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A density theorem about some system.

GIULIANO BRATTI (*).

Introduction.

Let A be an open subset of R^n ; suppose $P = P(x, D)$, $Q = Q(x, D)$ linear partial differential operators with $C^\infty(A)$ coefficients.

DEFINITION 1). We say that the system

$$(+)\quad \{ Pu = f, \quad Qu = 0 \}, \quad f \in C^\infty(A) \quad \sim$$

is $C^\infty(A)$ -locally solvable in A if for every $p \in A$ there is a neighborhood, V_p , of p and a function $u_p \in C^\infty(V_p)$ such that the (+) is satisfied in V_p .

DEFINITION 2). If B is an open subset of A , we say that the above system (+) is $C^\infty(B)$ -globally solvable if for every $f \in C^\infty(A)$ for which (+) is locally solvable in A , there is a function $u \in C^\infty(B)$ such that (+) is satisfied in B .

In (2) there is the following conjecture :

let $(B_n)_{n \in N}$ be a sequence of open subsets of A such that : $B_n \subset B_{n+1} \subset \sqcup B_n = B$ and the (+) is $C^\infty(B_n)$ -globally solvable for every $n \in N$. Then (+) is $C^\infty(B)$ -globally solvable.

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It is already known, (4), *the conjecture is false* in the case in which P and Q have constant coefficients and Q is semi-elliptic, but the conjecture is still open when Q is an elliptic operator.

It seems to the A. that to solve the above conjecture it is important to have some example of system like (+) without $C^\infty(A)$ -globally solutions for $f \in C^\infty(A)$ for which (+) is $C^\infty(A)$ -locally solvable.

First of all, by a *Lojasiewicz-Malgrange's theorem*, see (1), it is easy to show that: if P and Q are prime between them, the subspace of $C^\infty(A)$ of the functions for which the system (+) is $C^\infty(A)$ -locally solvable is $\ker Q|_A = \{f \in C^\infty(A) : Qf = 0\}$.

The object of this paper is that to characterize the open subset A of R^n for which there are systems like (+), with Q elliptic, such that:

$$P(\ker Q|_A) \text{ is not } C^\infty(A)\text{-dense in } \ker Q|_A$$

1) Let A be an open subset of R^n and let $b(A)$ be its boundary.

Let G be the subset of $b(A)$ so defined:

$G = \{p \in b(A) : \text{the connexe component, } Z_p, \text{ of } R^n - A \text{ with } p \in Z_p \text{ is compact}\}$

$P = P(D)$ and $Q = Q(D)$ are linear partial differential operators, with constant coefficient; Q will always be elliptic.

LEMMA a). If we put: $Z_A = \sqcup_{p \in G} Z_p$ and $L = A \sqcup Z_A$, we have: L is an open set. Proof. It is sufficient to see that every compact component, Z , of $R^n - A$, is such that: $Z \cap b(A) \neq \emptyset$. Then the proof. of the Lemma a) is in (5), pag. 235.

LEMMA b). Let n be a distribution with compact support: $n \in E'(\mathbf{R}^n)$. If $m = Q(D)n$ has its support in A , then $n \in E'(L)$.

Proof. If $p \notin A$ and Z_p , the connexe component of $R^n - A$ with $p \in Z_p$, is not bounded then there exists a neighbourhood of Z_p in which n is an analytic function. Because n has compact support, in such neighbourhood n must be zero. This shows that: if $p \in \text{supp}(n)$ and $p \notin A$, then $p \in Z_A$.

THEOREM. If $n \in E'(\mathbf{R}^n)$ and is orthogonal to all exponential solutions of the equation $Pu = 0$, then there exists $m \in E'(\mathbf{R}^n)$ such that: $n = P(-D)m$.

Proof. See Lemmas 3.4.1 and 3.4.2. of (3) pagg. 77/78.

LEMMA c). Let $g \in C_c^\infty(L)$ be a function such that: $P(-D)g \in C_c^\infty(A)$. If $P(-D)g$, with P hypoelliptic, is orthogonal to $\ker P_{|A}$, then: if $p \in \text{supp}(g) \cap Z_A$, $g(p) = 0$.

Proof. If δ_p is the Dirac distribution at the point p , the distribution $E_p * \delta_p$ is in $\ker P_{|A}$ if E_P is a fundamental solution of P : $PE_P = \delta$. Then: $\langle (E_P * \delta_p)_{|A}, P(-D)g \rangle = \langle \delta_p, g \rangle = g(p) = 0$.

DEFINITION 3). We say that a compact subset K of L disjoins Z_A if there exists a partition of G , $G = G_1 + G_2$, $G_1 \neq \emptyset$, and an open subset B of L such that $\sqcup_{p \in G_1} Z_p \subset K \subset B$ and $B \cap (\sqcup_{p \in G_2} Z_p) = \emptyset$.

DEFINITION 4). We say that an open subset A of R^n has the b -propriety if (or $Z_A = \emptyset$ or) there is no compact K of L which disjoins Z_A .

THEOREM. The following two propositions, p_1 and p_2 , are equivalent:

p_1) A is an open subset of R^n which has the b -propriety;

p_2) for every couple, (P, Q) , of partial differential operators with constant coefficients, prime between them, with Q elliptic, we have:

$$P(\ker Q_{|A}) \text{ is } C^\infty(A)\text{-dense in } \ker Q_{|A}.$$

Proof.

FROM p_1 to p_2). Suppose there exists P prime with Q such that $P(\ker Q_{|A})$ is not $C^\infty(A)$ -dense in $\ker Q_{|A}$; we will show that absurd.

From the Hahn-Banach theorem, we have: there exists $m \in E'(A)$ such that m is not orthogonal to $\ker Q_{|A}$ but m is orthogonal to $P(\ker Q_{|A})$.

By the precedent theorem, there exists, then, a distribution $n \in E'(R^n)$ such that: $P(-D)m = Q(-D)n$. Because P and Q are prime between them, there exists $n_0 \in E'(R^n)$ with: $m = Q(-D)n_0$, and, from lemma b), $n_0 \in E'(L)$.

Let K be the support of n_0 ; we will show that K disjoins Z_A , so we will have the absurd.

In fact: it can't be: $K \cap Z_A = \emptyset$, because, otherwise, $n_0 \in E'(A)$ and so m would be orthogonal to $\ker Q_{|A}$.

Let G_1 be the subset of G , $G_1 \neq \emptyset$ with: if $p \in G_1$, $Z_p \cap K \neq \emptyset$, (so that $Z_p \subset K$); we will show that there exists an open subset B of L with: $K \subset B$ and $B \cap (\sqcup_{p \in G - G_1} Z_p) = \emptyset$.

Of course, this is the case if $G - G_1 = \emptyset$. Otherwise, let $(B_n)_{n \in N}$ a sequence of open subsets of L such that $B_n \supset B_{n+1}$ and $\bigcap_n B_n = K$.

Suppose that $x_n \in B_n \cap \bigcup_{p \in G - G_1} Z_p$ for every $n \in N$; we can suppose, directly, $\lim_n x_n = x_0$, with, of course, x_0 in K .

It is impossible that infinite terms of the sequence (x_n) are in the same component Z_q , $q \in G - G_1$; in fact if it is so, we have $x_0 \in Z_q \cap K$; absurd.

It is easy to see that $x_0 \in b(A)$, because every segment (x_n, x_{n+1}) has a point of A ; it comes out that n_0 must be an analytic function in a neighbourhood V of x_0 . In such V there is a point $x_n \in Z_{q_n}$ with $q_n \in G - G_1$. Because $Z_{q_n} \cap K = \emptyset$, in a neighbourhood of x_n , n_0 is zero; so we can suppose n_0 equal to zero in all V . Absurd, because x_0 belongs to $\text{supp}(n_0)$.

FROM p_2 to p_1). If K is a compact subset of L and K disjoint Z_A , let g be a function in $C_c^\infty(B)$, with $g = 1$ on B' with: $K \subset B' \subset \bar{B}' \subset B$. If $h = Q(-D)g$, $h \in C_c^\infty(A)$ if $Q(0) = 0$; for the lemma c) above, h can't be orthogonal to $\ker Q|_A$.

But: if $P = P(D)$ is an operator prime with Q and $P(0) = 0$, h is orthogonal to $P(\ker Q|_A)$ because $P(-D)h = Q(-D)P(-D)g$ and $P(-D)g \in C_c^\infty(A)$.

This completes the proof.

The above theorem permits the construction of system like (+) without $C^\infty(A)$ -global solution. So, for the system.

$$(^o) \{ D_x u = f, \quad D_x u + i D_y u = 0 \}$$

in the set $A \subset \mathbb{R}^2$ so defined: $|x| < 1$, $|y| < 1$, $x^2 + y^2 \neq 0$, for the reason that $Z_A = (0, 0)$, there is a function, $f_0 \in \ker (D_x + i D_y)|_A$ for which there is no global solution in A ; on the other hand, by the Lojasiewicz-Malgrange theorem, (*), it is easy to show that there is a sequence, $(B_n)_{n \in N}$, of subset of A , such that:

(*) The theorem is the following: if $A(D)$ is the differential matrix $A(D) = \|a_{i,j}(D)\|$, $1 \leq i \leq p$, $1 \leq j \leq q$, $u \in E^q(A)$, $f \in E^p(A)$, respectively p and q times product of $E(A)$, the space of indefinitely differentiable functions over A , the system $A(D)u = f$ has a solution if and only if: for every $v = (v_1, \dots, v_p)$, v_i polynomial, for which $v(x)A(x) = 0$, we have $v(D)f = 0$, if A is convex.

a) $B_n \subset B_{n+1} \subset \sqcup_n B_n = A$; b) for every $n \in N$ there is an open subset $B'_n \subset A$ such that: $B_n \subset B'_n$ and the system $(\circ)$ is $C_c^\infty(B'_n)$ — globally solvable.

Of course, *this example is very near to show the De Giorgi's conjecture is false also in the case: Q is elliptic.*

2) I like to end this paper giving *an abstract condition* to have $P(\ker Q_{/A}) = \ker Q_{/A}$.

We put, over $C^\infty(A)$, the following T_P -topology:

V is a neighbourhood of zero in the T_P -topology if: $V \supset W + \ker P_{/A}$, for some W neighbourhood of zero in the usual topology of $C^\infty(A)$.

So we have: if A has the b -propriety, P and Q are linear partial differential operators, prime between them, and Q is elliptic,

THEOREM. *The following two proposition, $q_1)$ and $q_2)$, are equivalent:*

$q_1)$ $P(\ker Q_{/A}) = \ker Q_{/A}$;

$q_2)$ $\ker (Q \circ P)_{/A} = \ker Q_{/A} + \ker P_{/A}$; $\ker (Q \circ P)_{/A}$ is a complete subspace of $C^\infty(A)$ with the T_P -topology and $P: \ker (Q \circ P)_{/A} \rightarrow P(\ker (Q \circ P)_{/A})$ is an open mapping.

Proof.

$q_1) \Rightarrow q_2)$. The first part of $q_2)$ is simple. For the second part, we have: $\ker (Q \circ P)_{/A}$ is a closed subspace of $C^\infty(A)$ with the T_P -topology, so:

$(\ker Q_{/A})^\wedge \subset \ker (Q \circ P)_{/A}$. On the other hand, $\ker Q_{/A} + \ker P_{/A} \subset (\ker Q_{/A})^\wedge$. Because $P: \ker Q_{/A} \rightarrow \ker Q_{/A}$ is an open mapping, (it is a surjective map between Frechet spaces), we have:

if W is an usual neighbourhood of zero in $C^\infty(A)$, $P(W \wedge \ker Q_{/A})$ is open in $\ker Q_{/A}$, so: $P(W + \ker P_{/A}) \wedge \ker (Q \circ P)_{/A} \supset P(W \wedge \ker Q_{/A})$.

$q_2) \Rightarrow q_1)$. It is sufficient to see that in the diagram

$$\begin{array}{ccc} \ker Q \circ P_{/A} & \xrightarrow{P} & P(\ker Q \circ P_{/A}) \\ \downarrow p & \nearrow \tilde{P} & \\ \ker Q \circ P_{/A} & & \\ \hline \ker P_{/A} & & \end{array}$$

the quotient is a Frechet space, so $P(\ker (Q \circ P)_{/A})$ is a Frchet space.

But the last one is also a dense subspace of $\ker Q_{/A}$; so : $P(\ker Q_{/A}) = \ker Q_{/A}$.

Remark 1) It is very easy to see that : if A is $P(-D)$ - convex the topological part of q_2) it is always true. It comes out :

If A is $P(-D)$ - convex, (and it has the b -propriety, which is not a consequence if P is elliptic !), *the necessary and sufficient condition to have :*

$$P(\ker Q_{/A}) = \ker Q_{/A}$$

is : $\ker (Q \circ P)_{/A} = \ker P_{/A} + \ker Q_{/A}$.

Remark 2) The $P(-D)$ - convexity of A , is not, of course, a necessary condition to have the above result, as we can see by the system (°) in A like that, without the points : $x = o$, $o \leq y$.

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