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The Elimination of Descriptions from A. Bressan's Modal Language ML^ν on which the Logical Calculus MC^ν is Based.

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1. Introduction.

In A. Bressan's book [B] (1) the formal modal language ML^ν is both defined and studied, and the logical calculus MC^ν is based on it.

In this paper the elimination of the iota descriptor from ML^ν is dealt with.

We mean the elimination of some given signs from a formal language L in a way that involves a translation of L into a sublanguage L' of L devoid of those signs.

It is still an open problem how to define the notion of translation in a fully satisfactory way. Below we emphasize some distinctive features of translations, which are not quite sufficient for a full characterization of them. Our informal definition fits, nevertheless, the aims of this work; besides, I think that the particular translation we shall work out is acceptable and also complies with more careful definitions.

By a translation of L into L' we mean a pair of correspondences $p \mapsto p'$, $M \mapsto M'$ between the (closed) statements of L and L' and, respectively, between the models of L and L', such that:

(a) the correspondence $p \mapsto p'$ is an effective one;

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(1) By [B] we refer to [1].

(*b*) the same proposition (in an intensional sense) is characterized, or even expressed, by p at M and by p' at M'.

Let K be a logical calculus based on a language L from which an elimination of symbols has been performed by translating it into L'. Then, the main question to be tackled is that of basing a calculus K' on L' so that a statement p is a theorem in K if and only it its correspondent p' is a theorem in K'.

As far as our elimination of the iota descriptor is concerned, the solution of this problem will substantially differ from the well-known one of the analogous extensional problem. In fact, the axiom schemes of MC^ν that somehow define the description operator (cf. As. 3.18′) cannot simply be dropped when that operator is removed: they require to be replaced by a new scheme (As. 3.17) asserting the existence of those objects designated by the descriptions of ML^ν.

In conclusion, we outline our procedure of elimination.

Among the various methods to deal with descriptions explained by Carnap in [3] (pp. 33 to 39), Bressan adopts the one that is due to Frege and can be adapted to the theory of types. Namely, he assumes that in all possible cases in which a description does not fulfill the uniqueness condition, it makes reference to a particular extension, chosen at the outset once and for all. This method is the one that makes the treatment of descriptions simpler and more uniform.

Indeed, in a modal language it may well happen that a description meets the uniqueness condition in some, but not in all, possible cases. This remark prevents us from choosing Hilbert's method, according to which one is allowed to use a description only after it has been proved to fulfill the uniqueness condition.

Moreover, the adoption of Russel's method, which assigns the same meaning to an atomic formula $A\big((\imath x)B(x)\big)$ and the matrix

$$(\exists y)\{A(y)\ B(y)\ (x)[B(x)\supset y=x]\},$$

would invalidate some instances of the axiom of specification having the form

$$(y)A(y)\supset A\big((\imath x)B(x)\big)\ .$$

According to Frege's point of view, in a type-free extensional language an atomic formula $A\big((\imath x)r\big)$ must be assigned the same meaning as the matrix

$$(1)\qquad (\exists y)\{A(y)\wedge[(\exists_1 x)r\quad (x)(r\supset y=x)\vee\sim(\exists_1 x)r\ x=a]\}$$

where $(\exists_1 x)r$ means « there exists exactly one x such that r » and where « a » is a constant which designates a particular object acting as the « nonexisting object ».

In order to apply Frege's method as closely as possible to ML^ν (a modal language endowed with an infinity of types), one must replace (1) by the following matrix:

$$(2) \qquad (\exists y)\{A(y) \wedge N[(\exists_1 x)r\ (x)(r \supset y = x) \vee \sim (\exists_1 x)r\ x = a_t]\}$$

where a_t is a constant which designates the nonexisting object of the same type t as the variables x, y in every possible case.

As a matter of fact, according to Bressan's treatment of descriptions, $A\big((\imath x)r\big)$ turns out to have the same meaning in ML^ν as (2). Moreover, there are conventions which bind the choice of nonexisting objects of complex types to those having individual types. Therefore, in order to remove the description operator from ML^ν it is enough to require a particular constant of each individual type to designate the nonexisting object of its own type in the largest sublanguage ML^ν_λ (2) of ML^ν devoid of descriptions (see n. 4).

The calculus based on ML^ν_λ will include, in addition to the axioms of MC^ν that are well-formed in ML^ν_λ, also the new axiom scheme hinted above. Furthermore, I think it probably vital to replace the axiom of MC^ν concerning the existence of functions (cf. As. 3.16′, p. 10) by a stronger one (cf. As. 3.16, p. 10). Such a need would increase the difference between extensional and modal cases, as far as the elimination of the iota descriptor is concerned.

2. Preliminaries.

In the whole of the present work we use n, m, i, j, h, and k as metavariables running over nonzero natural numbers, unless otherwise stated. Let ν be a nonzero natural number arbitrarily fixed once and for all.

In [B], pp. 10 to 12, Bressan defines the modal language ML^ν, which is based on a *type system*. The type of *matrices* is 0; there are

(2) For typographical reasons we are using λ instead of a crossed reversed iota, that would be more expressive.

terms of the *individual types* $1, \ldots, \nu$. In addition there are other terms, called *relators* and *functors*, whose types are written in the forms $(t_1, \ldots, t_n)$ and $(t_1, \ldots, t_n: t_0)$ respectively—see below.

The set τ^ν of the *term types of* ML^ν is defined recursively as follows:

DEFINITION 2.1. (*a*) $\{1, \ldots, \nu\} \subset \tau^\nu$,

(b) if $t_i \in \tau^\nu$ for $i = 0, 1, \ldots, n$ then the $(n+1)$-tuples $\langle t_1, \ldots, t_n, 0\rangle$ and $\langle t_1, \ldots, t_n, t_0\rangle$—to be denoted, respectively, by $(t_1, \ldots, t_n)$ and by $(t_1, \ldots, t_n: t_0)$—belong to τ^ν.

The *primitive symbols of* ML^ν *are*:

the *variables* v_{tn} *with* $t \in \tau^\nu$;
the *constants* c_{tn} with $t \in \tau^\nu$;
the *connectives* $\sim$ and $\wedge$;
the *symbols for universal and modal quantification* $\forall$ *and* N;
the *descriptor* ι;
the *identity symbol* $=$;
the *parentheses*) and (and the *comma* ,.

Now we want to define the set $\mathcal{E}^\nu_t$ of the *well-formed expressions of* ML^ν *having the type* t, *for every* $t \in \tau^\nu \cup \{0\}$. All sets $\mathcal{E}^\nu_t$ are simultaneously characterized by the following recursive definition:

DEFINITION 2.2. (*a*) $v_{tn} \in \mathcal{E}^\nu_t$ and $c_{tn} \in \mathcal{E}^\nu_t$, for all $t \in \tau^\nu$.

(*b*) If $\Delta_1 \in \mathcal{E}^\nu_t$, $\Delta_2 \in \mathcal{E}^\nu_t$, and $t \in \tau^\nu$, then $\Delta_1 = \Delta_2 \in \mathcal{E}^\nu_0$.

(*c*) If $p \in \mathcal{E}^\nu_0$, $q \in \mathcal{E}^\nu_0$, and $t \in \tau^\nu$, then $\sim p \in \mathcal{E}^\nu_0$, $(p \wedge q) \in \mathcal{E}^\nu_0$, $Np \in \mathcal{E}^\nu_0$, $(\forall v_{tn})p \in \mathcal{E}^\nu_0$, and $(\iota v_{tn})p \in \mathcal{E}^\nu_t$.

(*d*) If $t, t_1, \ldots, t_n \in \tau^\nu$, $\Delta_i \in \mathcal{E}^\nu_{t_i}$ for $i = 1, \ldots, n$, $R \in \mathcal{E}^\nu_{(t_1, \ldots, t_n)}$, and $\Phi \in \mathcal{E}^\nu_{(t_1, \ldots, t_n: t)}$, then $R(\Delta_1, \ldots, \Delta_n) \in \mathcal{E}^\nu_0$ and $\Phi(\Delta_1, \ldots, \Delta_n) \in \mathcal{E}^\nu_t$.

Sometimes we shall use $x, y, z, x_i, y_i, z_i, F, G, f$, and g to express variables of ML^ν; particularly, F and G will be relators, while f and g wil be functors. $p, q, r, s, p_i, q_i, r_i, s_i$ will stand for matrices.

Generally, for the sake of brevity, the symbols $\forall$ and $\wedge$ will be dropped. As usual, square brackets] and [or braces } and { will often replaces round parentheses.

By means of the sign $=_D$ [$\equiv_D$] we shall introduce into ML^ν abbreviating terms [matrices]. For instance, by setting

$$(\exists x_1, \ldots, x_n)p \equiv_D \sim (x_1) \ldots (x_n) \sim p$$

$$\Diamond p \equiv_D \sim N \sim p$$

we introduce into ML^ν *existential quantifiers* and the symbol $\diamond$ to be read as « *it is possible that* ».

The connectives $\vee, \supset, \equiv$ and the sign $\bigwedge_{i=1}^{k}$ are defined as customary and rules for omission of parentheses are given. According to these rules, the signs $\sim$, N, $(\forall x)$, $(\imath x)$, $\wedge$, $\vee$, $\supset$, and $\equiv$ have decreasing cohesive powers in the written order.

Terms of the form $(\imath x)p$ are called *descriptions* while $(\imath x)$ is named a *description operator* and p the *scope* of $(\imath x)$ in $(\imath x)p$. The scopes of quantifiers and connectives have similar definitions.

In order to be able to attach meanings to the well-formed expressions of ML^ν, we consider $\nu+1$ domains $D_1, \ldots, D_\nu$, Γ together with ν objects $a_1^\nu, \ldots, a_\nu^\nu$ such that $a_t^\nu \in D_t$ for $t = 1, \ldots, \nu$.

The members of Γ, which are called *Γ-cases*, must be two at least, in order that ML^ν be really modal as an interpreted language.

By the following recursive definition, the set E_t^ν of the *extensions of type t* and the set QI_t^ν of the *quasi intensions of type t* are simultaneously defined for $t \in \tau^\nu\ R\ \{0\}$:

DEFINITION 2.3. (*a*) E_0^ν is the set $\{0, 1\}$ of truth values (1 stands for « true » and 0 stands for « false »).

(*b*) $E_t^\nu = D_t$ for $t = 1, \ldots, \nu$.

(*c*) QI_t^ν is the class of all functions from Γ into E_t^ν for $t \in \tau^\nu \cup \{0\}$.

(*d*) If $t, t_1, \ldots, t_n \in \tau^\nu$, then $E_{(t_1,\ldots,t_n)}^\nu$ is the class of all subsets of the Cartesian product $QI_{t_1}^\nu \times \ldots \times QI_{t_n}^\nu$, while $E_{(t_1,\ldots,t_n:t)}^\nu$ is the class of all functions from $QI_{t_1}^\nu \times \ldots \times QI_{t_n}^\nu$ into E_t^ν.

In each class E_t^ν with $t \in \tau^\nu$, we choose an object a_t^ν, to be called *improper extension of type t*:

a_t^ν is already given for $t \in \{1, \ldots, \nu\}$;
let a_t^ν be empty for $t = (t_1, \ldots, t_n)$;
let the range of the function a_t^ν be $\{a_{t_0}^\nu\}$ for $t = (t_1, \ldots, t_n\colon t_0)$.

DEFINITION 2.4. We say that V [M] is a value assignment [a model] for ML^ν, if it is a function from the class of variables [constants] of ML^ν into quasi intensions and

$$V(v_{tn}) \in QI_t^\nu \quad [M(c_{tn}) \in QI_t^\nu] \qquad \text{for } t \in \tau^\nu .$$

DEFINITION 2.5. Let v_{tn} be any variable of ML^ν. We denote by

$\sim_{v_{tn}}$ the equivalence relation holding for two value assignements if and only if they differ at most in v_{tn}.

DEFINITION 2.6. In connection with both a model M and a value assignment V, each well-formed expression Δ of ML^ν will now be assigned a *quasi-intensional designatum*, i.e. a quasi intension $\widetilde{\mathrm{des}}_{MV}(\Delta)$ (in short $\tilde{\Delta}$) having the same type as Δ.

For every γ, $\tilde{\Delta}(k)$ is defined by the following designation rules:

if Δ is	*then $\tilde{\Delta}(\gamma)$ is*
v_{tn}	$V(v_{tn})(\gamma)$
c_{tn}	$M(c_{tn})(\gamma)$
$\Delta_1 = \Delta_2$	1 if and only if $\tilde{\Delta}_1(\gamma) = \tilde{\Delta}_2(\gamma)$
$\sim p$	$1 - \tilde{p}(\gamma)$
$(p \wedge q)$	$\tilde{p}(\gamma) \cdot \tilde{q}(\gamma)$
Np	$\min\limits_{\gamma \in \Gamma}(\tilde{p}(\gamma))$
$(\forall v_{tn})p$	$\min\limits_{V' \sim_{v_{tn}} V}(\widetilde{\mathrm{des}}_{MV'}(p)(\gamma))$
$(\imath v_{tn})p$	the extension η for which there exists a V' such that $V' \sim_{v_{tn}} V$, $V'(v_{tn})(\gamma) = \eta$, $\widetilde{\mathrm{des}}_{MV'}(p)(\gamma) = 1$ in case there is precisely one of such extensions; a_t^ν otherwise
$R(\Delta_1, \ldots, \Delta_n)$	1 if and only if $\langle \tilde{\Delta}_1, \ldots, \tilde{\Delta}_n \rangle \in \tilde{R}(\gamma)$
$\Phi(\Delta_1, \ldots, \Delta_n)$	$\tilde{\Phi}(\gamma)(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n)$.

We say that a matrix p of ML^ν is *logically true* if $\widetilde{\mathrm{des}}_{MV}(p)(\gamma) = 1$ for every choice of $D_1, \ldots, D_\nu, \Gamma, a_1^\nu, \ldots, a_\nu^\nu, M, V$ and all $\gamma \in \Gamma$.

On ML^ν is based Bressan's logical calculus MC^ν. Every theorem of MC^ν is a logically true matrix.

3. The modal logical calculus K.

For the sake of simplicity in treating our subject-matter, we shall disregard four axiom schemes of MC^ν which have no concern with our elimination problem (As. 12.20, As. 12.23, As. 25.1, As. 45.1 on [B], pp. 46, 48, 95, 184). The first of them is the choice axiom; the second asserts the existence of a contingent proposition and the third that of absolute predicates of a particular kind.

The fourth is an axiom which concerns natural numbers.

The sign $\imath$ occurs only in the two last of the aforementioned axioms, and in a non-essential way.

Let us start considering a logical calculus, to be called K, which includes as theorems all the axioms of MC^{ν} save the four referred to previously. The results we shall reach will fit as well to MC^{ν} as to K.

Just like every other logical calculus that we shall consider, K is fully characterized by a set of axiom schemes. Some of the axiom schemes of MC^{ν} will be given in K a modified formulation, more convenient for dealing with the questions we are interested in.

We adopt only one primitive inference rule: *modus ponens* (or MP). However, in order to shorten proofs, we shall avail ourselves of all the inference rules of usual practice, such as rule C (or the rule of existential specification) and rule G (or the rule of universal generalization). The employment of such rules can be legitimized on the basis of MP solely and, besides, it agrees with natural ways of carrying out deductions (see [B], pp. 134 to 145, 171 to 174).

Let T be a logical calculus; then by the notation

$$p_1, \dots, p_n \vdash_T q$$

(to be read as « $p_1, \dots, p_n$ *yield* q *in* T », or simply as « q *is a theorem in* T » for $n = 0$) we mean that q can be deduced from the hypotheses $p_1, \dots, p_n$ by use of MP and the axioms of T.

We shall simplify $\vdash_T$ into $\vdash$ where no misunderstanding may arise.

We now present the axiom schemes of K. Now and then in listing axioms, we shall display some theorems which can be derived from them, the be used later on.

The symbol (N) occurring in axioms, stands for a finite (possibly empty) sequence of universal quantifiers and N's; its scope is the whole expression following (N).

A first set of axioms characterizes the modal propositional calculus S5 (cf. [6], pp. 56 to 76; [4], pp. 67 to 75):

As. 3.1. $(N)\ p \supset pp$

As. 3.2. $(N)\ pq \supset p$

As. 3.3. (N) $(p \supset q) \supset [\sim (qr) \supset \sim (rp)]$

As. 3.4. (N) $N(p \supset q) \supset (Np \supset Nq)$

As. 3.5. (N) $Np \supset p$

Let p, q be matrices and y be a variable. We remember that the formula Np is said to be *modally closed*; the same qualification is deserved by either $(\forall y)p$ or $\sim p$ [by $(p \wedge q)$] if and only if p [each of p and q] is modally closed.

As. 3.6. (N) $p \supset Np$, where p is modally closed.

Before surveying the axioms needed for the calculus with quantifiers, we want to make a brief summary of some common terminology.

An occurrence of a variable x is said to be *bound* in the well-formed expression Δ if it belongs to an occurrence of either of the operators $(\forall x)$ and $(\imath x)$ or to its scope. Other occurrences, which are not bound, are called *free.*

Let $A(x)$ be a matrix and Δ be a term. We say that Δ *is free for* x *in* $A(x)$ if in $A(x)$ there are no free occurrences of x placed in the scope of an occurrence of either $(\forall y)$ or $(\imath y)$ for any of the variables y occurring free in Δ. In such a case only, we shall denote by $A(\Delta)$ the formula obtained from $A(x)$ by replacing x with Δ in all of its free occurrences.

A matrix q in which x does not occur, as well as a matrix having the form $(\forall x)p$ is said to be *closed with respect to* x. The same denomination applies to each of the formulas $(\forall y)p$ (with y distinct from x), $Np, \sim p$ [to $(p \wedge q)$] if and only if p [each of p and q] is closed with respect to x. More generally, we say that a well-formed expression Δ is closed with respect to x if x does not occur free in Δ. Δ is said to be *closed* if it is closed with respect to all variables.

Below are the axioms on universal quantification:

As. 3.7. (N) $(x)(p \supset q) \supset [(x)p \supset (x)q]$

As. 3.8. (N) $(x)A(x) \supset A(\Delta)$

As. 3.9. (N) $p \supset (x)p$, where p is closed with respect to x.

From the axioms concerning identity it will be possible to deduce that identity is reflexive symmetric and transitive. Moreover, two terms which are strictly identical must be reciprocally replaceable. We employ the sign $=^{\cap}$ for strict identity. That is, if Δ_1 and Δ_2 are terms having the same type, we set:

DEFINITION 3.2. $\Delta_1 =^\cap \Delta_2 \equiv_D N\Delta_1 = \Delta_2$ (likewise we adopt for strict equivalence and for strict implication the signs $\equiv^\cap$, $\supset^\cap$ (cf. [B], p. 22)).

Here are the axioms on identity:

As. 3.10. $(N)\ x = x$

As. 3.11. $(N)\ x = z\ y = z \supset x = y$

As. 3.12. $(N)\ x =^\cap y \supset [A(x) \equiv A(y)]$.

We introduce below the notations $(\exists_1^\Delta x)r$, $(\exists_1 x)r$, $(\exists_1^\cap x)r$ to be read in the order as

« *Δ equals the only x such that r* »,

« *there exists exactly one x such that r* »,

« *there exists a strictly unique x such that r* ».

The first of these notations will be of use to us in As. 17 and as a a preliminary step toward the definition Def. 7 of $(\exists_1^{\cap\Delta} x)r$. This last notation will allow us to make easier the enunciation of Prop. 2, As. 18, Prop. 4.1-2, Lem. 4.1, and Def. 4.2.

DEFINITION 3.3. $(\exists_1^\Delta x)r \equiv_D (\exists x)r\ \ (x)(r \supset x = \Delta)$, where Δ is a term closed with respect to x, and has the same type as x.

DEFINITION 3.4. Let y be the first variable distinct from x which does not occur free in r and has the same type as x. Then we set

$$(\exists_1 x)r \equiv_D (\exists y)(\exists_1^y x)r$$

$$(\exists_1^\cap x)r \equiv_D (\exists y)(x)(r \equiv x =^\cap y)\,.$$

Observe that the formulas we have used to define the expressions $(\exists_1 x)r$, $(\exists_1^\cap x)r$ differ from those employed by Bressan (cf. [B], pp. 36, 52) only formally, while they are sintactically equivalent to them, as is easily checked on the basis of the axioms presented so far.

In addition note that the occurrence of y explicitly written in the expression $(\exists_1^y x)r$ is free, while all occurrences of x are bound; the other occurrences of variables are free or bound in $(\exists_1^y x)r$ according to whether they are free or bound in r. In short, the quantifier $(\exists_1^y x)$ acts on variables like the operator $\int_0^y \dots dx$ which, however, applies to functions of x and not to matrices.

Now we give four axioms concerning relations and functions. Among them, the first two are named intensionality principles:

As. 3.13. $(N)\ F = G \equiv (x_1) \dots (x_n)[F(x_1, \dots, x_n) \equiv G(x_1, \dots, x_n)]$

As. 3.14. $(N)\ f = g \equiv (x_1) \dots (x_n) f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$

As. 3.15. $(N)\ (\exists F)(x_1) \dots (x_n)\{[\Diamond F(x_1, \dots, x_n) \equiv NF(x_1, \dots, x_n)$
$[F(x_1, \dots, x_n) \equiv p]\}$

As. 3.16. $(N)\ (x_1) \dots (x_n)(\exists_1^\cap y) Np \supset (\exists f)(x_1) \dots (x_n)(\exists y)$
$N[f(x_1, \dots, x_n) = y \wedge p]$.

In the axioms As. 13 to 16 we intend $y, x_1, \dots, x_n, F, G, f, g$ to stand for distinct variables, having in the order the types $t_0, t_1, \dots, t_n$, $t, t, \vartheta, \vartheta$, where $t = (t_1, \dots, t_n)$ and $\vartheta = (t_1, \dots, t_n\colon t_0)$. The matrix p must be closed with respect to F in As. 15, and with respect to f in As. 16.

In MC^ν, As. 16 has the following simpler formulation (cf. [B], p. 45):

As. 3.16′. $(N)\ (\exists f)(x_1) \dots (x_n) f(x_1, \dots, x_n) = \Delta,$

where Δ is a term closed with respect to f.

We can deduce As. 16′ from As. 16 easily. To this end the theorems

$$(\exists_1^\cap y) \quad y =^\cap \Delta$$

and

$$f(x_1, \dots, x_n) =^\cap \Delta \equiv (\exists y) N[f(x_1, \dots, x_n) = y \wedge y = \Delta]\,,$$

where Δ is a term closed with respect to f and y, are to be derived by exclusive use of the axioms As. 1 to 15. Then, by identifying p with $y = \Delta$ in As. 16, the following scheme is proved

As. 3.16″. $(N)\ (\exists f)(x_1) \dots (x_n) f(x_1, \dots, x_n) =^\cap \Delta.$

Thence as 16′ follows rapidly.

The proof of As. 16″ from As. 16′ (see [B], pp. 166 to 168) that I know depends on the axioms concerning the description operator (see As. 18′). By making use of those axioms it is also easy to deduce As. 16 from As. 16″: assuming Δ to be $(\imath y) Np$ in As. 16″, one has merely to demonstrate the theorem

$$(\exists_1^\cap y) Np \supset [y =^\cap (\imath y) Np \equiv Np] .$$

To the aims of this paper it is essential to be able to assert As. 16 without applying to the axioms on the description operator. For this reason, we postulate this scheme instead of As. 16′ or As. 16″ that are weaker.

The scheme we are about to present can also be deduced by use of As. 18′ whereas it does not depend from the axiom schemes considered so far. It is therefore important to emphasize it.

As. 3.17. $(N)\ (\exists y) N[(\exists_1^y x) r \vee \sim (\exists_1 x) r\ y = z]$,

where x, y, z are distinct variables of the same type and where r is closed with respect to y.

The independence of As. 3.17 from axioms As. 1 to 16 is easily verified at least in case y is a variable having an individual or functor type: interpret the symbol $=$ as strict identity whenever it is placed between two individual terms or functors and keep the usual interpretation of the remaining symbols. This nonstandard interpretation preserves the logical truth of all among the axioms so far presented, save the last.

In developing the semantical analysis of ML^ν, we assumed that a description must denote the so-called *improper extension* of its own type in all Γ-cases in which it does not fulfill the exact uniqueness condition. For example, the description a_t^* defined below will denote the improper extension of type t in all Γ-cases.

DEFINITION 3.5. $a_t^* =_D (\imath v_{t1}) v_{t1} \neq v_{t1}$, for $t \in \tau^\nu$.

We call *degenerate descriptions* the terms $a_1^*, \ldots, a_\nu^*$.

We call *pure formula* a well-formed expression of ML^ν where no non-degenerate description occurs.

Let us introduce the notation $\mathrm{Impr}_t(\Delta)$ to be read as « Δ *equals the improper object of type, t* ». We choose the definiens of $\mathrm{Impr}_t(\Delta)$ in such a way that it is pure whenever Δ is a pure formula. Keeping mind on the conventions about improper extensions we have made (cf. [B], p. 19) we give the following metalinguistic recursive definition:

DEFINITION 3.6.

$$\mathrm{Impr}_t(\Delta) \equiv_D \Delta = a_t^* ,$$

if Δ is a term of type $t \in \{1, \dots, \nu\}$

$$\mathrm{Impr}_t(\Delta) \equiv_D (x_1) \dots (x_n) \sim \Delta(x_1, \dots, x_n),$$

if Δ is a term of type $t = (t_1, \dots, t_n)$

$$\mathrm{Impr}_t(\Delta) \equiv_D (x_1) \dots (x_n) \, \mathrm{Impr}_{t_0}(\Delta(x_1, \dots, x_n)),$$

if Δ is a term of type $t = (t_1, \dots, t_n : t_0)$.

In the second and the third of the above clauses, x_i is the first variable of type t_i which does not occur free in Δ and is distinct from $x_1, \dots, x_{i-1}$, for $i = 1, \dots, n$.

By making use of the axioms As. 1 to 16 alone it is possible to prove the following proposition:

PROPOSITION 3.1. For all $t \in \tau^\nu$, if y and z are distinct variables of type t, we have

(*a*) $\vdash (\exists z) N \, \mathrm{Impr}_t(z)$

(*b*) $\vdash \mathrm{Impr}_t(z) y = z \supset \mathrm{Impr}_t(y)$

(*c*) $\vdash \mathrm{Impr}_t(z) \, \mathrm{Impr}_t(y) \supset y = z$

(*d*) $\vdash (\exists z)(y)[\mathrm{Impr}_t(y) \equiv^\cap y = z]$ (in short $\vdash (\exists_1^\cap y) \, \mathrm{Impr}_t(y)$).

PROOF. Note that theorem (*d*) is easily deduced from (*a*), (*b*), and (*c*). In fact, (*b*) and (*c*) yield

$$\vdash N \, \mathrm{Impr}_t(z) \supset (y)[\mathrm{Impr}_t(y) \equiv^\cap y = z]$$

whence, by help of (*a*), (*d*) is soon reached.

For $t \in \{1, \dots, \nu\}$ the proof of (*a*) to (*d*) is trivial.

For $t = (t_1, \dots, t_n)$, (*b*) and (*c*) are drawn from the intensionality principle As. 13, while (*a*) is derived from As. 15 (in which p is replaced by $x_1 \neq x_1$).

In case $t = (t_1, \dots, t_n : t_0)$ the theorems (*b*) and (*c*) on t, y, and z follow from their analogues on t_0, y_0, and z_0. We can derive (*a*) by proving that

$$\vdash (\exists z_0)(y_0)[\mathrm{Impr}_{t_0}(y_0) \equiv^\cap y_0 = z_0]$$

and by making use of As. 16″ in the form

$$(\exists z) \qquad (x_1) \dots (x_n) z(x_1, \dots, x_n) =^{\cap} z_0 .$$

By means of the equivalence theorem (see [B], p. 172), the clause (d) in Prop. 1, and As. 17, the following assertion is soon proved.

COROLLARY 3.1. Let x and y be distinct variables of type t and r be closed with respect to y. Then

$$\vdash (\exists y) N[(\exists_1^y x) r \vee \sim (\exists_1 x) r \, \mathrm{Impr}_t(y)] .$$

We want to express concisely a certain very long formula:

DEFINITION 3.7. Let Δ be a term and x a variable of type t; and Δ be closed with respect to x. Then we set

$$(\exists_1^{\cap\Delta} x) r \equiv_D N[(\exists_1^{\Delta} x) r \vee \sim (\exists_1 x) r \, \mathrm{Impr}_t(\Delta)]$$

so that $(\exists_1^{\cap\Delta} x) r$ can be read as « Δ *necessarily equals the* x *such that* r ».
By means of the new notation, we can rewrite Cor. 1 as follows:

$$\vdash (\exists y)(\exists_1^{\cap y} x) r .$$

That corollary can also be stenghtened into the following proposition:

PROPOSITION 3.2. Let the same hypotheses of Cor. 1 hold. Then

$$\vdash (\exists_1^{\cap} y)(\exists_1^{\cap y} x) r$$

(i.e. « there exists a strictly unique y that in every Γ-case equals the x such that r »).

PROOF. As a preliminary step prove that

$$\vdash [(\exists_1^y x) r \vee \sim (\exists_1 x) r \, \mathrm{Impr}_t(y)] [(\exists_1^z x) r \vee \sim (\exists_1 x) r \, \mathrm{Impr}_t(z)] \supset y = z .$$

By modal quantification of the two members in the implication above, and by help of Cor. 1 the assertion is easily obtained.
The importance to us of Prop. 2 chiefly resides in the metatheorem stated below:

PROPOSITION 3.3. Let q be a matrix closed both modally and with respect to the variables $x_1, \dots, x_k$. Let

$$A(p_1, \dots, p_n) \quad \text{and} \quad A\big((\exists y)(p_1 q), \dots, (\exists y)(p_n q)\big)$$

be matrices constructed in the same way, by means of $\sim$, $\wedge$, N, $(\forall x_1), \dots, (\forall x_k)$ starting out from the p_i's and from the matrices $(\exists y)(p_i q)$ respectively. Then

$$\vdash (\exists_1^\cap y) q \supset \big[A\big((\exists y)(p_1 q), \dots, (\exists y)(p_n q)\big) \equiv (\exists y)\big(A(p_1, \dots, p_n) q\big)\big].$$

PROOF. The assertion results from the series of lemmas listed below:

LEMMA 3.1. $\vdash (\exists_1^\cap y) q \supset [(\exists y)(pq) \equiv (y)(q \supset p)]$

LEMMA 3.2. $\vdash (\exists_1^\cap y) q \subset [\sim (\exists y)(pq) \equiv (\exists y)(\sim pq)]$

LEMMA 3.3. $\vdash (\exists_1^\cap y) q \supset [(\exists y)(p_1 q)(\exists y)(p_2 q) \equiv (\exists y)(p_1 p_2 q)]$

LEMMA 3.4. If q is modally closed, then

$$\vdash (\exists_1^\cap y) q \supset [N(\exists y)(pq) \equiv (\exists y)(Npq)]$$

LEMMA 3.5. If q is closed with respect to the variable x (which need not be distinct from y) then

$$\vdash (\exists_1^\cap y) q \supset [(x)(\exists y)(pq) \equiv (\exists y)((x)pq)].$$

We exhibit a proof of Lem. 1, whence Lem. 2 to 5 easily follow.

PROOF. We want to prove the theorem

$$(1) \qquad (\exists_1^\cap y) q (y)(q \supset p) \supset (\exists y)(pq)$$

and that the hypotheses

$$(2) \qquad (\exists_1^\cap y) q$$

$$(3) \qquad (\exists y)(pq)$$

yield

$$(4) \qquad (y)(q \supset p).$$

Theorem (1) is deduced from the trivial one

(5) $$(\exists_1^\cap y)q \supset (\exists y)q\,.$$

From (2) and the definition of $(\exists_1^\cap y)q$, by use of rule C,

(6) $$(y)(q \supset y =^\cap x)$$

is obtained.

(7) $$(\exists y)(py =^\cap x)$$

follows from (3) and (6).

We finally reach (4) by help of the easy lemma

(8) $$(\exists y)(py =^\cap x) \supset (y)(y =^\cap x \supset p)$$

and (7) and (6).

Here we finally arrive at the axioms on the description operator. These are formulated in [B] as follows:

As. 3.18′.

$$(N)r(\exists_1 x)r \supset (\imath x)r = x$$

$$(N) \sim (\exists_1 x)r \supset (\imath x)r = a_t^*$$

$$(N) \sim a^*_{(t_1,\dots,t_n)}(x_1, \dots, x_n)$$

$$(N) a^*_{(t_1,\dots,t_n:t_0)}(x_1, \dots, x_n) = a^*_{t_0}$$

where:

x is a variable of type t and r is a matrix;

$t, t_0, t_1, \dots, t_n \in \tau^\nu$;

$x_1, \dots, x_n$ are distinct variables of the respective types $t_1, \dots t_n$.

By use of Def. 7 we can replace the four schemes As. 18′ by a single equivalent axiom:

As. 3.18.

$$(N)(\exists_1^{\cap(\imath x)r} x)r\,,$$

that is

As. 3.18.

$$(N)(\exists x)r\ (x)\big(r \supset x = (\imath x)r\big) \vee \sim (\exists_1 x)r\,\mathrm{Impr}_t\big((\imath x)r\big)\ .$$

We take as axioms for the calculus K on ML^ν all formulas that are included in some of the schemes As. 1 to 18.

4. The elimination of the descriptor $\imath$ from ML^ν.

In this section we shall describe an effective procedure that with every matrix p of ML^ν associates a pure formula p' sintactically equivalent with p in K (hence in MC^ν). Moreover, we shall choose p' in such a way that all occurrences of degenerate descriptions in it—if any—belong to contexts having the form $\mathrm{Impr}_t(y)$.

Provided that every degenerate description a_t^* be identified with c_{t1}, the transformation $p \mapsto p'$ will turn out to be a translation of ML^ν into its own largest sublanguage ML_λ^ν devoid of the symbol $\imath$ (a weak translation, in that it operates only on matrices and not on terms). We shall base on ML_λ^ν such a logical calculus that the relation of deducibility of q from $p_1, \dots, p_n$ is invariant under the aforementioned translation.

Definition 4.1. Let $\varDelta$ be a well-formed expression of ML^ν and let $(\imath x)r$ be a description. An occurrence of $(\imath x)r$ in $\varDelta$ is called maximal if it is placed outside the scope of every description operator. All descriptions which have maximal occurrences in $\varDelta$ are said to be maximal in $\varDelta$.

We point out that whenever $\varDelta$ is a term, all maximal occurrences in $\varDelta$ are placed outside the scope of universal quantifiers (belonging to $\varDelta$) too. The same holds, therefore, when $\varDelta$ is an *atomic formula*, *i.e.* a matrix having either of the forms $R(\varDelta_1, \dots, \varDelta_n)$ and $\varDelta_1 = \varDelta_2$.

Proposition 4.1. Let x and y be two distinct variables having the same type, and $A\big((\imath x)r\big)$ be an atomic formula closed with respect to y, in which the description $(\imath x)r$ is maximal.

If $A(y)$ is a matrix which is obtained from $A\big((\imath x)r\big)$ by replacing $(\imath x)r$ with y in one or more of its maximal occurrences, then

$$\vdash A\big((\imath x)r\big) \equiv (\exists y)[A(y)(\exists_1^{\cap y} x)r]\ .$$

PROOF. The term $(\imath x)r$ is closed with respect to y, because it is maximal in $A((\imath x)r)$. When the free occurrences of y in $A(y)$ are replaced by $(\imath x)r$ no confusion of bound variables arises and the result of this substitution is just $A((\imath x)r)$. By remembering As. 3.12 we note that

$$\vdash (y)\{(\imath x)r =^{\cap} y \supset [A((\imath x)r) \equiv A(y)]\},$$

hence

$$\vdash (y)\{[A(y)(\imath x)r =^{\cap} y] \supset A((\imath x)r)\}. \tag{1}$$

From Prop. 3.2 and As. 3.18 we deduce that

$$\vdash (\imath x)r =^{\cap} y \equiv (\exists_1^{\cap v} x)r. \tag{2}$$

By (1) and (2) the following theorem is easily checked:

$$\vdash (\exists y)[A(y)(\exists_1^{\cap v} x)r] \supset A((\imath x)r).$$

This is a half of the assertion under inspection.

To prove the converse implication we note that

$$A((\imath x)r) \vdash A((\imath x)r)(\imath x)r =^{\cap} (\imath x)r \vdash (\exists y)[A(y)(\imath x)r =^{\cap} y].$$

PROPOSITION 4.2. Let p be an atomic formula and $(\imath x_1)r_1, \ldots, (\imath x_k)r_k$ be all maximal descriptions in p. For $i = 1, \ldots, k$ let y_i be a variable of the same type as x_i, that has no free occurrences in p and is distinct from $y_1, \ldots, y_{i-1}$, and x_i. If q is obtained from p by replacing $(\imath x_i)r_i$ with y_i in all of its maximal occurrences (for $i = 1, \ldots, k$), then

$$\vdash p \equiv (\exists y_1, \ldots, y_k)\Big[q \bigwedge_{i=1}^{k} (\exists_1^{\cap y_i} x_i)r_i\Big].$$

PROOF. Let p_0 be p itself; and for every $h \in \{1, \ldots, k\}$ let p_h be the matrix obtained from p_{h-1} by replacing $(\imath x_h)r_h$ with y_h in all of its maximal occurrences.

By Prop. 1

$$\vdash p_{h-1} \equiv (\exists y_h)[p_h(\exists_1^{\cap y_h} x_h)r_h] \quad \text{for } h = 1, \ldots, k$$

so that

$$\vdash p \equiv (\exists y_1)\Big((\exists y_2)(\ldots (\exists y_k)\big(p_k(\exists_1^{\cap y_k} x_k)r_k\big) \ldots (\exists_1^{\cap y_2} x_2)r_2\big)(\exists_1^{\cap y_1} x_1)r_1\Big).$$

Hence the assertion follows easily, when one notes that p_k is q and

$$\vdash (\exists y_{h-1})\Big\{(\exists y_h, y_{h+1}, \ldots, y_k)\Big[q \bigwedge_{i=h}^{k} (\exists_1^{\cap \nu_i} x_i) r_i\Big] (\exists_1^{\cap \nu_{h-1}} x_{h-1}) r_{h-1}\Big\} \equiv$$
$$\equiv (\exists y_{h-1}, y_h, \ldots, y_k)\Big[q \bigwedge_{i=h-1}^{k} (\exists_1^{\cap \nu_i} x_i) r_i\Big],$$

for $h = k, k-1, \ldots, 2$.

In fact, since p is closed with respect to $y_h, y_{h+1}, \ldots, y_k$, the same holds for its maximal description $(\imath x_{h-1}) r_{h-1}$ and therefore also for $(\exists_1^{\cap \nu_{h-1}} x_{h-1}) r_{h-1}$, because y_{h-1} is distinct from $y_h, y_{h+1}, \ldots, y_k$.

Our last theorem allows us to define a transformation $p \mapsto p'$ which turns every matrix of ML^ν into a sintactically equivalent pure formula. In view of a proposition (Lem. 1) to be stated soon, it is useful to present this transformation as a particular instance of a more general one involving a term Δ.

Definition 4.2. Let Δ be a term of ML^ν. For every matrix p we denote by p^Δ the pure formula constructed according to the following recursive rule:

(*a*) If p is an atomic formula whose maximal descriptions ordered according to their first occurrences in p are $(\imath x_1) r_1, \ldots, (\imath x_k) r_k$, then

$$p^\Delta \equiv_D \begin{cases} p & \text{if } k = 0 \\ (\exists y_1, \ldots, y_k)\Big[p^* \bigwedge_{i=1}^{k} (\exists_1^{\cap \nu_i} x_i) r_i^\Delta\Big] & \text{if } k > 0 \end{cases}$$

where p^* is obtained from p by replacing each description $(\imath x_i) r_i$ with y_i in all of its maximal occurrences and where y_i is the first variable of the same type as x_i that has no free occurrences in either p or Δ and is distinct from $y_1, \ldots, y_{i-1}$, and x_i (for $i = 1, \ldots, k$).

(*b*) If p has the form $(q \wedge r)$, $\sim q$, Nq, or $(y)q$, then p^Δ is $(q^\Delta \wedge r^\Delta)$, $\sim q^\Delta$, Nq^Δ, or $(y)q^\Delta$ respectively.

Note that p^Δ equals $p^{\Delta'}$ if Δ and Δ' are arbitrary closed terms. Hence if Δ is closed p^Δ can be denoted simply by p'.

Prop. 2 yields trivially:

Proposition 4.3. For every term Δ and every matrix p

$$\vdash p^\Delta \equiv p\,.$$

Furthermore a variable occurs free in p^{Δ} if and only if it does so in p.

DEFINITION 4.3. Let ML_{λ}^{ν} be the language consisting of the well-formed expressions of ML^{ν} where the symbol $\imath$ does not occur. The models, value assignments and designation rules for ML_{λ}^{ν} are those for ML^{ν} except, of course, the designation rule for $\imath$.

DEFINITION 4.4. Let p be any matrix of ML^{ν}. We denote by p^{λ} the formula of ML_{λ}^{ν} which is obtained from p' by replacing in it first

$$c_{tm} \text{ with } c_{tm+1} \quad \text{for } t = 1, \dots, \nu$$

and then

$$a_t^* \text{ with } c_{t1} \quad \text{for } t = 1, \dots, \nu\,.$$

For every model M there is exactly one model M^{λ} such that

$$M^{\lambda}(c_{t1})(\gamma) = a_t^{\nu} \quad \text{in every } \Gamma\text{-case } \gamma \text{ for } t = 1, \dots, \nu$$

$$M^{\lambda}(c_{tm+1}) = M(c_{tm}) \quad \text{for } t = 1, \dots, \nu$$

$$M^{\lambda}(c_{tm}) = M(c_{tm}) \quad \text{for } t \notin \{1, \dots, \nu\}\,.$$

On the basis of Prop. 3 and Def. 4, it is clear that

PROPOSITION 4.4. For every matrix p of ML^{ν} and every value assignement V

$$\widetilde{\operatorname{des}}_{MV}(p) = \widetilde{\operatorname{des}}_{MV}(p') = \widetilde{\operatorname{des}}_{M^{\lambda}V}(p^{\lambda})\,.$$

Prop. 4 asserts—in some sense—that the transformation $p \mapsto p^{\lambda}$ is a translation of ML^{ν} into ML_{λ}^{ν}.

DEFINITION 4.5. Let K' be the logical calculus based on ML^{ν} whose axioms are those of K that are pure formulas; and let K^{λ} be the logical calculus based on ML_{λ}^{ν} whose axioms are those of K' in which $a_1^*, \dots, a_{\nu}^*$ do not occur (*i.e.* the axioms of K without descriptions).

By use of Prop. 3, the following assertion is immediately derived.

PROPOSITION 4.5. If $p_1', \dots, p_n' \vdash_{K'} q'$, then $p_1, \dots, p_n \vdash_{K} q$.

Note that the axiom scheme As. 3.18 is superfluous in K'. Indeed, a formula included in this scheme is pure only in the case when the description $(\imath x)r$ occurring in it is degenerate; but then it can be deduced easily from the remaining axiom schemes. Since no axiom in K' concerns degenerate descriptions, it is intuitive that these descriptions will behave in deductions just like constants. As a matter of fact, the following assertion is easily proved

PROPOSITION 4.6. $p'_1, \dots, p'_n \vdash_{\overline{K'}} q'$ if and only if $p^\lambda_1, \dots, p^\lambda_n \vdash_{\overline{K^\lambda}} q^\lambda$

hence

PROPOSITION 4.7. If $p^\lambda_1, \dots, p^\lambda_n \vdash_{\overline{K^\lambda}} q^\lambda$, then $p_1, \dots, p_n \vdash_{\overline{K}} q$.

The rest of this section is devoted to proving the converse of Prop. 5:

PROPOSITION 4.8. If $p_1, \dots, p_n \vdash_{\overline{K}} q$, then $p'_1, \dots, p'_n \vdash_{\overline{K'}} q'$

which, by Prop. 6, yields

PROPOSITION 4.9. If $p_1, \dots, p_n \vdash_{\overline{K}} q$, then $p^\lambda_1, \dots, p^\lambda_n \vdash_{\overline{K^\lambda}} q^\lambda$.

We point out at once that the proofs of Prop. 3.1, 2, 3 we gave with regard to K, work step by step also for K'.

Since $(p \supset q)' \equiv_D (p' \supset q')$, in order to prove Prop. 8 it is enough to check that the transformation $p \mapsto p'$ sends the axioms of K into theorems of K'. In fact, it turns every axiom scheme different from As. 3.8, 12, 18 into its analogue for K'.

The following lemma will enable us to treat the remaining three cases quickly.

LEMMA 4.1. Let $(\imath z_1)s_1, \dots, (\imath z_h)s_h$ be the maximal descriptions in the term Δ, which is free for x in $A(x)$. We denote by $A^\Delta(x)$ the matrix $A(x)^\Delta$.

For $j = 1, \dots, h$ let y_j be a variable not occurring in either $A(x)$ or Δ, which has the same type as z_j and is distinct from $y_1, \dots, y_{j-1}$, and z_j.

If Δ^* is obtained from Δ by replacing each $(\imath z_j)s_j$ with y_j in all of its maximal occurrences, then

$$\text{(1)} \qquad \vdash_{\overline{K'}} A(x)' \equiv A^\Delta(x)$$

$$\text{(2)} \qquad \vdash_{\overline{K'}} A(\Delta)' \equiv (\exists y_1, \dots, y_h)\Big[A^\Delta(\Delta^*) \bigwedge_{j=1}^{h} (\exists_1^{\frown y_j} z_j) s'_j\Big] .$$

PROOF. Since $A^\Delta(x)$ can be obtained from $A(x)'$ by replacing some bound occurrences of certain variables by other variables, (1) holds trivially.

Theorem (2) is also trivial when $A(x)$ is an atomic formula in which the symbol $\imath$ does not occur.

If $A(x)$ is not an atomic formula, then it has one of the following forms:

$$\big(B(x) \wedge C(x)\big)\,, \qquad \sim B(x)\,, \qquad NB(x)\,, \qquad (\forall y)B(x)\,.$$

In these cases theorem (2) on $A(x)$ follows from its analogues for $B(x)$ and $C(x)$, taking into account Prop. 3.3.

Now the only remaining case is the one in which $A(x)$ is an atomic formula whose maximal descriptions are $(\imath x_1)r_1(x), \ldots, (\imath x_k)r_k(x)$.

For $i = 1, \ldots, k$ let y_{h+i} be the first variable of the same type as x_i that is distinct from $y_1, \ldots, y_{h+i-1}, x_i$, and x and has no free occurrences in either $A(x)$ or Δ. Let $A^*(x)$ be the matrix obtained from $A(x)$ by replacing each $(\imath x_i)r_i(x)$ with y_{h+i} in all of its maximal occurrences. We denote by $(\exists Y)$ the string of quantifiers $(\exists y_1) \ldots (\exists y_h)$. In addition we set:

$$p_i \equiv_D (\exists_1^{\cap y_{h+i}} x_i) r_i^\Delta(\Delta^*)\,, \qquad \text{for } i = 1, \ldots, k;$$

$$q_j \equiv_D (\exists_1^{\cap y_j} z_j) s_j'\,, \qquad \text{for } j = 1, \ldots, h;$$

$$p \equiv_D \bigwedge_{i=1}^{k} p_i; \qquad q \equiv_D \bigwedge_{j=1}^{h} q_j\,.$$

Below are listed the main steps in the proof of (2).

(3) $$r_i(x)' \equiv r_i^\Delta(x)\,,$$

for $i = 1, \ldots, k$ (inductive hypothesis);

(4) $$r_i(\Delta)' \equiv (\exists Y)[r_i^\Delta(\Delta^*)q]\,,$$

for $i = 1, \ldots, k$ (inductive hypothesis);

(5) $$\bigwedge_{j=1}^{h} (\exists_1^{\cap} y_j) q_j$$

(on the basis of Prop. 3.2);

(6) $$A^\Delta(x) \equiv (\exists y_{h+1}, \ldots, y_{h+k})\Big[A^*(x) \bigwedge_{i=1}^{k} (\exists_1^{\cap y_{h+i}} x_i) r_i(x)'\Big]$$

(by Def. 2);

$$(7) \qquad A^{\Delta}(\Delta^*) \equiv (\exists y_{h+1}, \dots, y_{h+k})[A^*(\Delta^*)p]$$

(by steps (6) and (3));

$$(8) \qquad A(\Delta)' \equiv (\exists y_{h+1}, \dots, y_{h+k})(\exists Y)\Big[A^*(\Delta^*) \bigwedge_{i=1}^{k} (\exists_1^{\cap v_{h+i}} x_i) r_i(\Delta)' q\Big]$$

(by Def. 2);

$$(9) \qquad (\exists Y)\Big[\bigwedge_{i=1}^{k} p_i q\Big] \equiv \bigwedge_{i=1}^{k} (\exists Y)(p_i q)$$

(by Prop. 3.3 and (5));

$$(10) \qquad (\exists Y)(p_i q) \equiv (\exists_1^{\cap v_{h+i}} x_i)(\exists Y)[r_i^{\Delta}(\Delta^*)q]$$

(by Prop. 3.3, taking into account the remark at the end);

$$(11) \qquad \bigwedge_{i=1}^{k} (\exists_1^{\cap v_{h+i}} x_i) r_i(\Delta)' \equiv (\exists Y)(pq)$$

(by steps (4), (10) and (9));

$$(12) \qquad (\exists Y)[A^*(\Delta^*)(\exists Y)(pq)q] \equiv (\exists Y)[A^*(\Delta^*)pq]$$

(by iterated use of Prop. 3.3 and (5));

$$(13) \qquad A(\Delta)' \equiv (\exists Y)\{(\exists y_{h+1}, \dots, y_{h+k})[A^*(\Delta^*)p]q\}$$

(by steps (8), (11) and (12)).

From (13) and (7), (2) is fast deduced.

Remark. If x_i occurs freely in $(\exists_1^{\cap v_j} z_j)s_j'$ for some $j \in \{1, \dots, h\}$, then it does so also in Δ; then x cannot occur free in $r_i(x)$, because otherwise the replacement of x with Δ in all of its free occurrences in $(\imath x_i)r_i(x)$, hence in $A(x)$, would cause confusion of bound variables against the assumption.

From the preceding lemma we deduce three corollaries which complete the proof of Prop. 8.

Corollary 4.1. If p is an instance of the axiom scheme

$$(N)(x)A(x) \supset A(\Delta) \qquad \text{(cf. As. 3.8)}$$

of K, then

$$\vdash_{\overline{K'}} p' .$$

Proof. Keeping the notations used in Lem. 1 we list the main steps of the demonstration

(1) $$(x)A^{\Delta}(x) \supset A^{\Delta}(\Delta^*)$$

(2) $$(x)A^{\Delta}(x) \supset (y_1) \dots (y_h)A^{\Delta}(\Delta^*)$$

(3) $$(\exists y_1, \dots, y_h) \bigwedge_{j=1}^{h} (\exists_1^{\cap v_j} z_j)s'_j \qquad \text{(cf. Cor. 3.1)}$$

(4) $$(x)A^{\Delta}(x) \supset (\exists y_1, \dots, y_h)\Big[A^{\Delta}(\Delta^*) \bigwedge_{j=1}^{h} (\exists_1^{\cap v_j} z_j)s'_j\Big]$$

(5) $$(x)A(x)' \supset A(\Delta)' \qquad \text{(by Lem. 1)} .$$

The following corollary holds trivially:

Corollary 4.2. If p is an instance of the axiom scheme

$$(N)x =^{\cap} y \supset [A(x) \equiv A(y)] \qquad \text{(cf. As. 3.12)}$$

of K, then

$$\vdash_{\overline{K'}} p' .$$

Corollary 4.3. If p is an instance of the axiom scheme

$$(N)(\exists z)s\ (z)\big(s \supset z = (\imath z)s\big) \vee \sim (\exists_1 z)s\ \mathrm{Impr}_t\big((\imath z)s\big)$$

of K (cf. As. 3.18), then

$$\vdash_{\overline{K'}} p' .$$

Proof. To conform to the notations employed in Lem. 1 let x be a variable of type t distinct from z, and set

$$A(x) \equiv_D N[(\exists z)s\ (z)(s \supset z = x) \vee \sim (\exists_1 z)s\,\mathrm{Impr}_t(x)]\,, \qquad \varDelta =_D (\imath z)s\,,$$

so that $\varDelta^*$ turns out to be a certain variable y.

We easily verify that $\vdash_{\overline{K'}} \mathrm{Impr}_t(x)' \equiv \mathrm{Impr}_t(x)$, hence $\vdash_{\overline{K'}} A^{\varDelta}(x) \equiv \equiv (\exists_1^{\cap x} z)s^{\varDelta}$, so that by use of Lem. 1 we have $\vdash_{\overline{K'}} A(\varDelta)' \equiv (\exists y)(\exists_1^{\cap y} z)s^{\varDelta}$. By Cor. 3.1, we conclude that $\vdash_{\overline{K'}} A(\varDelta)'$, which is the thesis.

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