha \qquad (a, b > 0;\ \alpha \in (0,1))\,;$$

thus the result follows. The case $n < 4$ can be dealt with in a similar way, due to the inequality

$$(\mathrm{d}_2) \qquad |b^\alpha - a^\alpha| \leqslant c_\alpha\{|b-a|^\alpha + a^{\alpha-1}|b-a|\} \quad (a, b > 0;\ \alpha \geqslant 1, c_\alpha > 0)\,.$$

4. Continuity properties of the operator $Z(t)$.

We shall prove that the principal eigenvalue of $Z(t)$ converges to a strictly positive limit when $t \to +\infty$; in this respect, it is worth studying continuity properties of the following map:

$$(3) \qquad a \to \mu(a) = \min\left\{-(A\chi, \chi) + \int_\Omega a(x)\chi^2(x)\, dx \,|\, \chi \in H_0^1(\Omega),\ |\chi|_2 = 1\right\}.$$

The following result will be of use in the sequel [3].

[3] We are indebted to P. Marcellini for this proof.

PROPOSITION 1. *The map* (3) *is continuous on the positive cone of* $L^{n/2}(\Omega)$ *if* $n>2$ (*and of* $L^s(\Omega)$, *for any* $s\geqslant 1$, *if* $n\leqslant 2$).

PROOF. Let us limit ourselves to the case $n>2$. The map

$$a \to -(A\chi,\chi)+\int_\Omega a(x)\chi^2(x)\,dx \qquad (a\geqslant 0)$$

being continuous on $L^{n/2}(\Omega)$ for any $\chi\in H^1_0(\Omega)$, it follows that μ is upper semicontinuous in $L^{n/2}(\Omega)$; we wish to prove that μ is lower semicontinuous as well in $L^{n/2}(\Omega)$, namely that $\overline{\lim\limits_{n\to\infty}}\,\mu(a_n)=\mu(a)$ for any sequence $\{a_n\}$ converging to a in $L^{n/2}(\Omega)$. This requires several steps:

α) suppose the sequence $\{\mu(a_n)\}$ to be bounded (otherwise there is nothing to be proved), and denote by χ_n the first eigenfunction of the operator $(-A+a_n)$, namely

$$-(A\chi_n,\chi_n)+\int_\Omega a_n(x)\chi_n^2(x)\,dx=\mu(a_n)\,,$$

(where $\chi_n\in H^1_0(\Omega)$, $|\chi_n|_2=1$). Due to the positivity of a_n and the ellipticity of the operator A, the sequence $\{\chi_n\}$ is bounded in $H^1_0(\Omega)$, thus in $L^{2n/(n-2)}(\Omega)$;

β) as a consequence, there exist $\tilde\chi\in H^1_0(\Omega)$ and a subsequence $\{\chi_{n_k}\}$ strongly converging to $\tilde\chi$ in $L^2(\Omega)$. Moreover, the subsequence $\{\chi^2_{n_k}\}$ converges weakly to $\tilde\chi^2$ in $L^{n/(n-2)}(\Omega)$; in fact, for any $\psi\in C_0^\infty(\Omega)$ we have:

$$\left|\int_\Omega \chi^2_{n_k}(x)\psi(x)\,dx-\int_\Omega\tilde\chi^2(x)\psi(x)\,dx\right|\leqslant 2|\psi|_\infty|\chi_{n_k}-\tilde\chi|_2\,;$$

γ) it follows from α), β) that

$$\int_\Omega a_{n_k}(x)\chi^2_{n_k}(x)\,dx\xrightarrow{k\to\infty}\int_\Omega a(x)\tilde\chi^2(x)\,dx\,;$$

then we have:

$$\mu(a)\leqslant-(A\tilde\chi,\tilde\chi)+\int_\Omega a(x)\tilde\chi^2(x)\,dx\leqslant\underline{\lim}_{k\to\infty}(-(A\chi_{n_k},\chi_{n_k}))+$$
$$+\lim_{k\to\infty}\int_\Omega a(x)\chi^2_{n_k}(x)\,dx\leqslant\underline{\lim}_{k\to\infty}\mu(a_{n_k})\leqslant\overline{\lim_{n\to\infty}}\,\mu(a_n)\,.$$

As $\overline{\lim_{n\to\infty}}\, \mu(a_n) = \mu(a)$ for any sequence $\{a_n\}$ converging to a in $L^{n/2}(\Omega)$, the conclusion follows.

We shall make use of the above result when dealing with the following map:

$$\text{(4)} \qquad \begin{cases} \mu\colon [0, +\infty) \to \mathbb{R}\,, \\ t \to \mu(t) = \min\{-\left(Z(t)\chi, \chi\right) | \chi \in H_0^1(\Omega),\ |\chi|_2 = 1\}\,; \end{cases}$$

we shall also be concerned with the quantity:

$$\text{(5)} \qquad \mu_\infty = \min\left\{-(A\chi, \chi) - \lambda + (1+\alpha)\int_\Omega |\varphi|^\alpha(x)\chi^2(x)\,dx | \chi \in H_0^1(\Omega),\ |\chi|_2 = 1\right\}$$

namely, with the principal eigenvalue of the F-derivative of the right-hand side in (1) evaluated at the equilibrium solution. The following result plays a central rôle in proving asymptotical properties of system (1) [3]: we give the proof for convenience of the reader.

THEOREM 2. *μ_∞ is strictly positive.*

PROOF. As φ is a positive equilibrium solution of (1), the elliptic operator $A + \lambda - |\varphi|^\alpha$ has φ as a positive eigenfunction with eigenvalue zero, which is thus the principal eigenvalue. On the other hand, μ_∞ is the principal eigenvalue of $A + \lambda - (1+\alpha)|\varphi|^\alpha$, whence the result.

We can now prove the above mentioned convergence property of $\mu(\cdot)$.

PROPOSITION 2. *Let $\alpha < 4/n$ if $n > 2$ (and $\alpha < 2$ if $n \leqslant 2$). Then there exists a X-neighbourhood N of φ such that, for any $\xi \in N$, the map $t \to \mu(t)$ corresponding to the solution $u(t;\xi)$ converges to μ_∞ as $t \to +\infty$.*

PROOF. It is known that, for any ξ in a suitable X-neighbourhood of φ, the solution $u(t;\xi)$ converges to φ in $L^2(\Omega)$ [3]. Due to inequalities (d_1), (d_2) above, it follows that $|u(t;\xi)|^\alpha$ converges to $|\varphi|^\alpha$ in $L^{2/\alpha}(\Omega)$, hence in $L^{n/2}(\Omega)$ (if $n > 2$), or in $L^s(\Omega)$, for any $s \geqslant 1$ (if $n \leqslant 2$). Then the conclusion follows from Proposition 1.

5. Proof of the main result.

Let us prove a preliminary convergence result for the time derivative $u_t(t;\xi)$.

PROPOSITION 3. *Let $\alpha < 4/n$ if $n > 2$ (and $\alpha < 2$ if $n \leqslant 2$). Then there exists a X-neighbourhood N of φ such that, for any $\xi \in N$, $|u_t(t;\xi)|_2 \to 0$ as $t \to +\infty$.*

PROOF. Pick first $\xi \in N \cap D(f)$. Taking the scalar product in $L^2(\Omega)$ of both sides of equation (ii') by $u_t(t;\xi)$ gives

$$\frac{1}{2}\frac{d}{dt}|u_t(t;\xi)|_2^2 \leqslant -\mu(t)|u_t(t;\xi)|_2^2 \qquad (t\text{-a.e. in } [0,+\infty)) .$$

According to Proposition 2, there exists $\tau > 0$ such that, for any $t > \tau$, $\mu(t) \geqslant \mu_\infty/2 > 0$. Then we have:

$$\begin{aligned}|u_t(t;\xi)|_2^2 &\leqslant |f(\xi)|_2^2 \exp\Big(-2\int_0^t \mu(s)\,ds\Big) = \\ &= |f(\xi)|_2^2 \exp\Big(-2\int_0^\tau \mu(s)\,ds\Big)\exp\Big(-2\int_\tau^t \mu(s)\,ds\Big) \leqslant \\ &\leqslant |f(\xi)|_2^2 \exp\Big(-2\int_0^\tau \mu(s)\,ds\Big)\exp\big(-\mu_\infty(t-\tau)\big),\end{aligned}$$

whence the conclusion follows. The general case can be dealt with in the same way, due to the regularization property of the operator A.

We can prove now Theorem 1. Introducing the new unknown function

$$v(t;\xi) = u(t;\xi) - \varphi ,$$

it follows from (1):

$$-Av = -v_t + \lambda v - \{|u|^\alpha u - |\varphi|^\alpha \varphi\} .$$

Taking the scalar product in $L^2(\Omega)$ of both sides by $v(t;\xi)$ we get,

t-almost everywhere:

$$\begin{aligned}(6)\qquad \eta|Dv(t;\xi)|_2^2 &\leqslant |v_t(t;\xi)|_2\,|v(t;\xi)|_2 + \lambda|v(t;\xi)|_2^2 + \\ &+ \Big(|v(t;\xi)|,\ \big|\big(|u(t;\xi)|^\alpha u(t;\xi) - |\varphi|^\alpha\varphi\big)\big|\Big) \leqslant \\ &\leqslant |v_t(t;\xi)|_2\,|v(t;\xi)|_2 + \lambda|v(t;\xi)|_2^2 + \\ &+ (1+\alpha)\big(|\varphi|^\alpha, v^2(t;\xi)\big) + |v(t;\xi)|_{2+\alpha}^{2+\alpha},\end{aligned}$$

where use has been made of inequality (d_2).

For α satisfying the above restrictions, Sobolev's embedding theorem gives:

$$|v(t;\xi)|_{2+\alpha}^{2+\alpha} \leqslant c|Dv(t;\xi)|_2^\beta\,|v(t;\xi)|_2^\gamma,$$

where c, β, γ are suitable positive constants such that $\beta < 2$, and $\beta + \gamma > 2$ [3]. It follows that

$$|v(t;\xi)|_{2+\alpha}^{2+\alpha} \leqslant \zeta|Dv(t;\xi)|_2^2 + K(c)\,\zeta^{-\beta/(2-\beta)}\,|v(t;\xi)|_2^{2\gamma/(2-\beta)},$$

where $K(c) = c^{2/(2-\beta)}(1-\beta/2)$, and ζ is any positive real number. Introducing the above inequality into estimate (6) we get:

$$\begin{aligned}\eta(1-\zeta)|Dv(t;\xi)|_2^2 &\leqslant |v_t(t;\xi)|_2\,|v(t;\xi)|_2 + \\ &+ [\lambda + (1+\alpha)|\varphi|_\infty^\alpha]\,|v(t;\xi)|_2^2 + K(c)\zeta^{-\beta/(2-\beta)}\,|v(t;\xi)|_2^{2\gamma/(2-\beta)}.\end{aligned}$$

Let us now choose $\xi \in N$, N denoting a suitable X-neighbourhood of φ, and $\zeta \in (0,1)$; observe moreover that $H_0^1(\Omega)$ is embedded into $L^{2+\alpha}(\Omega)$. Then the conclusion follows as both $v(t;\xi)$ and $v_t(t;\xi)$ converge to zero in $L^2(\Omega)$ as $t \to +\infty$.

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