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On the Compactness of Minimal Spectrum.

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0. Introduction.

Let $A$ be a commutative ring with 1; denote by $\text{Spec}(A)$ the set of all prime ideals of $A$ equipped with the hull-kernel topology, by $\text{Min}(A)$ the subspace consisting of minimal prime ideals. Henriksen and Jerison [HJ] found some sufficient conditions for the compactness of $\text{Min}(A)$; subsequently, Quentel [Q] discovered an equivalent condition. Here we give another characterization of the compactness of $\text{Min}(A)$, which seems to give more light to the topological situation; this characterization, among other things, allows us to show that the class of (weakly) Baer rings coincides with the class of rings such that: 1) their minimal spectrum is compact; and 2) every prime ideal contains a unique minimal prime ideal.

We shall always deal with rings without non-zero nilpotents; but of course all purely topological results are independent of this hypothesis.

1. All rings are commutative and with 1. $\text{Spec}(A)$ denotes the set of prime ideals of $A$, equipped with the Zariski topology; i.e. $\text{Spec}(A)$ has as a base of open sets the sets $D(a) = \text{Spec}(A) - V(a) = \{P \in \text{Spec}(A): a \notin P\}$. Thus, the subspace $\text{Min}(A)$ of minimal prime ideals has $\{D^0(a) = D(a) \cap \text{Min}(A): a \in A\}$ as a base of open sets.

For the sake of simplicity, we assume that $A$ is semiprime (that is, $A$ has no non-zero nilpotents); however, it will be clear that all results

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obtained here hold in the general case, with some obvious modification (e.g., the nilradical of $A$ in place of the zero ideal).

The sets $D^0(a)$ are clopen in $\text{Min}(A)$; for, denoting by $\text{Ann}(a)$ the annihilator of $a$, we have $D^0(a) = \text{Min}(A) - V^0(a) = V^0(\text{Ann}(a))$ (if $I$ is an ideal of $A$, $V(I)$ is its hull in $\text{Spec}(A)$, and $V^0(I) = V(I) \cap \text{Min}(A)$), (see [HJ]). Thus $\text{Min}(A)$ is a space with a clopen basis, and, being $T^0$, it is also a Hausdorff space.

**Lemma.** Let $A$ be a semiprime ring, $P$ a prime ideal of $A$. The following are equivalent:

i) $P$ is a minimal prime.

ii) For every $a \in P$, $\text{Ann}(a) \not\subseteq P$.

iii) For every finitely generated ideal $I$ contained in $P$, $\text{Ann}(I) \not\subseteq P$.

Thus, if $A$ is semiprime and $I$ is finitely generated, $\text{Ann}(I) = 0$ iff $V^0(I) = \emptyset$.

**Proof.** The equivalence of i) and ii) is proved in [HJ,1.1]. iii) implies ii): trivial, ii) implies iii): let $a_1, \ldots, a_n \in P$ generate $I$; for each $i = 1, \ldots, n$ choose $b_i \in \text{Ann}(a_i) - P$; then $b = b_1 \ldots b_n \in \text{Ann}(I) - P$.

Plainly, iii) shows that no minimal prime ideal can contain a finitely generated ideal whose annihilator is zero; conversely, if $I$ is finitely generated and $\text{Ann}(I)$ contains a non-zero element $b$, then, since $A$ is semiprime, there exist some minimal prime ideal which does not contain $b$; every such prime necessarily belongs to $V^0(I)$.

**Theorem.** Let $A$ be a semiprime ring. The following are equivalent:

1) The family of sets $\{V^0(a): a \in A\}$ is a subbase for the topology of $\text{Min}(A)$.

2) $\text{Min}(A)$ is a compact space.

3) For every element $a \in A$, there exists a finite number of elements $a_1, \ldots, a_n \in A$ such that $aa_i = 0$ for each $i = 1, \ldots, n$ and $\text{Ann}(a_1, \ldots, a_n, a) = 0$.

**Proof.** 1) implies 2). By Alexander's subbase theorem it is enough to show that, if $B$ is a subset of $A$ such that $\bigcap_{a \in B} D^0(a) = \emptyset$, then there exists a finite number of elements in $B$, say $a_1, \ldots, a_n$ such that $\bigcap_{i=1}^n D^0(a_i) = \emptyset$. Let us observe that $\bigcap_{a \in B} D^0(a)$ coincides with the
set of minimal prime ideals disjoint from $B$. If $S$ is the multiplicative set generated by $B$, a prime ideal doesn’t meet $S$ if and only if it doesn’t meet $B$. Now, zero belongs to $S$ for, otherwise, there would exist a prime ideal, and then a minimal prime one, disjoint from $B$. But if zero belongs to $S$, there exist $a_1, \ldots, a_n \in B$ such that their product is zero, and so $\bigcap_{i=1}^{n} D^0(a_i) = D^0(a_1 \ldots a_n) = D^0(0) = \emptyset$.

2) implies 3). If $\text{Min}(A)$ is a compact space, then $V^0(a)$ is an open compact set, and therefore it is a finite union of basic open sets, that is $V^0(a) = D^0(a_1) \cup \ldots \cup D^0(a_n)$. Since $D^0(a_i)$ is contained in $V^0(a)$ for each $i = 1, \ldots, n$, every minimal prime ideal contains $aa_i$, and so $aa_i = 0$ for each $i$. Moreover, the above relation implies that $V^0(a_1, \ldots, a_n, a) = \emptyset$; by the Lemma, $\text{Ann}(a_1, \ldots, a_n, a) = 0$.

3) implies 1). Choose a basic open set $D^0(a)$. Let $a_1, \ldots, a_n$ be the elements given by 3). By the Lemma, the ideal $I = (a_1, \ldots, a_n, a)$ is contained in no minimal prime; this implies that $D^0(a) \supseteq V^0(a_1) \cap \ldots \cap V^0(a_n)$; but since $aa_i = 0$ for every $i = 1, \ldots, n$, equality actually holds.

Remark 1. Condition 3) is due to Quentel [Q, Proposition 4].

Remark 2. Condition 1) allows us to state Theorem 3.4 of [H.J] in the following way:

a) The following conditions on a ring $A$ without non zero nilpotents are equivalent:

a) $\text{Min}(A)$ is compact and, for every $x, y \in A$, there exists $z \in A$ such that $\text{Ann}(x) \cap \text{Ann}(y) = \text{Ann}(z)$.

b) The family of sets $\{V^0(x) : x \in A\}$ is a base for the open sets of $\text{Min}(A)$.

c) For each $x \in A$ there exists $x' \in A$ such that $\text{Ann}(\text{Ann}(x')) = \text{Ann}(x)$. Thus, the assumption of compactness of $\text{Min}(A)$ in condition b) is redundant.

Notice also that condition 3) of the Theorem may be written as follows:

3 bis) For each $a \in A$ there exist $a_1, \ldots, a_n$ such that

$$\text{Ann}(a) = \text{Ann}(\text{Ann}(a_1, \ldots, a_n)),$$

which thus appears as a weakening of condition c) in the above theorem.
2. In paper [K], Kist proves the equivalence of the following conditions:

\textit{a}) There exists a continuous function of Spec(\(A\)) onto Min(\(A\)) which is the identity on Min(\(A\)).

\textit{b}) \(A\) is a Baer ring, that is the annihilator ideal of each element in \(A\) is generated by an idempotent.

This implies that, in a Baer ring, every prime ideal contains a unique minimal prime ideal and that Min(\(A\)) is compact. We shall prove that these two last conditions characterize the Baer rings. First, we need two Lemmas:

\textbf{Lemma \(\alpha\).} Let \(P\) be a prime ideal of \(A\) and let \(O_P\) be the intersection of the prime ideals contained in \(P\). Then \(O_P\) coincides with the ideal of the elements of \(A\) whose annihilator is not contained in \(P\).

(For a proof, one may look at [DMO, p. 460]).

\textbf{Lemma \(\beta\).} Let \(A\) be a semiprime ring. The following are equivalent:

\textit{i}) Every prime ideal contains a unique minimal prime ideal.

\textit{ii}) If \(a, b\) are elements of \(A\) such that \(ab = 0\), then \(\text{Ann}(a) + \text{Ann}(b) = A\).

\textit{iii}) For every \(a, b \in A\), \(\text{Ann}(a) + \text{Ann}(b) = \text{Ann}(ab)\).

\textbf{Proof.} \(i\) implies \(ii\). If \(i\) holds, then for every maximal ideal \(M\), \(O_M\) is the unique minimal prime contained in \(M\). If \(\text{Ann}(a) + \text{Ann}(b)\) is contained in \(M\), then, since \(O_M\) is prime, either \(a\) or \(b\) belong to \(O_M\): this is absurd for the characterization of \(O_M\) given in Lemma \(\alpha\).

\(ii\) implies \(iii\). Of course \(\text{Ann}(a) + \text{Ann}(b)\) is contained in \(\text{Ann}(ab)\). If \(x\) belongs to \(\text{Ann}(ab)\), then \((xa)b = x(ab) = 0\) and so there exist \(y \in \text{Ann}(xa)\) and \(z \in \text{Ann}(b)\) such that \(1 = y + z\), hence \(x = xy + xz\), with \(xy \in \text{Ann}(a)\) and \(xz \in \text{Ann}(b)\).

\(iii\) implies \(i\). If \(P\) is a prime ideal, let us see that \(O_P\) is prime, too. In fact, if \(ab\) belongs to \(O_P\), there exists an element \(x \in \text{Ann}(ab)\) that doesn't belong to \(P\). According to \(iii\), \(x = y + z\), with \(y \in \text{Ann}(a)\) and \(z \in \text{Ann}(b)\); hence either \(y \notin P\), or \(z \notin P\); by Lemma \(\alpha\), this is equivalent to \(a \in O_P\) or \(b \in O_P\).

Now we can state the following theorem.
Theorem. Let $A$ be a semiprime ring. The following are equivalent:

1) $A$ is a Baer ring.

2) Every prime ideal contains a unique minimal prime ideal and $\text{Min}(A)$ is a compact space.

3) $\text{Min}(A)$ is a retract of $\text{Spec}(A)$, that is there exists a continuous function $\varphi$ of $\text{Spec}(A)$ onto $\text{Min}(A)$ which is the identity on $\text{Min}(A)$.

Proof. 1) implies 2). Trivially if $A$ is a Baer ring, condition 3) of Theorem 1 is satisfied and then $\text{Min}(A)$ is a compact space. Let us see that every prime ideal contains a unique minimal prime ideal, proving that condition ii) of Lemma $\beta$ holds. Let $a, b$ be elements such that $ab = 0$ and let $e, f$ be the idempotents which generate $\text{Ann}(a)$ and $\text{Ann}(b)$, respectively. Since $b \in \text{Ann}(a) = (e)$, there exists $c \in A$ such that $b = ce$, hence $be = ce^2 = ce = b$ and so $(1 - e)b = 0$. Then $(1 - e) \in \text{Ann}(b) = (f)$, so that $\text{Ann}(a) + \text{Ann}(b) = (e) + (f) = A$.

2) implies 3). Let $\varphi$ be the map from $\text{Spec}(A)$ to $\text{Min}(A)$ defined by $\varphi(P) = \mathcal{O}P$. Since $\text{Min}(A)$ is compact, to prove that $\varphi$ is a continuous function it is enough to show that $\varphi^{-1}[\mathcal{D}^0(a)]$ is a closed set (Theorem 1). This is trivial because, from the characterization of $\mathcal{O}_P$ given by the Lemma $\alpha$, we have $\varphi^{-1}[\mathcal{D}^0(a)] = V(\text{Ann}(a))$.

3) implies 1). First we prove that, if $Q$ is a minimal prime ideal contained in a prime ideal $P$, then $Q$ is the image of $P$ by the retraction $\varphi$. In fact $P \in \text{cl}_{\text{Spec}(A)}\{Q\}$, that is contained in $\varphi^{-1}[\varphi(Q)]$, so that $\varphi(P) = \varphi(Q) = Q$. Hence the retraction maps a prime ideal into the unique minimal prime ideal contained in it; therefore $V(\text{Ann}(a)) = \varphi^{-1}[\mathcal{D}^0(a)]$ is a clopen set; then $\text{Ann}(a)$ is a direct summand in $A$, because in a semiprime ring an ideal $I$ is a direct summand if and only if $V(I)$ is a clopen set.

Remark 1. A Baer ring $A$ is necessarily semiprime: assume $x$ nilpotent, and let $n$ be the smallest non negative integer such that $x^n = 0$. We want to show that $n = 1$, i.e. $x = 0$. For, otherwise, we have $\text{Ann}(x) \subseteq \text{Ann}(x^{n-1})$; since $A$ is Baer, $\text{Ann}(x) = (e)$, $\text{Ann}(x^{n-1}) = (f)$, with $e, f$ idempotents; since $x \in \text{Ann}(x^{n-1})$ then $x = xf$, which implies $x^{n-1} = x^{n-1}f = 0$, contradicting the minimality of $n$.

Remark 2. The ring $A = K[x, y]/(xy)$, where $K$ is a field and $x, y$ are indeterminates over $K$, is a ring whose minimal spectrum is compact, but it is not a Baer ring. $A$ is a noetherian ring; then $\text{Min}(A)$
is finite, hence compact (and discrete). It is easy to see that $A$ is a semiprime ring with no non trivial idempotents. Using the fact that $K[x, y]$ is a unique factorization domain, it can be shown that $\text{Ann}(x + (xy))$ is generated by $(y + (xy))$, so that $A$ is not a Baer ring.

**Remark 3.** If $X$ is a topological space, $C(X)$ denotes the ring of all real valued continuos functions on $X$; $X$ is said to be an $F$-space when every prime ideal of $C(X)$ contains a unique minimal prime ideal [GJ, 14.25]. $X$ is said to be basically disconnected if the closure of every cozero-set is an open set [GJ, 1H]. One can easily prove that $C(X)$ is a Baer ring if and only if $X$ is basically disconnected. There exist $F$-spaces $X$ that are not basically disconnected, for instance $\beta R - R$ [GJ, 6M, 14.0]. Hence there are rings in which every prime ideal contains a unique minimal prime ideal, without being Baer rings.

**REFERENCES**


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