

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

ATTILIO LE DONNE

On prime ideals of $C(X)$

Rendiconti del Seminario Matematico della Università di Padova,
tome 58 (1977), p. 207-214

http://www.numdam.org/item?id=RSMUP_1977__58__207_0

© Rendiconti del Seminario Matematico della Università di Padova, 1977, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques*
<http://www.numdam.org/>

On prime ideals of $C(X)$

ATTILIO LE DONNE (*)

Introduction.

In the paper [DO] De Marco and Orsatti have studied the mapping $\sigma: P \mapsto P \cap C^*(X)$ of $\mathfrak{S}(C(X))$ into $\mathfrak{S}(C^*(X))$, the spectra of the prime ideals of $C(X)$ and $C^*(X)$ respectively.

If M is a maximal ideal of $C(X)$, $\sigma(M) = M \cap C^*$ is comparable with every prime ideal $P^* \subset M^*$ if M^* is the unique maximal ideal of C^* containing σM . σ subordinates a bijection preserving inclusion between the prime ideals of $C(X)$ contained in M and the prime ideals of $C^*(X)$ contained in $M \cap C^*(X)$.

σM is the minimum prime ideal of $C^*(X)$ comparable with every prime ideal contained in M^* iff M has the same property in $C(X)$; in this case M will be called ramified. We generalize the definition at every prime ideal (necessarily z -ideal) of $C(X)$: that is, a prime ideal P of $C(X)$ is ramified if it is the minimum prime ideal comparable with every prime ideal contained in P . We give several equivalent conditions for a prime z -ideal to be ramified; we produce a result, due to De Marco and independent from the remaining work, stating that every maximal fixed ideal of a space satisfying the first axiom of countability is ramified.

The main result of this paper is the theorem stating that every prime z -ideal in a metric space is ramified.

I. X denotes a T_2 completely regular topological space, $C(X)$ the ring of continuous functions, βX the Stone — Čech compactifi-

(*) Current address: Seminario Matematico Università di Padova, via Belzoni 7-Padova.

Lavoro eseguito nell'ambito dei gruppi di Ricerca Matematica del C.N.R.

ation (βX is a subspace of $\text{Spec}(C(X))$, the set of prime ideals of $C(X)$ with spectral topology). For every $p \in \beta X$, M^p will be the maximal ideal associated to it. For every ideal P of $C(X)$, $Z[P]$ is the z -filter of P . If \mathfrak{F} is a z -filter, $Z^\leftarrow[\mathfrak{F}]$ is the z -ideal of \mathfrak{F} .

PROPOSITION. *Let Q be a prime ideal of $C(X)$. Let $R(Q)$ denote the ideal sum of all the minimal prime ideals contained in P ; $R(Q)$ is smallest among the prime ideals comparable with every prime ideal contained in Q .*

PROOF. Let $P \subset Q$ be prime. P contains a minimal ideal P_α . As either P or $R(Q)$ contain P_α , they are comparable. If now P is comparable with every prime contained in Q , then it contains the minimal prime ideals contained in Q , then $P \supset R(Q)$.

DEFINITION 1. Let Q be a prime ideal of $C(X)$. We say Q is *ramified* if $Q = R(Q)$.

DEFINITION 2. Let $p \in \beta X$. We say p is *ramified* if M^p is ramified.

DEFINITION 3. Let $p \in \beta X$. We say p is *totally ramified* if every prime z -ideal Q is ramified, with $Q \subset M^p$.

Analogously X is said to be ramified (r. totally ramified) if every $p \in \beta X$ is ramified (r. totally ramified).

COROLLARY. *A ramified prime ideal is necessarily a z -ideal.*

PROOF. A minimal prime ideal is a z -ideal. [GJ 14.7].

LEMMA. *If I and J are z -ideals then $Z[I + J] = (Z[I], Z[J])$ (the z -filter generated by $Z[I]$ and $Z[J]$) and every element of $Z[I + J]$ is the intersection of two elements of $Z[I]$ and $Z[J]$.*

PROOF. Trivial.

THEOREM. *Let Q be a non-minimal prime z -ideal. The following conditions are equivalent:*

- (a) Q is ramified;
- (b) for every $Z \in Z[Q]$ there exist $Z_1, Z_2 \in Z[Q]$ such that $Z_1 \cap Z_2 \subset Z$; and, if $Z' \in Z(X)$ is such that $Z' \supset Z_1 \setminus Z_2$ or $Z' \supset Z_2 \setminus Z_1$, then $Z' \in Z[Q]$;
- (c) for every $Z \in Z[Q]$ there exist $Z_1, Z_2 \in Z[Q]$ such that $Z_1 \cap Z_2 \subset Z$ and $Q \in \text{cl}_{\text{Spec}(C(X))} (Z_1 \setminus Z_2) \cap \text{cl}_{\text{Spec}(C(X))} (Z_2 \setminus Z_1)$;

(d) every cozero set $A = X \setminus Z$ with $Z \in Z[Q]$ is contained in a $B = X \setminus Z_0$ with $Z_0 \in Z[Q]$ and B non C^* -embedded in $B \cup \{Q\} \subset \subset \text{Spec}(C(X))$;

(e) Q is generated by the functions of Q that change their sign in every zero set of $O_Q = \bigcap \{P_\alpha : P_\alpha \text{ minimal prime ideals contained in } Q\}$.

PROOF.

(b) \Leftrightarrow (c).

A base of neighborhoods of Q in $\text{Spec}(C(X))$ is produced by the sets $V_f = \{P \in \text{Spec}(C(X)) : P \not\ni f\}$ with $f \notin Q$. But $V_f \cap X = \{p \in X : M_p \not\ni f\} = \{p \in X : f(p) \neq 0\} = X \setminus Z(f)$ (with $Z(f) \notin Z[Q]$). Then we have $Q \in cl_{\text{Spec}(C(X))}(Z_1 \setminus Z_2)$ iff for each $Z' \notin Z[Q]$ it is $(X \setminus Z') \cap (Z_1 \setminus Z_2) \neq \emptyset$ i.e. iff for each $Z' \notin Z[Q]$ it is $Z' \not\ni Z_1 \setminus Z_2$.

(a) \Rightarrow (b).

Call P_α ($\alpha \in I$) the minimal prime ideals. Put $Z \in Z[Q] = Z[\sum_{P_\alpha \subset Q} P_\alpha]$. Let $\alpha_1, \dots, \alpha_n$ a set of indexes minimal for the property: $Z \in Z[P_{\alpha_1} + \dots + P_{\alpha_n}]$. If $n > 1$, $Z = Z_1 \cap Z_2$ with $Z_1 \in Z[P_{\alpha_1} + \dots + P_{\alpha_{n-1}}]$ and $Z_2 \in Z[P_{\alpha_n}]$ by lemma. Then Z_1, Z_2 satisfy the hypothesis of (b). In fact they belong to $Z[Q]$ and if $Z' \in Z(X) \setminus Z[Q]$ is such that, for example, $Z' \supset Z_1 \setminus Z_2$ i.e. $Z' \cup Z \supset Z_1 \in Z[P_{\alpha_1} + \dots + P_{\alpha_{n-1}}]$ being $P_{\alpha_1} + \dots + P_{\alpha_{n-1}}$ a prime z -ideal, then $Z \in Z[P_{\alpha_1} + \dots + P_{\alpha_{n-1}}]$ against the hypothesis. Analogously if $Z' \cup Z \supset Z_2 \in Z[P_{\alpha_n}]$ and $Z' \notin Z[Q]$ then $Z \in Z[P_{\alpha_n}]$. If now $n = 1$ i.e. if $Z \in Z[P_\alpha]$, $P_\alpha \subset Q$ being Q not minimal there exists $P_\beta \neq P_\alpha$, $P_\beta \subset Q$. Let $Z_0 = Z \cap Z' \cap Z''$ with $Z' \in Z[P_\alpha] \setminus Z[P_\beta]$ and $Z'' \in Z[P_\beta] \setminus Z[P_\alpha]$: then $Z_0 \in Z[P_\alpha + P_\beta] \setminus (Z[P_\alpha] \cup Z[P_\beta])$, and we can apply to Z_0 the previous argument.

(b) \Rightarrow (a).

Put $Z \in Z[Q]$ and let $Z_1, Z_2 \in Z[Q]$ satisfy the property (b) for Z . Let $I = Z^{\leftarrow}[(Z_1)]$ (where (Z_1) is the z -filter generated by Z_1) and $S = \{h \cdot k : \text{with } Z(h) = Z_0 \text{ and } k \notin Q\} \cup (C(X) \setminus Q)$ with $Z_0 = Z_1 \cap Z_2$. S is closed under multiplication and disjoint from I ; in fact $I \subset Q$ and if $h \cdot k \in I$ with $Z(h) = Z_0$ and $k \notin Q$, we have $Z(h \cdot k) \supset Z_1$, $Z(h) \cup Z(k) \supset Z_1$, $Z(k) \supset Z_1 \setminus Z_0$ and for (b) we have $Z(k) \in Z[Q]$. Then there is an ideal Q_1 containing I , disjoint from S and maximal with respect to this property: namely $Z[Q_1] \ni Z_1$, $Z[Q_1] \not\ni Z_0$ and $Q_1 \in Q$ (such an ideal is prime and a z -ideal because

$Z^* [Z[Q_1]]$ has the same property). Doing the same for Z_2 , we obtain a prime z -ideal Q_2 such that $Z[Q_2] \ni Z_2, Z[Q_2] \not\ni Z_0$ and $Q_2 \subset Q$. Then $Z_0 = Z_1 \cap Z_2 \in Z[Q_1 + Q_2] = (Z[Q_1], Z[Q_2])$ so that Q is ramified.

(b) \Rightarrow (d).

Let $A = X \setminus Z$ with $Z \in Z[Q]$; then there exist Z_1, Z_2 satisfying the property (b). Put $B = X \setminus Z_0$ with $Z_0 = Z_1 \cap Z_2$. Being $Z_0 \subset Z$, it is $B \supset A$; now $Z_1 \setminus Z_0$ and $Z_2 \setminus Z_0$ are disjoint zero sets of B , but their closure in $B \cup \{Q\}$ contains the point Q .

(d) \Rightarrow (c).

Let $Z \in Z[Q]$. Put $A = X \setminus Z$ and take $B = X \setminus Z_0$ with $Z_0 \in Z[Q]$ satisfying (d). Then there is a bounded function on B , not extensible to $B \cup \{Q\}$, hence as x approximates Q , f has two limit points, i.e. there are two disjoint zero-sets Z'_1, Z'_2 of B , containing Q in their closure. With $Z_1 = Z'_1 \cup Z_0$ and $Z_2 = Z'_2 \cup Z_0$ we have (c).

(a) \Rightarrow (e).

It is sufficient to see that every $f \in P_\alpha \subset Q$, with P_α a minimal prime ideal and $f > 0$, is a sum of functions that change their sign on every zero-set of O_Q . If $f \notin O_Q$, there is $P_\beta \subset Q$ such that $f \notin P_\beta$; hence if $g \notin P_\alpha, g > 0$ is such that $fg = 0$ then $g \in P_\beta \subset Q$ and $f = (2f - g) - (f - g)$; and if $h \in O_Q$ it is $Z(f) \supset Z(h)$ and $Z(g) \supset Z(h)$. If $f \in O_Q$ let $f' \in P_\beta \setminus P_\alpha$ with $P_\alpha, P_\beta \subset Q$; put $g' \notin P_\alpha, f'g' = 0$; then $g' \in P_\beta$ and $f = (f + f' - g') - (f' - g')$, and, as f vanishes on every zero set of O_Q , we have (f).

(e) \Rightarrow (a).

Let $f \in Q$ be a function that changes its sign on every zero set of O_Q . Put $f = f^+ - f^-$ (where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$), it is $f^+ f^- = 0$ hence if $P_\alpha \subset Q$ we have, for example, $f^+ \in P$ but $f^+ \notin O_Q$ and then there is a $P_\beta \subset Q$ such that $f^+ \notin P_\beta$; hence $f^- \in P_\beta$ and $f \in P_\alpha + P_\beta \subset Q$. (q.e.d.).

In the special case where $Q = M^p$, (c) and (f) take the form :

(c') for each $Z \in Z[M^p]$ there exist $Z_1, Z_2 \in Z[M^p]$ such that $Z_1 \cap Z_2 \subset Z$ and $p \in cl_{\beta X}(Z_1 \setminus Z_2) \cap cl_{\beta X}(Z_2 \setminus Z_1)$

(e') M^p is generated by the functions of M^p that change their sign on every zero-set of O^p .

One may wonder whether in this theorem, if $Z \notin Z[O_Q]$, $Z_1 \cap Z_2 = Z$ can be assumed; this is true only if $Z \notin \bigcup_{P_\alpha \subset Q} Z[P_\alpha]$,

otherwise it may be not true. In fact let $\sigma \notin N$, put $X = N \cup \{\sigma\}$ with every point of N isolated, and let the neighborhoods of σ be the $\{\sigma\} \cup H_1 \cup H_2$, with $H_1 \in \mathcal{U}_1$ and $H_2 \in \mathcal{U}_2$ two distinct free ultrafilters on N . For σ there are three prime z -ideals:

$$M_\sigma = Z^\leftarrow(\{\sigma\}), \quad P_1 = Z^\leftarrow(\{\sigma\} \cup H_1 : H_1 \in \mathcal{U}_1),$$

$$P_2 = Z^\leftarrow(\{\sigma\} \cup H_2 : H_2 \in \mathcal{U}_2).$$

It is $P_1 + P_2 = M_\sigma$. P_1, P_2 are minimal and M_σ is ramified. If $f \in M_\sigma$ with $\text{coz } f \in \mathcal{U}_1 \setminus \mathcal{U}_2$ (or $\text{coz } f \in \mathcal{U}_2 \setminus \mathcal{U}_1$) then $\text{coz } f$ is C^* -embedded in the whole X .

We note that in this space every finitely generated ideal is generated by two functions.

2. PROPOSITION. (De Marco). *Let X satisfy the first axiom of countability. Then for every non-isolated point $p \in X$, there are non-maximal prime ideals P_1 and P_2 such that $P_1 + P_2 = M_p$.*

PROOF. Let $(x_n)_{n \in N}$ be a sequence of distinct points, converging to p in X ($x_n \neq p$). Select on $D = \{x_n : n \in N\}$ two distinct free ultrafilters $\mathcal{U}_1, \mathcal{U}_2$. For $i = 1, 2$, put:

$P_i = \{f \in C(X) : Z(f) \supset A \text{ for an } A \in \mathcal{U}_i\}$; P_i is a prime z -ideal of $C(X)$. In fact P_i is clearly a z -ideal and if $Z(fg) \supset A$ and $A \in \mathcal{U}_i$ we have $(Z(f) \cap D) \cup (Z(g) \cap D) \supset A$: this implies that either $Z(f) \cap D$ or $Z(g) \cap D$ belongs to \mathcal{U}_i ; hence, for example, $Z(f) \supset Z(f) \cap D \in \mathcal{U}_i$, then $f \in P_i$. Hence P_i is prime. For the continuity of the f , $f(p) = 0$ for every $f \in P_i$, hence P_i ($i = 1, 2$) is contained in M_p .

Suppose now that D is chosen as follows. Let $g \in C(X)$, $g \geq 0$ be such that $Z(g) = \{p\}$ (this is possible because in a $T_{3\frac{1}{2}}$ space every compact G_δ -set is a zero-set).

Let $V_1 \supset V_2 \supset \dots$ be a base of neighborhoods of p and (a_n) a real sequence constructed inductively in the following way:

$$0 < a_1, a_1 \in g[V_1]; \quad 0 < a_2 < a_1, a_2 \in g[V_2] \text{ etc.}$$

For every n , let $x_n \in V_n$ such that $g(x_n) = a_n$. Let D be the set of the x_n . Clearly $(x_n)_{n \in N}$ is formed of distinct points and converges to p . Take now $A_1 \in \mathcal{U}_1, A_2 \in \mathcal{U}_2$ such that $A_1 \cap A_2 = \emptyset$ and let $B_1 = g[A_1], B_2 = g[A_2]$. Finally let $\varphi_i \in C(R)$ ($i = 1, 2$) such that $Z(\varphi_i) = B_i \cup \{0\}$ and $\varphi_1 + \varphi_2 = 1$. Put $u_i = \varphi_i g$. Then $u_i \in P_i$ because $Z(u_i) \supset A_i$; and we have that $u_1 + u_2 = g((u_1 + u_2)(x)) = (\varphi_1 + \varphi_2)g(x) = g(x)$.

3. - *Theorem on metric spaces.*

LEMMA. Let X be a perfectly normal space, Q a prime z -ideal of $C(X)$. If $Z[Q]$ contains no nowhere dense set, then Q is minimal.

PROOF. Take $f \in Q$ and let $g \in C(X)$ be such that $Z(g) = cl_X(X \setminus Z(f))$; $g \notin Q$ because $\text{int}(Z(g) \cap Z(f)) = \emptyset$. Being $fg = 0$, f belongs to every prime ideal contained in Q . Hence Q is minimal.

THEOREM. Every metric space is totally ramified.

PROOF. Let Q be a non-minimal prime z -ideal of $C(X)$. For the lemma $Z[Q]$ contains a nowhere dense zero-set Z . For a lemma given by Hausdorff [W 4.39] there exists a discrete set $D \subset X \setminus Z$ with $cl_X D = D \cup Z$.

We want to find two disjoint subsets D_1 and D_2 of D such that $Z = cl D_1 \cap cl D_2$. Putting $Z_1 = Z \cup D_1$, $Z_2 = Z \cup D_2$ for (b) of theorem at n. 1 we have that Q is ramified.

Put $Y = D \cup Z$. For every $x \in Y$, define $\eta(x) = \min_{\varepsilon > \delta(x)} w(B(x, \varepsilon))$

where $\delta(x)$ is the distance of x from Z ; $B(x, \varepsilon)$ is the open ball in Y of center x and radius ε ; w is the weight.

It is $\aleph_0 \leq \eta(x) \leq |D|$. For each $H \subset Y$ and each infinite cardinal $\alpha \leq |D|$, put $H_\alpha = \{x \in H : \eta(x) = \alpha\}$. We prove that: $Z_\alpha \setminus cl \bigcup_{\beta < \alpha} Z_\beta \subseteq cl D_\alpha \setminus cl(D \setminus D_\alpha)$.

In fact if $z \in Z$ there exists $\varepsilon > 0$ such that $w(B(z, \varepsilon)) = \alpha$, if $d \in D$ with $d \in B(z, \varepsilon/2)$ it is $\varepsilon/2 > \delta(d)$, $B(d, \varepsilon/2) \subseteq B(z, \varepsilon)$ and hence $\eta(d) \leq \alpha$, i.e. $z \notin cl \bigcup_{\beta > \alpha} D_\beta$; now if $z \in cl \bigcup_{\beta < \alpha} D_\beta$ for each $\varepsilon' > 0$ there exists $d \in D$ with $\eta(d) < \alpha$ and $d \in B(z, \varepsilon'/2)$, hence there exists $\varepsilon'' \leq \varepsilon'/2$ such that $\varepsilon'' > \delta(d)$; $w(B(d, \varepsilon'')) = \eta(d) < \alpha$; then there is $z' \in Z$ with $z' \in B(d, \varepsilon'')$; we have $z' \in \bigcup_{\beta \leq \eta(d)} z_\beta$ and $z' \in B(z, \varepsilon')$; hence $z \in cl \bigcup_{\beta < \alpha} Z$.

Now, for a generalization of an exercise of [E 4c], there is a partition of Y_α with $Y_\alpha^i (i \in I_\alpha)$ clopen sets of Y_α that have a dense subset of cardinality not bigger than α .

Let us prove that for each $\alpha \leq |D|$, there exist two sets $D_\alpha^1, D_\alpha^2 \subset D_\alpha$ such that $D_\alpha^1 \cap D_\alpha^2 = \emptyset$ and $Z_\alpha \subseteq (cl \bigcup_{\beta < \alpha} Z_\beta) \cup cl D_\alpha^i$ for $i = 1, 2$; in fact $Z_\alpha = \bigcup_i (Y_\alpha^i \cap Z_\alpha)$; $Y_\alpha^i \cap Z_\alpha$ contains a dense

subset of cardinality α^i with $\alpha^i \leq \alpha$. Consider α^i as an ordinal (i.e. the minimum ordinal of cardinality α^i). We can write: $Y_\alpha^i \cap Z_\alpha = cl_{Z_\alpha} \{y_1^i, y_2^i, \dots, y_\nu^i, \dots\}_{\nu < \alpha^i \leq \alpha}$.

Now for each $y_\nu^i \notin cl \bigcup_{\beta < \alpha} Z_\beta$ take a sequence of distinct points, $(d_\nu^{in})_n \rightarrow y_\nu^i$ with $d_\nu^{in} \in D_\alpha \cap Y_\alpha^i$ such that $d_\nu^{in} \notin \{d_{\nu'}^{im} : \nu' < \nu, m \in N\}$. This is possible because y_ν^i has a neighborhood disjoint from $D \setminus D_\alpha$, and besides, if $\alpha > \aleph_0$, every neighborhood of it has weight $> \alpha$ and $card \{d_{\nu'}^{im} : \nu' < \nu, m \in N\} < \alpha$; if $\alpha = \aleph_0$ then there exists a neighborhood of y_ν^i disjoint from $\{d_{\nu'}^{im} : \nu' < \nu, m \in N\}$, as ν is in this case finite.

Put then $D_\alpha^1 = \{d_\nu^{i(2n)} : n \in N, \nu < \alpha^i, i \in I_\alpha\}$ and

$$D_\alpha^2 = \{d_\nu^{i(2n+1)} : n \in N, \nu < \alpha^i, i \in I_\alpha\}.$$

If now $D_1 = \bigcup_{\alpha \leq |D|} D_\alpha^1, D_2 = \bigcup_{\alpha \leq |D|} D_\alpha^2$, we have $D_1 \cap D_2 = \emptyset$ and $Z = cl D_1 \cap cl D_2$.

4. - *Some problems.*

PROBLEM 1. Is there a relation for a space X between being totally ramified and having particular ramified subsets ?

PROBLEM 2. Is it equivalent to ask that every cozero set of X be ramified and that X be totally ramified ?

We have only a partial answer for these problems, namely.

PROPOSITION: *Let X be ramified. Every cozero set of X is ramified iff: (i) for each prime z -ideal Q , between Q and $R(Q)$ there are no (prime) z -ideal; (ii) if Q has an immediate successor in the z -ideal then it is ramified ($Q = R(Q)$).*

PROOF. As a zero-set we consider for (i) a set A belonging to $Z[Q]$ but not to the z -filter of a prime between Q and $R(Q)$; for (ii), a set A belonging to the z -filter of the successor of Q but not to $Z[Q]$: if ι is the immersion of $X \setminus A$ in X we consider the lattice-isomorphism $\iota^\#$ of [GJ 4, 12].

REFERENCES

- [DO] G. DE MARCO A. ORSATTI, *Commutative rings in which every prime ideal is contained in a unique maximal ideal*, Proc. Amer. Math. Soc. - Vol. 30 Nov. 1971, 459-466.
- [E] R. ENGELKING, *Outline of General Topology*, New York, John Wiley and Sons, Inc. 1968.
- [GJ] L. GILLMAN and M. JERISON, *Rings of continuous functions*, Princeton, Van Nostrand, 1960.
- [W] RUSSEL C. WALKER, *The Stone-Čech Compactification*, Springer-Verlag 1974.

Manoscritto pervenuto in redazione l'11 ottobre 1977.