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On the Algebraic Compactness of Some Complete Modules.

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Introduction.

Let R be a commutative ring with unit.

An R-module M is algebraically compact if every finitely soluble family of linear equations over R in M has a simultaneous solution.

If R is a noetherian ring and Ω is the set of the maximal ideals of R, we can define over any R-module M the Ω-adic topology, by taking as a base of neighborhoods of 0 the submodels IM, where I is a finite intersection of powers of the maximal ideals.

If R is any ring and M is any R-module, we can define on M the R-topology, by taking as a base of neighborhoods of 0 the submodules rM with $0 \neq r \in R$.

Warfield [W_1] has proved that any algebraically compact R-module is complete in the Ω-adic topology, if R is a noetherian ring, and in the R-topology if R is any ring.

Moreover, Warfield has raised the problem to see if any complete Hausdorff module over a noetherian ring is necessarily algebraically compact.

In this work we answer in the affirmative to the question posed by Warfield and we characterize the neotherian rings R such that

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any R-module which is complete and Hausdorff in the R-topology is algebraically compact.

1. Complete modules in the Ω-adic topology.

Let R be a noetherian commutative ring with unit, M a topological R-module equipped with the Ω-adic topology.

We denote by $\hat{M}$ the Ω-adic completion of M.

(« Complete module » means « Hausdorff complete module »).

The purpose of this section is to prove that, for any R-module M, $\hat{M}$ is an algebraic compact R-module.

First of all, we recall that M is topologically isomorphic to the product $\prod_{\mathfrak{m}\in\Omega} \hat{M}_{\mathfrak{m}}$ where $\hat{M}_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-adic completion of the localization of M at $\mathfrak{m}$, so since the class of algebraically compact modules is closed under direct products, we shall have to settle the problem only with respect to the $\mathfrak{m}$-adic completion of a module.

By a suitable definition of pure submodule, Warfield has proved that the class of algebraically compact modules, coincides with the class of pure-injective modules. Therefore we now recall the principal definitions concerning the concept of purity and pure-injectivity.

DEFINITION 1. *Let R be a ring, $\mathcal{S}$ a class of R-modules.*

N is an $\mathcal{S}$-pure submodule of an R-module M, if every element of $\mathcal{S}$ is projective for the exact sequence:

$$0 \to N \to M \to M/N \to 0 .$$

An equivalent definition to definition 1 is the following: ([W$_2$])

DEFINITION 1'. *Let $\mathcal{S}$ be a class of R-modules.*

N is an $\mathcal{S}$-pure submodule of an R-module M if it is a direct summand of any module H such that: a) $N \leqslant H \leqslant M$, b) $H/N \in \mathcal{S}$.

Walker ([W$_2$]) has introduced the notion of $\mathcal{S}$-copure submodule by dualizing the definition 1' in the following way:

DEFINITION 2. *A submodule N of an R-module M is $\mathcal{S}$-copure in M, if for every submodule H of N such that $N/H \in \mathcal{S}$, N/H is a summand of M/H.*

We are interesting with a particular class of modules, namely we consider the class $\mathcal{F}$ of all finitely presented modules, so that we give the following definition:

DEFINITION 3. *A submodule of a module M is pure (copure) in M if it is $\mathcal{F}$-pure ($\mathcal{F}$-copure).*

Moreover we say that a module is pure-injective if it is injective for any pure exact sequence.

REMARK. If R is a noetherian ring; the class $\mathcal{F}$ is the class of all finitely generated modules.

LEMMA 1. *Let R be an artinian local ring.*

If N is a pure submodule of M, then it is also copure in M.

PROOF. Let H be a submodule of N such that $N/H \in \mathcal{F}$.

We have to prove that N/H is a summand of M/H.

Since N is pure in M, N/H is pure in M/H ([W$_2$], Theor. 2.1); moreover, since the maximal ideal of R is nilpotent, the Ω-adic topology over any R-module is the discrete topology, so N/H is a finitely generated and a complete module in the Ω-adic topology.

Then, by Theor. 3 of [W$_1$], N/H is pure-injective and therefore it is a summand of M/H. //

LEMMA 2. *Let R be a ring satisfying the hypotheses of the preceding lemma. If N is a copure submodule of a module M, then it is a summand of M.*

PROOF. Let:

$$\mathcal{F}(N) = \{N_\alpha \leqslant N : N/N_\alpha \in \mathcal{F}\}$$

$$N^{\mathcal{F}} = \bigcap_{N_\alpha \in \mathcal{F}(N)} N_\alpha$$

For each $0 \neq x \in N$, let H_x be a submodule of N maximal with respect to the property of not containing x.

It is easy to verify that the submodule generated by $x + H_x$ is simple and essential in N/H_x.

Therefore the injective envelope $E(N/H_x)$ of N/H_x is isomorphic to $E(R/\mathfrak{m}) = E$. Now, by [M] Theor. 3.4 and 3.11, $E = \bigcup_k E_k$ where E_k is an increasing sequence of finitely generate submodules of E with $E_k = \{x \in E : \mathfrak{m}^k x = 0\}$. Therefore, since $\mathfrak{m}^h = 0$, for a convenient integer h, we have $E = E_h$; then N/H_x is finitely generated since it is a submodule of the noetherian module E and then we have:

$$0 = \bigcap_{0 \neq x \in N} H_x \supseteq \bigcap_{\alpha} N_\alpha = N^{\mathcal{F}} .$$

Now, ([W$_2$], Corollary 2.9′) the group Copext (L, N) of the copure extensions of N by a generical module L, is the image of the homomorphism $f\colon \mathrm{Ext}(L, N^{\mathcal{F}}) \to \mathrm{Ext}(L, N)$ induced by the inclusion $N^{\mathcal{F}} \to N$. Then, since $N^{\mathcal{F}} = 0$, N is a summand of every module in which it is a copure submodule. //

THEOREM 1. *Let R be a noetherian ring, $\mathfrak{m}$ an element of Ω and M an R-module. For every $k \in \mathbf{N}$, $M/\mathfrak{m}^k M$ is an algebraically compact R-module.*

PROOF. $M/\mathfrak{m}^k M$ is an $R/\mathfrak{m}^k$-module, then by lemmas 1 and 2, it is an algebraically compact $R/\mathfrak{m}^k$-module. Moreover we can easily deduce from the definition of algebraically compactness, that $M/\mathfrak{m}^k M$ is also an R-module algebraically compact //.

Let M be an R-module, we denote by $B(M)$ the Bohr compactification of M, that is:

$$B(M) = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Hom}_{\mathbf{Z}}(M, K), K)$$

where K is the circle group ([W$_1$], § 3).

Let ω_M be the natural homomorphism of M in $B(M)$; then $\omega_M(M) = \tilde{M}$ is canonically isomorphic to M.

Warfield ([W$_1$], § 3), has proved that $B(M)$ is a topological compact R-module and that M is a pure (and dense) submodule of $B(M)$.

Now we have the following:

THEOREM 2. *Let R be a noetherian ring, $\mathfrak{m}$ a maximal ideal of R. The $\mathfrak{m}$-adic completion of any R-module M is an algebraically compact R-module.*

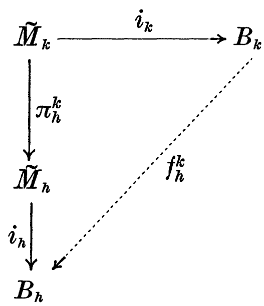
PROOF. Let $M_k = M/\mathfrak{m}^k M$ and π_h^k the natural homomorphisms $\pi_h^k\colon M_k \to M_h$ $(k > h,\ h, k \in \mathbf{N})$; then we have:

$$\hat{M} \approx \varprojlim_k \{M_k, \pi_h^k \quad k > h\}_{k \in \mathbf{N}}$$

Let B_k be the Bohr compactification of M_k for every $k \in \mathbf{N}$, $\tilde{M}_k$ the copy of M_k in B_k and let $\tilde{\pi}_h^k$ be the homomorphisms induced by the π_h^k.

Then M is isomorphic to $\varprojlim \{\tilde{M}_k, \tilde{\pi}_h^k\}$ since ω_k are natural isomorphism for every $k \in \boldsymbol{N}$.

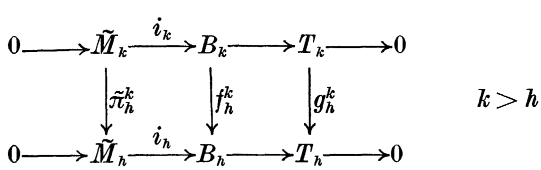
Therefore it will suffices to prove that $\varprojlim \{\tilde{M}_k, \tilde{\pi}_h^k\}$ is algebraically compact. Let's consider the following diagram:



The universal property of B_k, assures the existence of a unique continuous homormorphism f_h^k such that the diagram commutes.

Now, by Theor. 1, $\tilde{M}_k$ is an algebraically compact R-module, so $B_k = \tilde{M}_k \oplus T_k$ for every $k \in \boldsymbol{N}$.

Let us consider the following diagram:



with $f_h^k \circ i_k = i_h \circ \tilde{\pi}_h^k$. Since the rows are exact, there is a unique homomorphisms g_h^k such that the diagram commutes.

By the unicity of the g_h^k, the system $\{T_k;\, g_h^k \;\; k > h\}$ is an inverse system. Then we have:

$$\varprojlim_k \{\tilde{M}_k;\, \tilde{\pi}_h^k\} \oplus \varprojlim_k \{T_k;\, g_h^k\} \cong \varprojlim_k \{B_k;\, \tilde{\pi}_h^k \oplus g_h^k\} .$$

Now, $\varprojlim B_k$ is a compact module in the topology induced by the product topology of the B_k, then by [W₁] Theor. 2, $\varprojlim M_k$ is algebraically compact. //

2. Complete modules in the R-topology.

Let R be a noetherian commutative ring with unit.

The purpose of this section is to characterize the rings R such that any R-module which is complete and T_2 in the R-topology is algebraically compact.

First of all we consider the case in which the Ω-adic topology on R is the discrete topology.

This hypothesis implies that every R-module is discrete in the Ω-adic topology and so, any R-module is algebraically compact.

In the general case, that is, when the open ideals in the Ω-adic topology on R are always non zero, then the R-topology over any R-module M is finer than the Ω-adic topology.

Now, since R is a noetherian ring, any ideal rR, with $r \neq 0$, contains a finite intersection of powers of prime non zero ideals of R.

Therefore, if R has the following property:

(P) every non zero prime ideal of R is maximal

the Ω-adic topology and the R-topology coincides over any R-module.

Then the results contained in Section 1, allow us to say that (P) is a sufficient condition on R to insure that any complete and T_2 module in the R-topology is algebraically ompact.

(The converse has been stated by Warfield, as we have just noted).

Now, we shall prove that (P) is also a necessary condition.

Let us suppose that R has a non zero and non maximal prime ideal $\mathfrak{I}$.

Let $\mathfrak{m}$ be a maximal ideal of R containing $\mathfrak{I}$ and let T be the localization of $R/\mathfrak{I}$ at $\mathfrak{m}/\mathfrak{I}$; we consider the R-module $A = T[x]$ where x is a trascendental element over T.

Clearly the R-topology on A is discrete, so A is complete in such topology, but we shall prove that A is not algebraically compact.

Infact, if A were algebraically compact, Warfield's results would entail the completeness of A in the Ω-adic topology. But now, it is easy to verify that the Ω-adic topology on A is the same as the $\mathfrak{m}$-adic topology, so it suffices to find a nonconvergent Cauchy sequence of elements of A.

We denote by $\mathfrak{n}$ the maximal ideal of T.

The powers of $\mathfrak{n}$ give a strictly decreasing chain of ideals of T, since, by Krull Theorem, $\bigcap_i \mathfrak{n}^i = 0$ and by the hypotheses on $\mathfrak{I}$ we cannot have $\mathfrak{n}^i = 0$ for any $i \in \mathbf{N}$.

Let a_i be an element of $\mathfrak{n}^i \setminus \mathfrak{n}^{i+1}$ for every $i \in \boldsymbol{N}$, and let us consider the following elements of A:

$$s_k = \sum_{i=0}^{k} a_i x_i \qquad k \in \boldsymbol{N}.$$

Now it is easy to prove that $\{s_k\}_k$ is a Cauchy sequence of element of A which cannot converge to any element of A.

BIBLIOGRAPHY

[B] N. BOURBAKI, *Algèbre commutatice* (Ch. 3 and 4), Paris, Hermann.

[M] E. MATLIS, *Injectives modules over noetherian rings*, Pac. J. Math., **8**, No. 3 (1958), pp. 511-528.

[W1] R. B. WARFIELD, *Purity and algebraic compactness for modules*, Pac. J. Math., **28**, No. 3 (1969), pp. 699-719.

[W2] C. P. WALKER, *Relative homological algebra and abelian groups*, Illinois J. Math., **10** (1966), pp. 186-209.

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