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An estimate on convergence for an homogeneisation problem for variational inequalities

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An estimate on convergence for an homogeneisation problem for variational inequalities.

MARCO BIROLI (*)

1. - Introduction and results.

Let be $\Omega \subset R^n$ an open bounded set with smooth boundary Γ, $Y = \prod_{i=1}^n [o, y_i]$, $a_{ij}(y) \in L^\infty(Y)$ $i, j = 1, \ldots, n$, such that

$$\sum_{i,j=1}^n a_{ij}(y)\, \xi_i\, \xi_j \geq a \mid \xi \mid^2, \ a > 0 \ , \ a.\ e.\ \text{in } Y$$

Let be $\forall\ u, v \in H_0^1(\Omega)$

$$< A^\varepsilon u, v > = \int_\Omega \sum_{i,j=1}^n a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_j}(x) \frac{\partial v}{\partial x_i}(x)\, dx$$

(where $< , >$ is the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and $a_{ij}(y)$ are prolonged to R^n by periodicity) and

$$< A\, u\, , v > = \int_\Omega \sum_{i,j=1}^n q_{ij} \frac{\partial u}{\partial x_j}(x) \frac{\partial v}{\partial x_i}(x)\, dx$$

where q_{ij} are constants defined as in [3], [9], [10].

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Let be $f \in H^{-1}(\Omega)$, we have :

Th. 1 — *We consider the problems*

$$(1,1_\varepsilon) \qquad A^\varepsilon u = f$$

$$(1,1) \qquad A\, u = f$$

let be u_ε *and* u *the solution of* $(1, 1_\varepsilon)$ *and* $(1, 1)$, *we have*

$$\lim_{\varepsilon \to 0}{}^{*}\ u_\varepsilon = u \text{ in } H_0^1(\Omega)$$

Moreover if $a_{ij}(y) \in C^1(R^n), f \in W^{1,p}(\Omega), n < p \leq +\infty$, *we have*

$$(1,2) \qquad \| u_\varepsilon - u \|_{L^\infty} \leq C \, \| f \|_{w^{1,p}} \, \varepsilon .$$

The first part of Th. 1 is shown in [7] for the symmetric case and in [8], [12] in the general case by « local energy » method, the second part of Th. 1 is shown by « multiple scales » method in [8] [9] (*). We consider now the problems

$$(1,3_\varepsilon) \qquad \begin{array}{c} < A^\varepsilon u_\varepsilon \, , \, v\text{-}u_\varepsilon > \ \geq \ < f, \, v\text{-}u_\varepsilon > \\ \forall \, v \in H_0^1(\Omega) \quad , \quad v \leq \psi \\ u \in H_0^1(\Omega) \quad , \quad u \geq \psi \end{array}$$

$$(1,3) \qquad \begin{array}{c} < A\, u \, , \, v\text{-}u > \ \geq \ < f, v\text{-}u > \\ \forall \ v \in H_0^1(\Omega) \ , \quad v \leq \psi \\ u \in H_0^1(\Omega) \ , \quad u \leq \psi \end{array}$$

where ψ is a measurable function, such that $\{v \in H_0^1(\Omega), \, v \leq \psi\} \neq \Phi$.

Th. 2 *Let be* u_ε *and* u *solution of* $(1,3_\varepsilon)$ *and* $(1,3)$ *we have*

$$\lim_{\varepsilon \to 0}{}^{*} \ \ u_\varepsilon = u \quad \text{in } H_0^1(\Omega)$$

Moreover if $\psi \in C(\bar{\Omega}), f \in L^\infty(\Omega)$

$$\lim_{\varepsilon \to 0} u_\varepsilon = u \text{ in } C(\bar{\Omega}).$$

(*) Cf. also [1], [2].

The first part of Th. 2 is shown in [11] by the method of « local energy » (cfr. also [5] and [6]), the second part of Th. 2 is shown in [4] by penalisation method and Hölder-estimates. The aim of this work is show an estimate analogous to (1, 2) in the case of variational inequalities.

Th. 3. *Let be* $a_{ij}(y) \varepsilon C^1 (R^n)$, $f \in L^\infty(\Omega)$, $u_\varepsilon(u)$ *the solution of* $(1, 3_\varepsilon)$ $\big((1, 3)\big)$, α the De Giorgi's exponent, $n \geq 2$

(I) *If* $\|A^\varepsilon \psi\|_{L^\infty}, \|A \psi\|_{L^\infty} \leq C$, *we have*

$$\| u_\varepsilon - u\|_{L^\infty} \leq K \, \varepsilon^{\frac{\alpha}{n-2+3\alpha}}$$

(II) *If* $\psi \in W^{1,p}(\Omega)$, $n < p > +\infty$, *we have*

$$\|u_\varepsilon - u\|_{L^\infty} \leq K \, \varepsilon^{\frac{\alpha}{2(n-2+3\alpha)}}$$

The Th. 2 is shown by penalisation and « multiple scales » method. In § 2 we show some preliminary results, in § 3 we study the convergence for the penalised equations and in § 4 we give the proof of Th. 3.

Rem. 1 (*a*) The hypothesis of the part (I) are verified if $\psi = 0$:

(*b*) The result of Th. 3 can be generalized to the case $a_{ij}\left(x, \dfrac{x}{\varepsilon}\right)$

(*c*) We don't know if the estimates in Th. 3 are optimal;

2. - Some preliminary results.

We consider the penalised problem

$$A\,u + \frac{1}{\lambda}\,(u - \psi)^+ = f \ , \ \lambda > 0 \, . \tag{2,1}$$

Lemma 1. *The problem* (2.1) *has a unique solution* u^λ; *if* $A\,\psi \in L^\infty(\Omega)$ *we have*

$$\frac{1}{\lambda}\,\|(u^\lambda-\psi)^+\|_{L^\infty} \le \|f\|_{L^\infty} + \|A\psi\|_{L^\infty} \tag{2,2}$$

$$\|A\,u^\lambda\|_{L^\infty} \le 2\,\|f\|_{L^\infty} + \|A\psi\|_{L^\infty} \tag{2,3}$$

From (2,1) we have

$$\frac{1}{\lambda^P}\,\|(u^\lambda-\psi)^+\|^P_{L^p} \le \frac{1}{\lambda^{P-1}}\|(u^\lambda-\psi)^+\|^{P-1}_{L^p}\,(\|f\|_{L^p} + \|A\psi\|_{L^p})$$

then

$$\frac{1}{\lambda}\,\|(u^\lambda-\psi)^+\|_{L^p} \le C_1 + C_2\,\|A\,\psi\,\|_{L^p} \tag{2,4}$$

$$\|A\,u^\lambda\|_{L^p} \le C_1 + C_2\,\|A\,\psi\|_{L^p} \tag{2,5}$$

From (2,5) we have $u^\lambda \in C(\bar{\Omega})$.
We show now (2,2) « ab absurdum ».
We write

$$K = \|f\|_{L^\infty} + \|A\,\psi\|_{L^\infty}\,. \tag{2,6}$$

Let be

$$\Omega' = \left\{x \mid \frac{1}{\lambda}(u^\lambda-\psi)^+ > K\right\} \ne \varnothing$$

(2,7)

$$\Omega'_\eta = \left\{x \mid \frac{1}{\lambda}(u^\lambda-\psi)^+ > K+\eta\right\} \ne \varnothing \quad , \; \eta > 0$$

Ω' and Ω'_η are open set and $\Omega'_\eta \subset \bar{\Omega}'_\eta \subset \Omega'$.
Let be now Ω'' an open set with smooth boundary such that $\bar{\Omega}'_\eta \subset \subset \Omega'' \subset \bar{\Omega}'' \subset \Omega'$.

We have

$$A\,(u^\lambda - \psi) < 0 \text{ in } \Omega''$$

then from the maximum principle, [13],

$$\frac{1}{\lambda}\,(u^\lambda - \psi)^+ \leq K + \eta \quad \text{in } \Omega'' \supset \Omega'_\eta \tag{2,8}$$

From (2,8) we have

$$\Omega'_\eta = \varnothing\ , \tag{2,9}$$

then from (2,7) (2,9) we have a contradiction.

From (2,1) (2,2) we have also (2,3).

Lemma 2. *If* $f \in L^\infty(\Omega)$, $\psi \in W^{1,p}(\Omega)$, $n < p \leq +\infty$, *we have*

$$\|u^\lambda - u\|_{L^\infty} \leq K\,\lambda^{\frac{\alpha}{2[n-2(1-\alpha)]}}. \tag{2,10}$$

Moreover if $A\,\psi \in L^\infty(\Omega)$ *we have*

$$\|u^\lambda - u\|_{L^\infty} \leq K\,\lambda^{\frac{\alpha}{n-2(1-\alpha)}}. \tag{2,11}$$

We show at first (2.11).

We observe that if $A\,\psi \in L^\infty(\Omega)$ from the lemma 1 we have $\|A\,u^\lambda\|_{L^\infty}$, $\|A\,u\|_{L^\infty} \leq \tilde{C}$ and repeating the calculation of [6] pg. 76-88 or [13] 177-197 we have

$$\|\,u^\lambda\,\|_{C^\alpha}\,,\ \|\,u\,\|_{C^\alpha} \leq C$$

where C depend only on $\tilde{C}$.

From $(1,6_\varepsilon)$ (2,1) we have

$$< A\,u^\lambda\,,\, u - u^\lambda > \ \geq\ < f\,,\, u - u^\lambda >$$

$$< Au\,,\, u^\lambda - (u^\lambda - \psi)^+ - u > \ \geq\ < f\,,\, u^\lambda - (u^\lambda - \psi)^+ - u >$$

then

$$\text{2,12)} \qquad \| u^\lambda - u \|_{H_0^1} \leq \lambda^{1/2} \left(C_1 + C_2 \| A \psi \|_{L^\infty} \right)$$

From (2,12) we have

$$(2,13) \qquad \| u^\lambda - u \| L^{\frac{2n}{n-2}} \leq \lambda^{1/2} \left(C_1' + C_2' \| A \psi \|_{L^\infty} \right).$$

We prolongate u^λ and u by 0 and we denote again by u^λ and u these new functions.

From (2,13) we have

$$\int_{B(\bar{x},\varrho)} | u^\lambda (x) - u(x) | \, dx \leq \lambda^{1/2} \left(C_1' + C_2' \| A \psi \|_{L^\infty} \right) | B(\bar{x}, \varrho) |^{1 - \frac{n-2}{2n}}$$

then

$$\frac{1}{| B(\bar{x}, \varrho) |} \int_{B(\bar{x},\varrho)} | u^\lambda (x) - u(x) | \, dx \leq \lambda^{1/2} (C_1' +$$

$$+ C_2' \| A \psi \|_{L^\infty}) | B(\bar{x}, \varrho) |^{-\frac{n-2}{2n}} \leq \lambda^{1/2} \left(C_1'' + C_2'' \| A \psi \|_{L^\infty} \right) \varrho^{-\frac{n-}{2n}}$$

then there is $\tilde{x} \in B(\bar{x}, \varrho)$ such that

$$(2,14) \qquad | u^\lambda (\tilde{x}) - u(\tilde{x}) | \leq \lambda^{1/2} \left(C_1'' + C_2'' \| A \psi \|_{L^\infty} \right) \varrho^{-\frac{n-2}{2n}} .$$

From (2,14) we have

$$(2,15) \qquad | u^\lambda (x) - u(x) | \leq \lambda^{1/2} \left(C_1'' + C_2'' \| A \psi \|_{L^\infty} \right) \varrho^{-\frac{n-2}{2n}} + C \varrho^\alpha$$

for $x \in B(\bar{x}, \varrho)$.

We choose $\varrho = \lambda^{\frac{1}{n-.(1-\alpha)}}$ then

$$(2,16) \quad | u^\lambda (x) - u(x) | \leq \lambda^{\frac{}{n-2(1-\alpha)}} \left(C_1'' + C_2'' \| A \psi \|_{L^\infty} \right) + C \lambda^{\frac{\alpha}{n-2(1-\alpha)}}.$$

We show now (2,10).

From the lemma 2.1 of [7] (the hypothesis of symmetry is unnecessary for this lemma) if $\psi \in W^{1,p}(\Omega)$, $n < p \leq +\infty$, there is a sequence $\{\psi_\eta\}$ such that

$$(2{,}17) \qquad \|A\, \psi_\eta\|_{L^\infty} \leq C_3\, \eta^{-1/2}$$

$$(2.18) \qquad \|\psi_\eta - \psi\|_{L^\infty} \leq C_3\, \eta^{1/2}$$

where C_3 is a constant dependent only on $\|\psi\|_{W^{1,p}}$.
Let be u^λ_η (u_η) the solution of the problem

$$(2.19) \qquad A\, u^\lambda_\eta + \frac{1}{\lambda}\,(u^\lambda_\eta - \psi_\eta)^+ = f$$

$$((2{,}20) \qquad < A\, u_\eta, v - u_\eta > \ \geq \ < f, v - u_\eta >$$

$$\forall\, v \in H^1_0(\Omega)\ ,\ v \leq \psi_\eta$$

$$u_\eta \in H^1_0(\Omega)\ ,\ u_\eta \leq \psi_\eta).$$

From [12] pg. 22, part III, *we* frave

$$(2{,}21) \qquad \|u^\lambda_\eta - u^\lambda\|_{L^\infty} \leq C_5\, \eta^{1/2}$$

$$(2{,}22) \qquad \|u_\eta - u\|_{L^\infty} \leq C_5\, \eta^{1/2}\ .$$

From (2,16) (2,21) (2,22) we have

$$|u^\lambda(x) - u(x)| \leq |u^\lambda(x) - u^\lambda_\eta(x)| + |u^\lambda_\eta(x) - u_\eta(x)| + |u_\eta(x) - u(x)|$$

$$\leq \lambda^{\frac{\alpha}{n-2(1-\alpha)}}\,(C_1'' + C_2''\,\eta^{-1/2}) + C\,\lambda^{\frac{\alpha}{n-2(1-\alpha)}} + 2\,\eta^{1/2}$$

We choose $\eta = \lambda^{\frac{\alpha}{2[n-2(1-\alpha)]}}$ then

$$|u^\lambda(x) - u(x)| \leq C_6\, \lambda^{\frac{\alpha}{2[n-2(1-\alpha)]}}$$

Rem. 1 — We observe that the constants in the lemma 2 depend only on L^∞ — norm of the coefficients of A.

Lemma 3. *We consider the problems*

$$A^\varepsilon u_\varepsilon = f \ , \ A\, u = f$$

for $f \in L^\infty(\Omega)$ *we have*

$$\|u_\varepsilon - u\|_{L^\infty} \leq C\, \varepsilon^{1/2} \ .$$

We consider $\theta = -\Delta^{-1} f$, $\theta \in W^{2,p}(\Omega)$. $\forall p$, for the lemma 2.1 of [7] there is a sequence θ_η such that

$$\|\theta_\eta - \theta\|_{W^{1,\infty}} \leq C_1\, \eta^{1/2}$$

$$\|\Delta\, \theta_\eta\|_{W^{1,\infty}} \leq C_1\, \eta^{-1/2} \ .$$

Let be $f_\eta = -\Delta\theta_\eta$: we have

$$\|f_\eta - f\|_{W^{-1,\infty}}\, C_2\, \eta^{1/2}$$

$$\|f_\eta\|_{W^{1,\infty}} \leq C_2\, \eta^{-1/2} \ .$$

Let be u_ε^η (u^η) the solution of the problem

$$A^\varepsilon u_\varepsilon^\eta = f_\eta \qquad (A\, u^\eta = f_\eta) \,.$$

We have

$$\|u_\varepsilon - u\|_{L^\infty} \leq \|u_\varepsilon - u_\varepsilon^\eta\|_{L^\infty} + \|u_\varepsilon^\eta - u^\eta\|_{L^\infty} + \|u^\eta - u\|_{L^\infty} \leq$$
$$\leq 2\, C_1'\, \eta^{1/2} + C_2'\, \varepsilon\, \eta^{-1/2}$$

We choose $\eta = \varepsilon$ and we have the result.

3. – The penalised equation.

We suppose for the moment $f = 0$.

We consider the penalised problems

$$(3{,}1_\varepsilon) \qquad A^\varepsilon u_\varepsilon^\lambda + \frac{1}{\lambda}\, (u_\varepsilon^\lambda - \psi)^+ = 0$$

$$A\,u^{\lambda} + \frac{1}{\lambda}\,(u^{\lambda} - \psi)^{+} = 0\,. \tag{3.1}$$

Lemma 4. *We have*

$$\|u_{\varepsilon}^{\lambda} - u^{\lambda}\|_{L^{\infty}} \leq (K_1 + K_2\,\lambda^{-1})\;\|A\,\psi\|_{L^{p}}\,\varepsilon$$

$$n < p < +\infty\,.$$

We use for the proof the « multiple scales » method.

Let be $y = \dfrac{x}{\varepsilon}$ we have, [8] [9],

$$A^{\varepsilon} = \varepsilon^{-2}\,A_1 + \varepsilon^{-1}\,A_2 + A_3 \tag{3,2}$$

where

$$A_1 = -\sum_{ij-1}^{n} a_{ij}(y)\,\frac{\partial^2}{\partial y_i\,\partial y_j} - \sum_{ij=1}^{n}\left(\frac{\partial}{\partial y_i}\,a_{ij}(y)\right)\frac{\partial}{\partial y_j} =$$

$$= -\sum_{ij=1}^{n}\frac{\partial}{\partial y_i}\left(a_{ij}(y)\,\frac{\partial}{\partial y_j}\right)$$

$$\mathrm{A}_2 = -\sum_{ij=1}^{n} a_{ij}(y)\left(\frac{\partial^2}{\partial x_i\,\partial y_j} + \frac{\partial^2}{\partial x_j\,\partial y_i}\right) - \sum_{ij=1}^{n}\left(\frac{\partial}{\partial y_i}\,a_{ij}(y)\right)\frac{\partial}{\partial x_j} =$$

$$= -\sum_{i,j=1}^{n}\frac{\partial}{\partial y_j}\left(a_{ij}(y)\,\frac{\partial}{\partial x_i}\right) - \sum_{i,j=1}^{n} a_{ij}(y)\,\frac{\partial^2}{\partial x_i\,\partial y_j}$$

$$\mathrm{A}_3 = -\sum_{i,j=1}^{n} a_{ij}(y)\,\frac{\partial^2}{\partial x_i\,\partial x_j}$$

Let be

$$A_1\,\chi^{j}(y) = \sum_{i=4}^{n}\frac{\partial}{\partial y_i}\left(a_{ij}(y)\right)$$

$$\chi^{j}(y)\ Y\text{-periodic}\ ,\ \int_{Y}\chi^{j}(y)\,dy = 0$$

we have $\chi^{j}(y) \in C^2(\bar{\Omega})$.

Let be

$$\chi_1(x\,;y)=\sum_{i=1}^{n}\chi^i(y)\,\frac{\partial u^\lambda}{\partial x_i}\quad;\quad v_\varepsilon=u^\lambda+\varepsilon\,\chi_1+\varepsilon^2\,\chi_2$$

where $\chi_2(x\,;y)$ is not given for the moment.

We consider $w=u_\varepsilon^\lambda-v_\varepsilon$,

$$\begin{aligned}A^\varepsilon(w)+\frac{1}{\lambda}(u_\varepsilon^\lambda-\psi)^+-\frac{1}{\lambda}(u^\lambda+\varepsilon\,\chi_1+\varepsilon^2\,\chi_2-\psi)^+ = \\ =-\Big(A_1\,\chi_2+A_2\,\chi_1+A_3\,u^\lambda+\frac{1}{\lambda}(u^\lambda-\lambda)^+\Big)- \\ -\varepsilon\,(A_3\,\chi_1+A_2\,\chi_2)-\varepsilon^2\,A_3\,\chi_2+\frac{1}{\lambda}(u^\lambda-\psi)^+ \\ -\frac{1}{\lambda}(u^\lambda+\varepsilon\,\chi_1+\varepsilon^2\,\chi_2-\psi)^+\end{aligned}$$

We choose $\chi_2(x\,;y)$ such that

$$A_1\,\chi_2+A_2\,\chi_1+A_3\,u^\lambda+\frac{1}{\lambda}(u^\lambda-\psi)^+=0 \tag{3,4}$$

$\chi_2(x\,;y)$ Y-periodic in y.

We observe that we have

$$\int_Y\Big(A_2\,\chi_1+A_3\,u^\lambda+\frac{1}{\lambda}(u^\lambda-\psi)^+\Big)dy=0$$

then the problem (3,4) has a solution.

We observe that

$$\begin{aligned}\|A\,\psi\|_{L^p}\le C_1\Rightarrow\frac{1}{\lambda}\,\|(u^\lambda-\psi)^+\|_{L^p}\le C_1\Rightarrow \\ \Rightarrow\|u^\lambda\|_{w^{2,p}}\le C_2\cdot C_1\Rightarrow\frac{1}{\lambda}\,\|(u^\lambda-\psi)^+\|_{w^{1,p}}\le\frac{C_3\,C_1}{\lambda} \\ \Rightarrow\|u^\lambda\|_{w^{3,p}}\le\frac{C_3\cdot C_1}{\lambda}\quad(n<p\le+\infty)\,.\end{aligned}\tag{3,5}$$

From (3,5) we have

$$\|\chi_1\,(x\,;\,y)\|_{c^1(Y;w^{2,P})} \leq \frac{C_5}{\lambda}\,\|A\psi\|_{L^P}$$

then

$$\|A_3\,\chi_1\,(x\,;\,y)\|_{c(Y;L^P)} \leq \frac{C_5}{\lambda}\,\|A\psi\|_{L^P} \tag{3,6}$$

$$\|A_2\chi_1\,(x\,;\,y)\|_{c(Y,W^{1,P})} \leq \frac{C_7}{\lambda}\,\|A\psi\|_{L^P}\,. \tag{3,7}$$

From (3,4) (3,5) (3,7) we have

$$\|\chi_2\,(x\,;\,y)\|_{c^2(Y;w^{1,P})} \leq \frac{C_8}{\lambda}\,\|A\psi\|_{L^P} \tag{3,8}$$

and, by multiplication for $v \in W^{1,p'}(\Omega)$ ${}^1/_p + {}^1/_{p^1} = 1$, we can easily show

$$\left\|\varepsilon\,A_3\,\chi_2\left(x\,;\,\frac{x}{\varepsilon}\right)\right\|_{w^{-1,p}} \leq \frac{C_9}{\lambda}\,\|A\psi\|_{L^P}\,. \tag{3,9}$$

We observe now that

$$\begin{aligned}\left\|\frac{1}{\lambda}\,(u^\lambda - \psi)^+ - \frac{1}{\lambda}\,(u^\lambda + \varepsilon\,\chi_1 + \varepsilon^2\,\chi_2 - \psi)^+\right\|_{L^\infty} &\leq \\ \leq \frac{\varepsilon}{\lambda}\,\|\chi_1\|_{L^\infty} &+ \frac{\varepsilon^2}{\lambda}\,\|\chi_2\|_{L^\infty} \\ \leq C_{10}&\left(\frac{\varepsilon + \varepsilon^2}{\lambda}\right)\|\,A\,\psi\,\|_{L^\infty}\end{aligned} \tag{3,10}$$

From (3,6) (3,7) (3,8) (3,9) (3,10) we have

$$A^\varepsilon(w) + \frac{1}{\lambda}\,(u^\lambda - \psi)^+ - \frac{1}{\lambda}\,(u^\lambda + \varepsilon\,\chi_1 + \varepsilon^2\,\chi_2 - \psi)^+ = \theta \tag{3,11}$$

where

$$\|\theta\|_{w^{-1,P}} \leq \left(C_{11} + \frac{C_{12}}{\lambda}\right) \|A\psi\|_{L^P} \varepsilon$$

By the same method used in [10], [13] for the proof of L^∞ — — estimate in elliptic problems we have

$$\|w\|_{L^\infty} \leq \left(C'_{11} + \frac{C'_{12}}{\lambda}\right) \|A\psi\|_{L^P} \varepsilon \tag{3,12}$$

then from (3,12)

$$\|u^\lambda_\varepsilon - u^\lambda\|_{L^\infty} \leq \left(C''_1 + \frac{C'_{12}}{\lambda}\right) \|A\psi\|_{L^P} \varepsilon \,.$$

We consider now the case $f \neq 0$, $f \in L^P$, $n < p \leq +\infty$.
Let be

$$\sigma = A^{-1} f \qquad \sigma^\varepsilon = (A^\varepsilon)^{-1} f$$

we have

$$\|\sigma^2 - \sigma\|_{L^\infty} \leq C_{13}\, \varepsilon^{1/2}$$

We observe that the problems

$$A^\varepsilon u^\lambda_\varepsilon + \frac{1}{\gamma} (u^\lambda_\varepsilon - \psi)^+ = f \tag{3,13$_\varepsilon$}$$

$$A u^\lambda + \frac{1}{\lambda} (u^\lambda - \psi)^+ = f \tag{3,13}$$

are equivalent to the problems

$$A^\varepsilon v^\lambda_\varepsilon + \frac{1}{\lambda} \big(v^\lambda_\varepsilon - (\psi - \sigma^\varepsilon)\big)^+ = 0 \tag{3,14$_\varepsilon$}$$

$$A v^\lambda + \frac{1}{\lambda} \big(v^\lambda - (\psi - \sigma)\big)^+ = 0 \tag{3,14}$$

where

$$v^\lambda = u^\lambda - \sigma \qquad v^\lambda_\varepsilon = u^\lambda_\varepsilon - \sigma^\varepsilon . \tag{3,15}$$

We consider now the problem

$$A^\varepsilon w^\lambda_\varepsilon + \frac{1}{\lambda}\big(w^\lambda_\varepsilon - (\psi - \sigma)\big)^+ = 0 . \tag{3,14$'_\varepsilon$}$$

From [12] pg 22. part III we have

$$\|v^\lambda_\varepsilon - w^\lambda_\varepsilon\|_{L^\infty} \leq C_{13}\, \varepsilon^{1/2}$$

and from (3,12) we have

$$\|w^\lambda_\varepsilon - v^\lambda\|_{L^\infty} \leq \left(C_{14} + \frac{C_{15}}{\lambda}\right)(\|A\psi\|_{L^p} + \|f\|_{L^p})\, \varepsilon$$

then

$$\|u^\lambda_\varepsilon - u^\lambda\|_{L^\infty} \leq (C_{14} + \underset{\overline{\lambda}}{C_{15}})\, (\, \|A\, \psi\|_{L^p} + \|f\|_{L^p})\, \varepsilon + 3\, C_{13}\, \varepsilon^{1/2} . \tag{3,16}$$

4. - Prof. of Th. 3.

We consider the problem

$$< A^\varepsilon u_\varepsilon , v - u_\varepsilon > \; \geq \; < f, v - u_\varepsilon > \tag{4,1$_\varepsilon$}$$

$$\forall\, v \in H^1_0(\Omega) \qquad v \leq \psi$$

$$u_\varepsilon \in H^1_0(\Omega) \qquad v_\varepsilon \leq \psi$$

$$< A\, u, v - u > \; \geq \; < f, v - u > \tag{4,1}$$

$$\forall\, v \in H^1_0(\Omega) \qquad v \leq \psi$$

$$u \in H^1_0(\Omega) \qquad u \leq \psi$$

and the corresponding penalised problems

$$A^\varepsilon u^\lambda_\varepsilon + \frac{1}{\lambda}\, (u^\lambda_\varepsilon - \psi)^+ = f \tag{4,2}$$

$$A\,u^{\lambda}+\frac{1}{\lambda}\,(u^{\lambda}-\psi)^{+}=f \tag{4,2}$$

We suppose for the moment

$$\|A^{\varepsilon}\,\psi\|_{L^{\infty}}\ ,\ \|A\,\psi\|_{L^{\infty}}\leq C\,.$$

We have

$$\|u_{\varepsilon}^{\lambda}-u_{\varepsilon}\|_{L^{\infty}}\leq C_1\,\lambda^{\frac{1}{n-2(1-\alpha)}}$$

$$\|u^{\lambda}-u\|_{L^{\infty}}\leq C_1\,\lambda^{\frac{1}{n-2(1-\alpha)}}$$

$$\|\,u_{\varepsilon}^{\lambda}-u^{\lambda}\,\|_{L^{\infty}}\leq\left(C_2+\frac{C_3}{\lambda}\right)\varepsilon+C_4\,\varepsilon^{1/2}$$

then

$$\|u_{\varepsilon}-u\|_{L^{\infty}}\leq 2C_1\,\lambda^{\frac{1}{n-2(1-\alpha)}}+\left(C_2+\frac{C_3}{\lambda}\right)\varepsilon+C_4\,\varepsilon^{1/2}\,.$$

We choose $\lambda=\varepsilon^{\frac{n-2+2x}{n-2+3\alpha}}$ then

$$\|u_{\varepsilon}-u\|_{L^{\infty}}\leq C_5\,\varepsilon^{\frac{\alpha}{n-2+3\alpha}}\,.$$

Let be now $\psi\in W^{1,p}(\Omega)$, $p>n$; there is $\psi_{\varepsilon}^{\eta}$, ψ^{η} $(\eta>0)$. such that, [7],

$$\|\psi_{\varepsilon}^{\eta}-\psi\|_{L^{\infty}}\leq C_6\,\eta^{1/2}$$

$$\|\psi^{\eta}-\psi\|_{L^{\infty}}\leq C_6\,\eta^{1/2}$$

$$\|A^{\varepsilon}\,\psi_{\varepsilon}^{\eta}\|_{L^{\infty}}\leq C_7\,\eta^{-1/2}$$

$$\|A\,\psi^{\eta}\|_{L^{\infty}}\leq C_7\,\eta^{1/2}\ .$$

We consider the problems

$$(4,3_\eta) \qquad < A^\varepsilon u_\varepsilon^\eta, v - u_\varepsilon^\eta > \; \geq \; < f, v - u_\varepsilon^\eta >$$

$$\forall\, v \in H_0^1(\Omega) \;,\; v \leq \psi_\varepsilon^\eta$$

$$u_\varepsilon^\eta \in H_0^1(\Omega) \;,\; u_\varepsilon^\eta \leq \psi_\varepsilon^\eta$$

$$(4,3) \qquad < A\, u^\eta, v - u^\eta > \; \geq \; < f, v - u^\eta >$$

$$\forall\, v \in H_0^1(\Omega) \;,\; v \leq \psi^\eta$$

$$u^\eta \in H_0^1(\Omega) \;,\; u^\eta \leq \psi^\eta .$$

From the abowe we have

$$\|u_\varepsilon^\eta - u^\eta\|_{L^\infty} \leq \left(C_8 + \frac{C_9}{\eta^{1/2}}\right) \varepsilon^{\frac{\alpha}{n-2+3\alpha}}$$

$$\|u_\varepsilon^\eta - u_\varepsilon\|_{L^\infty} \;,\; \|u^\eta - u\|_{L^\infty} \leq C_7\, \eta^{1/2}$$

then

$$\|u_\varepsilon - u\|_{L^\infty} \leq (C_8 + C_{9/\eta^{1/2}})\, \varepsilon^{\frac{\alpha}{n-2+3\alpha}} + 2\, C_7\, \eta^{1/2}.$$

We choose $\eta = \lambda^{\frac{\alpha}{n-2+3\alpha}}$ then

$$\|u_\varepsilon - u\|_{L^\infty} \leq C_{10}\, \varepsilon^{\frac{\alpha}{2[n-2+3\alpha]}}$$

Rem. 1 — We observe that the dependence on n of the estimate of th. 3 derive from the dependence on n of the estimate in the lemma 2 ; then from (2,13) repeating our calculations we have

$$\|u_\varepsilon - (u + \varepsilon\, \chi_1)\|_{H_0^1} \leq C\, \varepsilon^{1/4}$$

if $\|A^\varepsilon\, \psi\|_{L^2} \;,\; \|A\, \psi\|_{L^2} \leq C .$

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