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Stirling pairs

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Stirling Pairs.

L. CARLITZ (*)

1. Introduction.

The Stirling numbers of the first and second kind can be defined by

$$(1.1) \quad x(x+1) \dots (x+n-1) = \sum_{k=0}^n S_1(n, k) x^k$$

and

$$(1.2) \quad x^n = \sum_{k=0}^n S(n, k) x(x-1) \dots (x-k+1),$$

respectively. Since $S_1(n, n-k)$ and $S(n, n-k)$ are polynomials in n of degree $2k$, it follows readily that

$$(1.3) \quad S_1(n, n-k) = \sum_{j=0}^{k-1} S'_1(k, j) \binom{n}{2k-j} \quad (k > 0)$$

and

$$(1.4) \quad S(n, n-k) = \sum_{j=0}^{k-1} S'(k, j) \binom{n}{2k-j} \quad (k > 0).$$

The coefficients $S'_1(k, j)$, $S'(k, j)$ were introduced by Jordan [8, Ch. 4] and Ward [12]; the present notation is that of [2]. The numbers are closely related to the *associated* Stirling numbers of Riordan [10, Ch. 4].

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The Stirling numbers and the associated Stirling numbers are related in various ways. We have

$$(1.5) \quad \begin{cases} S_1(n, n-k) = \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} S(j+k, j), \\ S(n, n-k) = \sum_{j=0}^k \binom{k-n}{k+j} \binom{k+n}{k-j} S_1(j+k, j), \end{cases}$$

and

$$(1.6) \quad \begin{cases} S'(n, k) = \sum_{j=0}^k (-1)^j \binom{n-j-1}{k-j} S'_1(n, j), \\ S'_1(n, k) = \sum_{j=0}^k (-1)^j \binom{n-j-1}{k-j} S'(n, j). \end{cases}$$

Also

$$(1.7) \quad \begin{cases} S_1(n, n-k) = \sum_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-j} S'(k, j), \\ S(n, n-k) = \sum_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-1} S'_1(k, j), \end{cases} \quad (k > 0).$$

The first half of (1.5) is due to Schläfli [11]; the second was proved by Gould [7].

The writer [3] has defined another triangular array of numbers that is closely related to $(S_1(n, k))$ and $(S(n, k))$. Analogous to (1.3) and (1.4) we have

$$(1.8) \quad S_1(n, n-k) = \sum_{j=1}^k B_1(k, j) \binom{n+j-1}{2k} \quad (k > 0),$$

and

$$(1.9) \quad S(n, n-k) = \sum_{j=1}^k B(k, j) \binom{n+j-1}{2k} \quad (k > 0).$$

The coefficients $B_1(k, j)$, $B(k, j)$ are positive integers and satisfy the recurrences

$$(1.10) \quad B_1(k, j) = jB_1(k-1, j) + (2k-j)B_1(k-1, j-1), \quad (k > 1)$$

and

$$(1.11) \quad \begin{aligned} B(k, j) &= \\ &= (k-j+1)B(k-1, j) + (k+j-1)B(k-1, j+1), \quad (k > 1), \end{aligned}$$

respectively. Moreover

$$(1.12) \quad B_1(k, j) = B(k, k - j + 1), \quad (1 \leq j \leq k).$$

The writer [1] proved (1.5) by making use of the representations

$$(1.13) \quad \begin{cases} S_1(n, n - k) = \binom{k - n}{k} B_k^{(n)}, \\ S(n, n - k) = \binom{n}{k} B_k^{(-n+k)}, \end{cases}$$

where $B_n^{(z)}$ is the Nörlund polynomial [9, Ch. 6] defined by

$$(1.14) \quad \left(\frac{x}{e^z - 1} \right)^z = \sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!}.$$

(The polynomial $B_n^{(z)}$ should not be confused with the Bernoulli polynomial $B_n(z)$.) Incidentally it follows from (1.13) that

$$(1.15) \quad S_1(-n + k, -n) = S(n, n - k).$$

In a recent paper [6] the writer showed that the above results can be generalized considerably in the following way. Let $\{f_k(z)\}$ denote a sequence of polynomials such that

$$(1.16) \quad \deg f_k(z) = k; \quad f_k(0) = 0, \quad (k > 0).$$

Put

$$(1.17) \quad \begin{cases} F_1(n, n - k) = \binom{k - n}{k} f_k(n), \\ F(n, n - k) = \binom{n}{k} f_k(-n + k). \end{cases}$$

Then all the above results generalize. The proof makes use of two arrays $(G_1(k, j))$, $(G(k, j))$ that generalize $(B_1(k, j))$, $(B(k, j))$, respectively. They are defined by

$$(1.18) \quad \begin{cases} F_1(n, n - k) = \sum_{j=1}^k G_1(k, j) \binom{n + j - 1}{2k}, \\ F(n, n - k) = \sum_{j=1}^k G(k, j) \binom{n + j - 1}{2k}, \end{cases}$$

and satisfy

$$(1.19) \quad G_1(k, j) = G(k, k - j + 1), \quad (1 \leq j \leq k).$$

Incidentally it follows from (1.17) that

$$(1.20) \quad F_1(-n + 1, -n) = F(n, n - k);$$

$F_1(x, y)$, $F(x, y)$ are defined for arbitrary x, y such that $x - y$ is equal to a nonnegative integer.

In order to generalize the orthogonality relations for the Stirling numbers, an additional condition on $\{f_k(z)\}$ is assumed, namely that

$$(1.21) \quad (\varphi(\chi))^z = \sum_{k=0}^{\infty} f_k(z) x^k / k!$$

for some

$$\varphi(x) = 1 + \sum_{n=1}^{\infty} c_n x^n / n!.$$

It is proved, using (1.21) that

$$(1.22) \quad \sum_{k=j}^n (-1)^{n-k} F_1(n, k) F(k, j) = \sum_{k=j}^n (-1)^{k-j} F(n, k) F_1(k, j) = \delta_{nj}.$$

In the present paper we generalize the definition (1.17) further. Let r be a fixed positive integer and define

$$(1.21) \quad \begin{cases} F_1^{(r)}(n, n - rk) = \binom{rk - n}{rk} f_k(n), \\ F^{(r)}(n, n - rk) = \binom{n}{rk} f_k(rk - n), \end{cases}$$

where $f_k(z)$ is a polynomial in z that satisfies (1.16). For $r = 1$, (1.21) reduces to (1.17). We call a pair of polynomials

$$(1.22) \quad g_{1,k}^{(r)}(n) = F_1^{(r)}(n, n - rk), \quad g_k^{(r)}(n) = F^{(r)}(n, n - rk)$$

a *Stirling pair* of order r , or, briefly, a Stirling pair. Clearly each of

the polynomials is of degree $(r + 1)k$ in n . Moreover, by (1.21), $g_{1,k}^{(n)}(z)$ and $g_k^{(r)}(z)$ are defined for arbitrary (complex) z .

We shall show that the results previously obtained in the case $r = 1$, when properly modified, hold for all $r \geq 1$. We show also that a condition similar to (1.21) suffices for orthogonality in the general case.

Finally we consider the possibility of recurrence of the type

$$(1.23) \quad F_1^{(r)}(n + 1, m) = F_1^{(r)}(n, m - 1) + p_r(n)F_1^{(r)}(n - r + 1, m)$$

and

$$(1.24) \quad F^{(r)}(n + 1, m) = F^{(r)}(n, m - 1) + q_r(m)F^{(r)}(n, m + r - 1).$$

For $r = 1$, $p_r(n) = n$, (1.23) reduces to the familiar recurrence for $S_1(n, m)$; for $r = 1$, $q_r(m) = m$, (1.24) reduces to the recurrence for $S(n, m)$. Recurrences for the numbers defined by the generating functions [5]

$$(1.25) \quad 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n T_1(n, m) \frac{x^n}{n!} z^m = \left(\frac{1+x}{1-x} \right)^{z/2},$$

$$(1.26) \quad 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n T(n, m) \frac{x^n}{n!} z^m = \exp(z \tanh x),$$

are of the form (1.23) and (1.24), respectively, with $r = 2$. Indeed it was the study of such arrays that motivated the generalization (1.21).

We show that (1.23) holds if and only if

$$(1.27) \quad p_r(n) = (-1)^r(r + 1) c_r \binom{n}{r},$$

where $f_1^{(r)}(z) = c_r z$. Similarly (1.24) holds if and only if

$$(1.28) \quad q_r(m) = -(r + 1) c_r \binom{m + r - 1}{r}.$$

Thus

$$q_r(n) = (-1)^{r-1} p_r(n + r - 1).$$

The two conditions (1.27), (1.28) are equivalent.

Moreover when (1.27) is satisfied we have

$$1 + \sum_{n=1}^{\infty} \sum_{m=1}^n F_1^{(r)}(n, m) \frac{x^n}{n!} z^m = \exp \{z\psi(x)\}$$

and

$$1 + \sum_{n=1}^{\infty} \sum_{m=1}^n F^{(r)}(n, m) \frac{x^n}{n!} z^m = \exp \{-z\omega(-x)\},$$

where

$$\psi(x) = \sum_{m=0}^{\infty} \frac{a_r^m x^{rm+1}}{rm+1}, \quad a_r = (-1)^r \frac{(r+1)c_r}{r!}$$

and

$$\psi(\omega(x)) = \omega(\psi(x)) = x, \quad \omega(0) = 0.$$

2. - We first prove the following generalized version of (1.5).

THEOREM 1. *For all integral $r \geq 1$, we have*

$$(2.1) \quad \begin{cases} F_1^{(r)}(n, n-k) = \sum_{j=0}^{rk} \binom{rk-n}{rk+j} \binom{rk+n}{rk-j} F^{(r)}(j+rk, j), \\ F^{(r)}(n, r-k) = \sum_{j=0}^{rk} \binom{rk-n}{rk+j} \binom{rk+n}{rk-j} F_1^{(r)}(j+rk, j). \end{cases}$$

PROOF. It suffices to prove the identity

$$(2.2) \quad \binom{z}{rk} f_k(rk-z) = \sum_{j=0}^{rk} \binom{rk-z}{rk+j} \binom{rk+z}{rk-j} \binom{-j}{rk} f_k(j+rk).$$

For $z = rk - n$, (2.2) reduces to the first of (2.1), for $z = n$, (2.2) reduces to the second of (2.1).

Each side of (2.2) is a polynomial in z of degree $(r+1)k$. Hence it is only necessary to show that (2.2) holds for $(r+1)k+1$ distinct values of z . For $z = t = 0, 1, \dots, rk-1$, the LHS of (2.2) is equal to zero; since

$$\binom{rk-t}{rk+j} \binom{-j}{rk} = 0 \quad (0 \leq t < k; 0 \leq j \leq k),$$

it follows that (2.2) holds for these values of z . For $z = rk$ we get

$$f_k(0) = \sum_{j=0}^{rk} \binom{0}{rk} \binom{(r+1)k}{rk-j} \binom{-j}{rk} f_k(j+rk),$$

which is evidently correct. Finally, for $z = -t$, where $1 \leq t \leq k$, the RHS of (2.2) reduces to the single term ($j = t$)

$$\binom{rk+t}{rk+t} \binom{rk-t}{rk-t} \binom{-t}{rk} f_k(t+rk) = \binom{-t}{rk} f_k(r+rk),$$

so that (2.2) holds for these values of z .

This completes the proof of (2.2) and therefore of Theorem 1.

It is of some interest to ask whether the polynomials in a Stirling pair can be equal. By (1.21) and (1.22), $g_{1,k}^{(r)}(z) = g_k^{(r)}(z)$ if and only if

$$(2.3) \quad \binom{rk-z}{rk} f_k(z) = \binom{z}{rk} f_k(rk-z).$$

Since

$$\binom{rk-z}{rk} = (-1)^{rk} \binom{z-1}{rk},$$

(2.3) reduces to

$$(-1)^{rk}(z-rk)f_k(z) = zf_k(rk-z),$$

so that

$$(-1)^{rk}(z-1)f_k(rkz) = zf_k(rk(1-z)).$$

Hence we have

$$(2.4) \quad \varphi_{k-1}(1-z) = (-1)^{rk-1} \varphi_{k-1}(z),$$

where

$$f_k(rkz) = rkz\varphi_{k-1}(z), \quad (k \geq 1).$$

Note that $\varphi_{k-1}(z)$ is a polynomial of degree $k-1$.

We may state

THEOREM 2. *The polynomials $g_{1,k}^{(r)}(z)$, $g_k^{(r)}(z)$ in the Stirling pair*

$$g_{1,k}^{(r)} = \binom{rk-z}{rk} f_k(z), \quad g_k^{(r)}(z) = \binom{z}{rk} f_k(rk-z)$$

are equal if and only if

$$f_k(rkz) = rkz\varphi_{k-1}(z), \quad (k \geq 1),$$

where $\varphi_{k-1}(z)$ is a polynomial of degree $k-1$ that satisfies (2.4).

For r odd, the condition (2.4), or, what is the same,

$$(2.5) \quad \varphi_k(1-z) = (-1)^k \varphi_k(z)$$

is a familiar one. An equivalent condition is

$$(2.6) \quad \tilde{\varphi}_k(-z) = (-1)^k \tilde{\varphi}_k(z),$$

where $\tilde{\varphi}_k(z) = \varphi(\frac{1}{2} + z)$.

The Bernoulli and Euler polynomials $B_n(z)$, $E_n(z)$ defined by

$$(2.7) \quad \frac{ze^{zx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{x^n}{n!}$$

and

$$(2.8) \quad \frac{2e^{zx}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{x^n}{n!},$$

respectively, are well known examples of polynomials satisfying (2.5). It therefore follows from Theorem 2 that, for r odd,

$$(2.9) \quad g_{1,k}^{(r)}(z) = g_k^{(r)}(z) = \binom{rk-z}{rk} \frac{z}{rk} B_{k-1}\left(\frac{z}{rk}\right) = \binom{z}{rk} \frac{rk-z}{rk} B_{k-1}\left(\frac{rk-z}{rk}\right)$$

and

$$(2.10) \quad g_{1,k}^{(r)}(z) = g_k^{(r)}(z) = \binom{rk-z}{rk} \frac{z}{rk} E_{k-1}\left(\frac{z}{rk}\right) = \binom{z}{rk} \frac{rk-z}{rk} E_{k-1}\left(\frac{rk-z}{rk}\right)$$

are instances of Stirling pairs consisting of equal polynomials.

Another example with $f_k(z) = \binom{z/r}{k}$ is

$$(2.11) \quad g_{1,k}^{(r)}(z) = g_k^{(r)}(z) = \binom{rk-z}{rk} \binom{z/r}{k} = \binom{z}{rk} \binom{(rk-z)/r}{k}.$$

For even r , (2.4) becomes

$$(2.12) \quad \varphi_{k-1}(1-z) = -\varphi_{k-1}(z).$$

For $k=1$, this implies $\varphi_0(z) \equiv 0$, so that $f_1(z) \equiv 0$. Hence, for even r , $g_{1,k}^{(r)}(z)$ and $g_k^{(r)}(z)$ cannot be equal for all k . However if we require only that $g_{1,2k}^{(r)}(z) = g_{2k}^{(r)}(z)$ then the previous discussion (for odd r) applies.

Clearly, by (1.21), $F_1^{(r)}(x, y)$, $F^{(r)}(x, y)$ are defined for arbitrary x, y such that $x-y = rk$, where k is any nonnegative integer. Indeed

$$(2.13) \quad \begin{cases} F_1^{(r)}(x, y) = \binom{-y}{rk} f_k(x), \\ F^{(r)}(x, y) = \binom{x}{rk} f_k(-y). \end{cases}$$

Moreover it follows from (2.13) that

$$(2.14) \quad F^{(r)}(x, y) = F_1^{(r)}(-y, -x), \quad (x-y = rk).$$

3. - We now consider

$$(3.1) \quad \begin{cases} F_1^{(r)}(n, n-rk) = \sum_{j=1}^k G_1^{(r)}(k, j) \binom{n+j-1}{(r+1)k}, \\ F^{(r)}(n, n-rk) = \sum_{j=1}^k G^{(r)}(k, j) \binom{n+j-1}{(r+1)k}, \end{cases}$$

where $G_1^{(r)}(k, j)$, $G^{(r)}(k, j)$ are independent of n . That such representations hold follows from (1.21); for the general situation see [4].

Inverting each of (3.1) we get

$$(3.2) \quad \begin{cases} G_1^{(r)}(k, k-j+1) = \\ \quad = \sum_{t=0}^j (-1)^t \binom{(r+1)k+1}{t} F_1^{(r)}(rk+j-t, j-t), \\ G^{(r)}(k, k-j+1) = \\ \quad = \sum_{t=0}^j (-1)^t \binom{(r+1)k+1}{t} F^{(r)}(rk+j-t, j-t). \end{cases}$$

We shall now show that

$$(3.3) \quad (-1)^{(r+1)k} G_1^{(r)}(k, j) = G^{(r)}(k, k-j+1), \quad (1 \leq j \leq k).$$

It follows from (1.21) and (3.1) that

$$(3.4) \quad \binom{n}{rk} f_k(rk-n) = \sum_{j=1}^k G^{(r)}(k, j) \binom{n+j-1}{(r+1)k}$$

and

$$(3.5) \quad \binom{rk-n}{rk} f_k(n) = \sum_{j=1}^k G_1^{(r)}(k, j) \binom{n+j-1}{(r+1)k}.$$

Both (3.4) and (3.5) are polynomial identities in n . Thus in (3.5) we may replace n by $rk-n$. This gives

$$\binom{n}{rk} f_k(rk-n) = \sum_{j=1}^k G_1^{(r)}(k, j) \binom{rk-n+j-1}{(r+1)k}.$$

Since

$$\binom{rk-n+j-1}{(r+1)k} = (-1)^{(r+1)k} \binom{n+k-j}{(r+1)k},$$

we get

$$\binom{n}{rk} f_k(rk-n) = (-1)^{(r+1)k} \sum_{j=1}^k G_1^{(r)}(k, j) \binom{n+k-j}{(r+1)k}$$

and therefore

$$(3.6) \quad \binom{n}{rk} f_k(rk-n) = (-1)^{(r+1)k} \sum_{j=1}^{k-j+1} G_1^{(r)}(k, j) \binom{n+j-1}{(r+1)k}.$$

Comparison of (3.6) with (3.4) yields (3.3).

We may state

THEOREM 3. *The coefficients $G_1^{(r)}(k, j)$, $G^{(r)}(k, j)$ defined by (3.1) satisfy the symmetry relation*

$$(3.7) \quad (-1)^{(r+1)k} G_1^{(r)}(k, j) = G^{(r)}(k, k-j+1), \quad (1 \leq j \leq k).$$

Theorem 3 can be used to give another proof of Theorem 1. However we shall not take the space to do so.

4. - The generalized versions of (1.3) and (1.4) are

$$(4.1) \quad F_1^{(r)}(n, n - rk) = \sum_{j=0}^{k-1} F_1^{(r)'}(k, j) \binom{n}{(r+1)k - j}$$

and

$$(4.2) \quad F^{(r)}(n, n - rk) = \sum_{j=0}^{k-1} F^{(r)'}(k, j) \binom{n}{(r+1)k - j},$$

respectively.

To invert (4.1) multiply both sides by x^{n-rk} and sum over k . Thus

$$\begin{aligned} \sum_{n=0}^{\infty} F_1^{(r)}(n + rk, n) x^n &= \sum_{j=0}^{k-1} F_1^{(r)'}(k, j) x^{k-j} \sum_{n=(r+1)k-j}^{\infty} \cdot \\ &\cdot \binom{n}{(r+1)k - j} x^{n-(r+1)k+j} = \sum_{j=0}^{k-1} F_1^{(r)'}(k, j) x^{k-j} (1-x)^{-(r+1)k+j-1} = \\ &= \sum_{j=1}^k F_1^{(r)'}(k, k-j) x^j (1-x)^{-rk-j-1} \end{aligned}$$

so that

$$\sum_{j=1}^k F_1^{(r)'}(k, k-j) x^j (1-x)^{-j} = (1-x)^{rk+1} \sum_{n=0}^{\infty} F_1^{(r)}(n + rk, n) x^n.$$

Put

$$z = \frac{x}{1-x}, \quad x = \frac{z}{1+z}, \quad 1-x = \frac{1}{1+z}$$

and we get

$$\sum_{j=1}^k F_1^{(r)'}(k, k-j) z^j = \sum_{n=0}^{\infty} F_1^{(r)}(n + rk, n) z^n (1+z)^{-n-rk-1}.$$

Expanding the right member and equating coefficients, we get

$$(4.3) \quad F_1^{(r)'}(k, k-j) = \sum_{t=0}^j (-1)^{j-t} \binom{j+rk}{j-t} F_1^{(r)}(t + rk, t).$$

Similarly we have

$$(4.4) \quad F^{(r)'}(k, k-j) = \sum_{t=0}^j (-1)^{j-t} \binom{j+rk}{j-t} F^{(r)}(t + rk, t).$$

By (4.3) and (3.1) we have

$$\begin{aligned} F_1^{(r)'}(k, k-j) &= \sum_{t=0}^j (-1)^{j-t} \binom{j+rk}{j-t} \sum_{s=1}^k G_1^{(r)}(k, s) \binom{t+rk+s-1}{(r+1)k} = \\ &= \sum_{s=1}^k G_1^{(r)}(k, s) \sum_{t=0}^j (-1)^{j-t} \binom{j+rk}{j-t} \binom{t+rk+s-1}{(r+1)k}. \end{aligned}$$

By Vandermonde's theorem the inner sum is equal to $\binom{s-1}{k-j}$, so that

$$(4.5) \quad F_1^{(r)'}(k, j) = \sum_{s=j+1}^k \binom{s-1}{j} G_1^{(r)}(k, s).$$

Similarly

$$(4.6) \quad F^{(r)'}(k, j) = \sum_{s=j+1}^k \binom{s-1}{j} G^{(r)}(k, s).$$

The inverse formulas are

$$(4.7) \quad G_1^{(r)}(k, t) = \sum_{s=t-1}^{k-1} (-1)^{s-t+1} \binom{s}{t-1} F_1^{(r)'}(k, s)$$

and

$$(4.8) \quad G^{(r)}(k, t) = \sum_{s=t-1}^{k-1} (-1)^{s-t+1} \binom{s}{t-1} F^{(r)'}(k, s).$$

Next, by (3.7) and (4.8), we have

$$\begin{aligned} F_1^{(r)}(n, n-rk) &= \sum_{t=1}^k G_1^{(r)}(k, t) \binom{n+t-1}{(r+1)k} = \\ &= \sum_{t=1}^k G_1^{(r)}(k, k-t+1) \binom{n+k-t}{(r+1)k} = (-1)^{(r+1)k} \sum_{t=1}^k G^{(r)}(k, t) \binom{n+k-t}{(r+1)k} = \\ &= (-1)^{(r+1)k} \sum_{t=1}^k \binom{n+k-t}{(r+1)k} \sum_{s=t-1}^{k-1} (-1)^{s-t+1} \binom{s}{t-1} F^{(r)'}(k, s) = \\ &= (-1)^{(r+1)k} \sum_{s=0}^{k-1} F^{(r)'}(k, s) \sum_{t=1}^{s+1} (-1)^{s-t+1} \binom{s}{t-1} \binom{n+k-t}{(r+1)k}. \end{aligned}$$

The inner sum is equal to

$$(-1)^s \binom{n+k-s-1}{(r+1)k-s},$$

so that

$$(4.9) \quad F_1^{(r)}(n, n - rk) = \sum_{j=1}^k (-1)^{rk-j} \binom{n+j-1}{rk+j} F^{(r)'}(k, k-j).$$

The companion formula is

$$(4.10) \quad F^{(r)}(n, n - rk) = \sum_{j=1}^k (-1)^{rk-j} \binom{n+j-1}{rk+j} F_1^{(r)'}(k, k-j).$$

Again, by (4.5) and (4.8),

$$\begin{aligned} F_1^{(r)'}(n, k) &= \sum_{s=k+1}^n \binom{s-1}{k} G^{(r)}(n, s) = \\ &= (-1)^{(r+1)n} \sum_{s=k+1}^n \binom{s-1}{k} G^{(r)}(n, n-s+1) = \\ &= (-1)^{(r+1)n} \sum_{s=1}^{n-k} \binom{n-s}{k} G^{(r)}(n, s) = \\ &= (-1)^{(r+1)n} \sum_{s=1}^{n-k} \binom{n-s}{k} \sum_{t=s-1}^{n-1} (-1)^{t-s+1} \binom{t}{s-1} F^{(r)'}(n, t) = \\ &= (-1)^{(r+1)n} \sum_{t=0}^{n-1} F^{(r)'}(n, t) \sum_{s=1}^{t+1} (-1)^{t-s+1} \binom{t}{s-1} \binom{n-s}{k}. \end{aligned}$$

The inner sum is equal to

$$\sum_{s=0}^t (-1)^{t-s} \binom{t}{s} \binom{n-s-1}{k} = (-1)^t \binom{n-t-1}{k-t}$$

and therefore

$$(4.11) \quad F_1^{(r)'}(n, k) = (-1)^{(r+1)n} \sum_{t=0}^{n-1} (-1)^t \binom{n-t-1}{k-t} F^{(r)'}(n, t).$$

Similarly

$$(4.12) \quad F^{(r)'}(n, k) = (-1)^{(r+1)n} \sum_{t=0}^{n-1} (-1)^t \binom{n-t-1}{k-t} F_1^{(r)'}(n, t).$$

To sum up the results of § 4 we state

THEOREM 4. *The coefficients $F_1^{(r)'}(n, n - rk)$, $F^{(r)'}(n, n - rk)$ defined by (4.1) and (4.2) satisfy (4.5), (4.6), (4.9), (4.10), (4.11) and (4.12).*

5. — For the results obtained above it sufficed to assume that the $\{f_k(z)\}$ were a sequence of polynomials in z satisfying

$$(5.1) \quad \deg f_k(z) = k; \quad f_k(0) = 0, \quad (k \geq 1).$$

In order to obtain orthogonality relations more is needed. We shall make use of a sufficient condition that is convenient for applications.

Let

$$(5.2) \quad \varphi^{(r)}(x) = 1 + \sum_{n=1}^{\infty} c_n x^{rn} / (rn)!$$

denote a function that is analytic in the neighborhood of $x = 0$ and such that $\varphi(0) = 1$. Put

$$(5.3) \quad (\varphi(x))^z = \sum_{k=0}^{\infty} f_k^{(r)}(z) x^{rk} / (rk)!$$

It is easily verified that the $\{f_k^{(r)}(z)\}$ are polynomials in z that satisfy (5.1). The Nörlund polynomials $B_n^{(z)}$ are given by $\varphi(x) = x/(e^x - 1)$ and $r = 1$.

We remark that a sequence of polynomials $\{f_k^{(r)}(z)\}$ satisfies (5.3) if and only if it satisfies.

$$(5.4) \quad \sum_{j=0}^k \binom{rk}{rj} f_j^{(r)}(y) f_{k-j}^{(r)}(z) = f_k^{(r)}(y+z) \quad (k = 0, 1, 2, \dots).$$

It follows from (5.3) that

$$z\varphi^{z-1}(x)\varphi'(x) = \sum_{k=0}^{\infty} f_k^{(r)}(z) x^{rk-1} / (rk-1)!,$$

$$(y+z)\varphi^{y+z-1}(x)\varphi'(x) = \sum_{k=0}^{\infty} f_k^{(r)}(y+z) x^{rk-1} / (rk-1)!.$$

Since

$$(y+z)\varphi^{y+z-1}(x)\varphi'(x) = \frac{y+z}{z} \varphi^y(x) \cdot z\varphi^{z-1}(x)\varphi'(x),$$

we get

$$(5.5) \quad f_k^{(r)}(y+z) = \frac{z}{y+z} \sum_{j=1}^k \binom{rk-1}{rj-1} f_{k-j}^{(r)}(y) f_j^{(r)}(z).$$

THEOREM 5. *Let*

$$(5.6) \quad \begin{cases} F_1^{(r)}(n, n - rk) = \binom{rk - n}{rk} f_k^{(r)}(n), \\ F^{(r)}(n, n - rk) = \binom{n}{rk} f_k^{(r)}(rk - n), \end{cases}$$

where the $\{f_k^{(r)}(z)\}$ are defined by (5.2) and (5.3). Then we have

$$(5.7) \quad \begin{aligned} \sum_{k=0}^j (-1)^{rk} F_1^{(r)}(n, n - rk) F^{(r)}(n - rk, n - rj) = \\ = \sum_{k=0}^j (-1)^{r(j-k)} F^{(r)}(n, n - rk) F_1^{(r)}(n - rk, n - rj) = \delta_{j,0}. \end{aligned}$$

PROOF. - Put

$$(5.8) \quad H(n, j) = \sum_{k=0}^j (-1)^{rk} F_1^{(r)}(n, n - rk) F^{(r)}(n - rk, n - rj).$$

By (5.6),

$$(5.9) \quad \begin{aligned} H(n, j) &= \sum_{k=0}^j (-1)^{rk} \binom{rk - n}{rk} f_k^{(r)}(n) \cdot \binom{n - rk}{r(j - k)} f_{j-k}^{(r)}(rj - n) = \\ &= \sum_{k=0}^j \binom{n - 1}{rk} \binom{n - rk}{r(j - k)} f_k^{(r)}(n) f_{j-k}^{(r)}(rj - n) = \\ &= \frac{1}{n} \binom{n}{rj} \sum_{k=0}^j (n - rk) \binom{rj}{rk} f_k^{(r)}(n) f_{j-k}^{(r)}(rj - n). \end{aligned}$$

By (5.4) and (5.5), for $j > 0$,

$$\begin{aligned} \sum_{k=0}^j (n - rk) \binom{rj}{rk} f_k^{(r)}(n) f_{j-k}^{(r)}(rj - n) &= n \sum_{k=0}^j \binom{rj}{rk} f_k^{(r)}(n) f_{j-k}^{(r)}(rj - n) - \\ - rj \sum_{k=1}^j \binom{rj - 1}{rk - 1} f_k^{(r)}(n) f_{j-k}^{(r)}(rj - n) &= n f_j^{(r)}(rj) - rj \cdot \frac{n}{rj} f_j^{(r)}(rj) = 0. \end{aligned}$$

Thus

$$(5.10) \quad H(n, j) = 0, \quad (j > 0).$$

For $j = 0$, it is evident from (5.8) that

$$(5.11) \quad H(n, 0) = 1.$$

This proves the first half of (5.7). The second half then follows as a corollary.

6. — We recall that the Stirling numbers $S_1(n, k)$, $S(n, k)$ satisfy the respective recurrences

$$(6.1) \quad \begin{cases} S_1(n+1, k) = S_1(n, k-1) + nS_1(n, k), \\ S(n+1, k) = S(n, k-1) + ks(n, k). \end{cases}$$

Also it is proved that in [5] that

$$(6.2) \quad T_1(n+1, k) = T_1(n, k-1) + n(n-1)T_1(n-1, k)$$

and

$$(6.3) \quad T(n+1, k) = T(n, k-1) - k(k+1)T(n, k+1),$$

where

$$(6.4) \quad 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n T_1(n, k) \frac{x^n}{n!} z^k = \left(\frac{1+x}{1-x} \right)^{z/2}$$

and

$$(6.5) \quad 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n T(n, k) \frac{x^n}{n!} z^k = \exp(z \operatorname{Tanh} x);$$

$(T_1(n, k))$ and $(T(n, k))$ are reciprocal arrays.

Another example from [5] is

$$(6.6) \quad U(n+2, k) = U(n, k-2) + k^2 U(n, k),$$

where

$$(6.7) \quad 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n U(n, k) \frac{x^n}{n!} z^k = \exp(z \sinh x).$$

Thus it is clear that, for $r > 1$, there are apparently numerous possible recurrences for $F_1^{(r)}(n, k)$ and $F^{(r)}(n, k)$. The instances (6.2), (6.3) and (6.6) illustrate the case $r = 2$.

To begin with, we consider the possibility of recurrences of the type

$$(6.8) \quad F_1^{(r)}(n+1, n-rk+1) = \\ = F_1^{(r)}(n, n-rk) + p_r(n)F_1^{(r)}(n-r+1, n-rk+1),$$

where $p_r(n)$ is independent of k .

By the first of (5.6), the recurrence (6.8) becomes

$$(6.9) \quad \binom{rk-n-1}{rk} f_k^{(r)}(n+1) = \\ = \binom{rk-n}{rk} f_k^{(r)}(n) + p_r(n) \binom{rk-n-1}{r(k-1)} f_{k-1}^{(r)}(n-r+1).$$

We now take $k=1$ in (6.9). Since $f_0(z)=1$, $f_1(z)=c_r z$, where c_r is independent of z , we get

$$(-1)^r p_r(n) = c_r(n+1) \binom{n}{r} - c_r n \binom{n-1}{r} = \\ = (r+1)c_r \left\{ \binom{n+1}{r+1} - \binom{n}{r+1} \right\}.$$

Hence

$$(6.10) \quad p_r(n) = (-1)^r (r+1) c_r \binom{n}{r}.$$

Thus we have proved that (6.10) is a *necessary* condition for the existence of the recurrence (6.8).

Now let

$$(6.11) \quad G_1^{(r)} \equiv G_1^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n F_1^{(r)}(n, m) \frac{x^n}{n!} z^m.$$

Since, by (6.8) and (6.10),

$$(6.12) \quad F_1^{(r)}(n+1, m) = \\ = F_1^{(r)}(n, m-1) + (-1)^r (r+1) c_r \binom{n}{r} F_1^{(r)}(n-r+1, m),$$

it follows from (6.11) that

$$\begin{aligned} D_x G_1^{(r)} &= z G_1^{(r)} + (-1)^r \frac{(r+1)c_r}{r!} \sum_{n,m} F_1^{(r)}(n-r+1, m) \frac{x^n z^m}{(n-r)!} = \\ &= z G_1^{(r)} + (-1)^r \frac{(r+1)c_r}{r!} x^r D_x G_1^{(r)}, \end{aligned}$$

where $D_x \equiv \partial/\partial x$. Thus we have

$$(6.13) \quad D_x G_1^{(r)} = \frac{z}{1 - a_r x^r}, \quad a_r = (-1)^r \frac{(r+1)c_r}{r!}.$$

For example, for $r=2$, $a_r=1$, (6.13) reduces to

$$D_x G_1^{(2)} = \frac{z}{1-x^2},$$

which yields

$$(6.14) \quad G_1^{(2)}(x, z) = \left(\frac{1+x}{1-x} \right)^{z/2}$$

in agreement with (6.4).

For the general case (6.13), we have

$$(6.15) \quad G_1^{(r)}(x, z) = \exp \left\{ z \sum_{m=0}^{\infty} \frac{a_r^m x^{m+1}}{mr+1} \right\}.$$

Conversely, given (6.15) we get the recurrence (6.8) with $p_r(n)$ satisfying (6.10). Thus (6.10) is both necessary and sufficient for the recurrence (6.8).

Note that substitution of (6.10) in (6.9) gives

$$(6.16) \quad f_k^{(r)}(n+1) = \frac{n-rk}{n} f_k^{(r)}(n) c_r \binom{rk}{r} f_{k-1}^{(r)}(n-r+1) \quad (n > 0).$$

By (6.11) and (5.6) we have

$$\begin{aligned} D_x G_1^{(r)}(x, z) &= \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} F_1^{(r)}(n+1, m) \frac{x^n}{n!} z^m = \\ &= \sum_{m,k=0}^{\infty} (-1)^{rk} f_k^{(r)}(m+rk+1) \frac{x^{m+rk} z^m}{m!(rk)!}. \end{aligned}$$

Hence (6.15) becomes

$$\sum_{m,k=0}^{\infty} (-1)^{rk} f_k^{(r)}(m + rk + 1) \frac{x^{m+rk} z^m}{m!(rk)!} = D_x \{ \exp(z\psi(x)) \},$$

so that

$$(6.17) \quad 1 + \sum_{m=1}^{\infty} \frac{x^m z^m}{m!} \sum_{k=0}^{\infty} (-1)^{rk} \frac{f_k^{(r)}(m + rk + 1)}{m + rk + 1} \frac{x^{rk}}{(rk)!} = \exp(z\psi(x)).$$

where

$$(6.18) \quad \psi(x) = \sum_{k=0}^{\infty} \frac{a_r^k x^{rk+1}}{rk + 1}.$$

Similarly we have

$$G^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n F^{(r)}(n, m) \frac{x^n}{n!} z^m = \sum_{m,k=0}^{\infty} \binom{m + rk}{rk} f_k(-m) \frac{x^{m+rk} z^m}{m!(rk)!}$$

and so

$$(6.19) \quad G^{(r)}(x, z) = \sum_{m=0}^{\infty} \frac{x^m z^m}{m!} \sum_{k=0}^{\infty} f_k(-m) \frac{x^{rk}}{(rk)!}.$$

7. - Parallel to (6.8) we consider the possibility of a recurrence of the type

$$(7.1) \quad F^{(r)}(n + 1, n - rk + 1) = \\ = F^{(r)}(n, n - rk) + q_r(n - rk + 1) F^{(r)}(n, n - r(k - 1)).$$

By the second of (5.6), (7.1) becomes

$$\binom{n + 1}{rk} f_k^{(r)}(rk - n - 1) = \\ = \binom{n}{kr} f_k^{(r)}(rk - n) + q_r(n - rk + 1) \binom{n}{r(k - 1)} f_{k-1}^{(r)}(r(k - 1) - n).$$

For $k = 1$, this reduces to

$$\binom{n + 1}{r} f_1^{(r)}(-n) = \binom{n}{r} f_1^{(r)}(r - n) + q_r(n - r + 1).$$

Hence

$$q_r(n-r+1) = -c_r n \binom{n+1}{r} + c_r(n-r) \binom{n}{r},$$

so that

$$(7.2) \quad q_r(n-r+1) = -(r+1)c_r \binom{n}{r}.$$

Thus (7.2) is a necessary condition for the existence of the recurrence (7.1).

Comparison of (7.2) with (6.10) gives

$$(7.3) \quad q_r(n-r+1) = (-1)^{r-1} p_r(n).$$

By (7.1) and (7.2) we have

$$(7.4) \quad \begin{aligned} F^{(r)}(n+1, m) &= \\ &= F^{(r)}(n, m-1) - (r+1)c_r \binom{m+r-1}{r} F^{(r)}(n, m+r-1). \end{aligned}$$

As above put

$$(7.5) \quad G^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n F^{(r)}(n, m) \frac{x^n}{n!} z^m.$$

Then

$$\begin{aligned} D_x G^{(r)}(x, z) &= \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} F^{(r)}(n+1, m) \frac{x^n}{n!} z^m = \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} \left\{ F^{(r)}(n, m-1) - (r+1)c_r \binom{m+r-1}{r} F^{(r)}(n, m+r-1) \right\} \cdot \\ &\cdot \frac{x^n}{n!} z^m = z G^{(r)}(x, z) - (r+1)c_r z \sum_{n=1}^{\infty} \sum_{m=1}^{n-r+1} \binom{m+r-1}{r} \cdot \\ &\cdot F^{(r)}(n, m+r-1) \frac{x^n}{n!} z^{m-1}. \end{aligned}$$

Since the double sum on the extreme right is equal to

$$\sum_{n=r}^{\infty} \sum_{m=r}^n \binom{m}{r} F^{(r)}(n, m) \frac{x^n}{n!} z^{m-r} = \frac{1}{r!} D_x^r G^{(r)}(x, z),$$

we get the partial differential equation

$$(7.6) \quad D_x G^{(r)}(x, z) = zG^{(r)}(x, z) + (-1)^{r-1} a_r z D_z^r G^{(r)}(x, z),$$

where, as in (6.13),

$$(7.7) \quad a_r = (-1)^r \frac{(r+1)e_r}{r!}.$$

Let $\omega(x)$ denote the inverse of $\psi(x)$ that vanishes at the origin:

$$(7.8) \quad \psi(\omega(x)) = x = \omega(\psi(x)),$$

where as above

$$(7.9) \quad \psi(x) = \sum_{m=0}^{\infty} \frac{a_r^m x^{r m + 1}}{r m + 1}.$$

Let $u = \psi(x)$, $x = \omega(u)$. Since

$$\psi'(x) = \frac{1}{1 - a_r x^r}$$

and $\psi'(x)\omega'(u) = 1$, it follows that

$$(7.10) \quad \omega'(u) = 1 - a_r \omega^r(u).$$

Now put

$$(7.11) \quad H(x, z) = \exp \{z\omega(x)\}.$$

This implies an expansion of the form

$$(7.12) \quad H(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n c(n, m) \frac{x^n}{n!} z^m.$$

Differentiation of (7.11) gives

$$D_x H(x, z) = z\omega'(x)H(x, z),$$

$$D_z^r H(x, z) = \omega^r(x)H(x, z),$$

Hence, by (7.10),

$$(7.13) \quad D_x H(x, z) = zH(x, z) - a_r z D_z^r H(x, z).$$

In (7.6) replace x by $-x$, z by $-z$. Then

$$(7.14) \quad D_x G^{(r)}(-x, -z) = zG^{(r)}(-x, -z) - a_r z D_z^r H(x, z).$$

Thus $H(x, z)$ and $G^{(r)}(-x, -z)$ satisfy the same partial differential equation.

Next, since by (7.12),

$$\begin{aligned} D_x H(x, z) &= \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} c(n+1, m) \frac{x^n}{n!} z^m, \\ D_z^r H(x, z) &= r! \sum_{n=r}^{\infty} \sum_{m=r}^n \binom{m}{r} c(n, m) \frac{x^n}{n!} z^{m-r}, \\ z D_z^r H(x, z) &= r! \sum_{n=r}^{\infty} \binom{m+r-1}{r} c(n, m+r-1) \frac{x^n}{n!} z^m. \end{aligned}$$

substitution in (7.13) yields the recurrence

$$(7.15) \quad \begin{aligned} c(n+1, m) &= \\ &= c(n, m-1) + (-1)^{r-1} (r+1) c_r \binom{m+r-1}{r} c(n, m+r-1). \end{aligned}$$

As for $F^{(r)}(n+1, m)$, by (7.4) we have

$$(7.16) \quad \begin{aligned} (-1)^{n-m+1} F^{(r)}(n+1, m) &= (-1)^{n-m+1} F^{(r)}(n, m-1) + \\ &+ (-1)^{r-1} (r+1) c_r \cdot (-1)^{n-m-r+1} F^{(r)}(n, m+r-1). \end{aligned}$$

Since $\omega(x)$ is of the form

$$\omega(x) = \sum_{m=0}^{\infty} \frac{b_m x^{m+1}}{mr+1}, \quad b_0 = 1,$$

it follows from (7.11) that

$$(7.17) \quad c(n, m) = 0, \quad (n \not\equiv m \pmod{r}).$$

Also it is clear from (7.15) that

$$(7.18) \quad c(n, n) = 0, \quad (n = 0, 1, 2, \dots).$$

We conclude that

$$c(n, m) = (-1)^{n-m} F^{(r)}(n, m)$$

and therefore

$$(7.19) \quad G^{(r)}(-x, -z) = \exp \{z(\omega(x))\}.$$

Combining the results of §§ 6, 7 we state the following

THEOREM 6. *The function $F_1^{(r)}(n, m)$ satisfies a recurrence of the form*

$$(7.20) \quad F_1^{(r)}(n+1, m) = F_1^{(r)}(n, m-1) + p_r(n) F_1^{(r)}(n-r+1, m)$$

if and only if

$$(7.21) \quad p_r(n) = (-1)^{r(r+1)} c_r \binom{n}{r},$$

where $f_1^{(r)}(z) = c_r z$. The function $F^{(r)}(n, m)$ satisfies a recurrence of the form

$$F^{(r)}(n+1, m) = F^{(r)}(n, m-1) + q_r(m) F^{(r)}(n, m+r-1)$$

if and only if

$$(7.22) \quad q_r(m) = (-1)^{r-1} p_r(m+r-1),$$

where $p_r(m)$ satisfies (7.21).

Moreover (7.20) and (7.21) are satisfied if and only if

$$G_1^{(r)}(x, z) \equiv 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n F_1^{(r)}(n, m) \frac{x^n}{n!} z^m = \exp \{z\psi(x)\},$$

where

$$\psi(x) = \sum_{n=0}^{\infty} \frac{a_r^n x^{r+1}}{r n + 1}, \quad a_r = (-1)^r \frac{(r+1) c_r}{r!}.$$

It then follows that

$$G^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^n F^{(r)}(n, m) \frac{x^r}{n!} z^m = \exp \{-z\omega(-x)\},$$

where $\omega(x)$ is the inverse of $\psi(x)$ that vanishes at the origin:

$$\psi(\omega(x)) = \omega(\psi(x)) = x.$$

8. – Put

$$(8.1) \quad \exp \{zf(x)\} = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n B_{nk} \frac{x^n}{n!} z^k,$$

where

$$(8.2) \quad f(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!}, \quad b_1 = 1.$$

Let $g(x)$ denote the inverse of $f(x)$:

$$(8.3) \quad g(x) = \sum_{n=1}^{\infty} c_n \frac{x^n}{n!}, \quad c_1 = 1$$

and put

$$(8.4) \quad \exp \{zf(x)\} = 1 + \sum_{n=1}^{\infty} C_{nk} \frac{x^n}{n!} z^k.$$

It is proved in [5] that (B_{nk}) and (C_{nk}) are reciprocal arrays:

$$(8.5) \quad \sum_{k=j}^n B_{nk} C_{kj} = \sum_{k=j}^n C_{nk} B_{kj} = \delta_{nj}.$$

We now apply this result to

$$G_1^{(r)}(x, z) = \exp \{z\psi(x)\},$$

and

$$G^{(r)}(x, z) = \exp \{-z\omega(-x)\}.$$

It follows at once from

$$G_1^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n F_1^{(r)}(n, k) \frac{x^n}{n!} z^k$$

and

$$G^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n F^{(r)}(n, k) \frac{x^n}{n!} z^k$$

that

$$(8.6) \quad \sum_{k=1}^n (-1)^{n-k} F_1^{(r)}(n, k) F^{(r)}(k, j) = \sum_{k=1}^n (-1)^{k-j} F^{(r)}(n, k) F_1^{(r)}(k, j) = \delta_{nj} .$$

We may state

THEOREM 7. *Let*

$$G_1^{(r)}(x, z) = \exp \{z\psi(x)\}$$

or, equivalently,

$$G^{(r)}(x, z) = \exp \{-z\omega(-x)\},$$

where

$$\psi(x) = \sum_{m=0}^{\infty} \frac{a_r^m x^{r m + 1}}{r m + 1}, \quad a_r = (-1)^r \frac{(r + 1) c_r}{r!}$$

and $\psi(\omega(x)) = \omega(\psi(x)) = x$, $\omega(0) = 0$. Then $(F_1^{(r)}(n, k))$, $(F^{(r)}(n, k))$ satisfy the orthogonality relations (8.6).

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