Luciano Battaia

A principle involving the variation of the metric tensor in a stationary space-time of general relativity

Rendiconti del Seminario Matematico della Università di Padova, tome 56 (1976), p. 139-145

<http://www.numdam.org/item?id=RSMUP_1976__56__139_0>
A Principle Involving the Variation of the Metric Tensor in a Stationary Space-Time of General Relativity.

LUCIANO BATTAlA (*)

SUMMARY - Within general relativity we introduce a variational principle, involving the variation of the metric tensor in a stationary spacetime, and concerning the equilibrium of an elastic body capable of couple stresses but not of heat conduction.

1. Introduction.

In this work we consider an elastic body C capable of couple stresses but not of heat conduction and we assume absence of electromagnetic phenomena.

Basing ourselves on a certain variational theorem involving the variation of the space-time metric, firstly formulated by Taub in [4] and extended by Schöpf and Bressan to the non-polar and polar cases respectively — cf. [3] and [2]—, we introduce a variational principle concerning the rest of a body C of the type above.

More in detail we prove that if $C_3$ is a certain 3-dimensional region of a stationary spacetime $S_4$, the equilibrium of the body $C$ in the stationary frame ($x$), is physically possible if and only if the functional

$$J = \int_{C_3} (R + 16\pi h \sigma^{-4} \sigma) \sqrt{-\bar{g}} \, dC_3,$$

(*) Indirizzo dell'A.: Seminario Matematico, Università di Padova - Via Belzoni 7 - 35100 Padova.
Lavoro eseguito nell’ambito dei gruppi di ricerca matematica del C.N.R.
is stationary with respect to certain variations of the metric, where $R$ is the bicontracted Riemann tensor, $h$ Cavedish's constant, $c$ the velocity of light in vacuum and $\rho$ the proper actual density of matter energy.

This theorem is an analogue of the relativistic variational principle proved in [1].

2. Preliminaries.

We follow the theory of continuous media in general relativity constructed by Bressan, cf. [2] (cf. also [1] and the references therein).

Let $C$ be a continuous body and $\mathcal{F}$ a process physically possible for $C$ in a space-time $S_4$ of general relativity. We shall consider only regular motions for $C$, e.g. without slidings and splittings; hence $C$ can be regarded as a collection of material points.

By $(x)$ we denote an admissible frame; by $g_{\alpha\beta} = g_{\alpha\beta}(x^\sigma)$ the metric tensor corresponding to $\mathcal{F}$; by $u^\alpha[A^\sigma]$ the four velocity [acceleration] of $C$ at the event point $\xi$.

Then we consider a particular process $\mathcal{F}^*$ physically possible for the universe containing $C$, the world-tube $W^*_C$ of $C$ in $\mathcal{F}^*$ and an admissible frame $(y)$. We call $S^*_y$ the intersection of $W^*_C$ with the hypersurface $y^0 = 0$. We use the co-ordinate $y^L$ of the intersection of $S^*_y$ with the world line of the point $P^*$ of $C$ as $L$-th material co-ordinate $(2)$.

We represent the arbitrary (regular) motion of $C$ in the system of co-ordinates $(x)$ by means of the functions

\begin{equation}
\varphi^\sigma = \hat{\varphi}^\sigma(t, y^1, y^2, y^3),
\end{equation}

where $t$ is an arbitrary time parameter.

If $T_{\cdot\cdot\cdot}$ is a double tensor field associated to the event point $x^\sigma$ and the material point $y^L$, we shall denote by $T_{\cdot\cdot\cdot,\alpha}$ the ordinary partial derivative, by $T_{\cdot\cdot\cdot}^{/\alpha}$ the covariant derivative and by $T_{\cdot\cdot\cdot}^{/\alpha}_{\cdot\cdot\cdot}$ the lagrangian spatial derivative based on the map (1) and introduced by Bressan.

We are interested in an elastic body $C$ capable of couple stress but not of heat conduction and we shall always assume that electromagnetic phenomena are absent.

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(1) Greek and Latin indices run over 0, 1, 2, 3 and 1, 2, 3 respectively.

(2) Capital and lower case latters represent material and space-time indices respectively.
We call $\rho$ the proper actual density of total internal energy, $X^{\alpha\beta}$ the stress tensor, $m^{\alpha\beta\gamma}$ the couple stress tensor and we shall assume as total energy tensor (3)

$$U^{\alpha\beta} = \rho u^\alpha u^\beta + X^{(\alpha\beta)} + 2m^{(\alpha\beta)}/\lambda + 2\nu^{(\alpha} u^{\beta)},$$

where

$$\nu^\alpha = 2m^{(\alpha\sigma)} u_{\sigma}/\lambda.$$

This assumption is proved to be physically acceptable in the case considered here (cf. [2]).

3. A theorem concerning the variation of the metric tensor in $S_4$.

We always consider a body $C$ of the type specified above and we suppose assigned in $S_4$ the motion (1) of $C$ and the metric tensor $g_{\alpha\beta} = g_{\alpha\beta}(x)$ (4).

We take into account a bounded 4-dimensional domain $C_4$ of $S_4$, where the motion of $C$ is of class $C^{(2)}$ and we call $\mathcal{F}C_4$ its boundary oriented outwards. Furthermore let $\delta g_{\alpha\beta}$ be an arbitrary variation of $g_{\alpha\beta}$, of class $C^{(2)}$ in $C_4$ and such that

$$\delta g_{\alpha\beta} = 0 = \delta g_{\alpha\beta}, \quad \text{on } \mathcal{F}C_4.$$

Consider the functional

$$I = \int_{C_4} (R + 16\pi\hbar c^{-4} \rho) \sqrt{-g} \, dC_4.$$

In [2] it is proved that for every variation $\delta g_{\alpha\beta}$ of the aforementioned type we have

$$\delta \int_{C_4} (R + 16\pi\hbar c^{-4} \rho) \sqrt{-g} \, dC_4 = -\int_{C_4} (A^{\alpha\beta} + 8\pi\hbar c^{-4} U^{\alpha\beta}) \delta g_{\alpha\beta} \sqrt{-g} \, dC_4.$$

(3) We use the notations $2T^{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}$; $2T_{(\alpha\beta)} = T_{\alpha\beta} - T_{\beta\alpha}$.

(4) For the constitutive equations of an elastic body $C$ capable of couple stresses in the absence of heat conduction an electromagnetic phenomena see [1].
4. A variational principle concerning equilibrium in a stationary frame.

Let now $S_4$ be stationary and $(x)$ a stationary frame. The metric tensor, that we always consider as assigned in $S_4$, satisfies

\begin{equation}
 g_{\alpha\beta,0} = 0.
\end{equation}

We call $C_3$ the intersection of the world tube $W_C$ of $C$ with the hypersurface $x_0 = 0$.

We identify the arbitrary parameter in the equations (1) of the motion with $x^0$ and denote by $x^r = \chi^r(y^\ell)$ the configuration of $C$ in $C_3$. We consider the following motion of $C$:

\begin{equation}
\begin{cases}
 x^0 = t \\
 x^r = x^r(t, y^\ell) = \chi^r(y^\ell),
\end{cases}
\end{equation}

hence $C$ is in equilibrium with respect to $(x)$.

Consider an arbitrary variation $\delta g_{\alpha\beta} = \delta g_{\alpha\beta}(x^1, x^2, x^3)$ of $g_{\alpha\beta}$ on $C_3$, of class $C^{(2)}$ and such that

\begin{equation}
 \delta g_{\alpha\beta} = 0 = \delta g_{\alpha\beta,\tau} \quad \text{on } \mathcal{F} C_3 \quad (\alpha, \beta = 0, 1, 2, 3).\end{equation}

Consider the functional

\begin{equation}
 J = \int_{C_3} \left( R + 16\pi\hbar c^{-4} \varrho \right) \sqrt{-g} \, dC_3.
\end{equation}

We shall prove that the rest (7) of the body $C$, with respect to the stationary frame $(x)$, is physically possible if and only if

\begin{equation}
 \delta J = 0
\end{equation}

for every variation of $g_{\alpha\beta}$ of the aforementioned type.

Let $a$ be a real positive number. We consider the following subsets of $W_C$

\begin{align*}
 C_4 &= \{ P \in W_C | |x^0| < a + 1 \}, \\
 C_i^a &= \{ P \in W_C | x^0 < a \}, \\
 C_i^+ &= \{ P \in W_C | a < x^0 < a + 1 \}, \\
 C_i^- &= \{ P \in W_C | (a + 1) < x^0 < -a \},
\end{align*}
where by $x^\alpha$ we mean the co-ordinates of the point $P$ of $S_4$.

Let then $\varphi(\xi)$ be an arbitrary function of the real variable $\xi$, of class $C^{(2)}$ in $[0, 1]$ and such that

$$
\begin{align*}
\varphi(1) = \varphi'(1) = \varphi'(0) = \varphi''(0) &= 0 \\
\varphi(0) &= 1 \\
\int_0^1 \varphi(\xi) \, d\xi &= 0 .
\end{align*}
$$

(11)

Consider the following variation $\delta g_{\alpha\beta}(x_0, x^1, x^2, x^3)$ in $C_4$

$$
\delta g_{\alpha\beta} = \begin{cases}
\delta g_{\alpha\beta}(x^1, x^2, x^3) & \text{in } C_4^a \\
\varphi(x_0 - a) \, \delta g_{\alpha\beta} & \text{in } C_4^+ \\
\varphi(-x_0 - a) \, \delta g_{\alpha\beta} & \text{in } C_4^- .
\end{cases}
$$

(12)

On the basis of (11) this variation is of class $C^{(2)}$ in $C_4$ and satisfies the conditions

$$
\delta g_{\alpha\beta} = 0 = \delta g_{\alpha\beta,y} \quad \text{on } \mathcal{F}C_4 .
$$

(13)

Hence the variational theorem enunciated in the previous paragraph can be applied:

$$
\delta \int_{C_4}(R + 16\pi \hbar c^{-4} \phi) \sqrt{-g} \, dC_4 = -\int_{C_4}(A_{\alpha\beta} + 8\pi \hbar c^{-4} \mathcal{U}_{\alpha\beta}) \delta g_{\alpha\beta} \sqrt{-g} \, dC_4 .
$$

(14)

The stationarity of spacetime and the equations (7) of equilibrium imply that $g_{\alpha\beta}, R, \mathcal{U}_{\alpha\beta,\gamma}$ do not depend on $x^\alpha$.

We have

$$
\delta \int_{C_4^+} R \sqrt{-g} \, dC_4 = \int_{C_4^+} \frac{\partial R}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} \, dC_4 = \int_{C_4^+} \varphi(x^0 - a) \, \frac{\partial R}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} \, dC_4 =
$$

$$
= \int_{C_4^a} \frac{\partial R}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} \, dC_3 \int_a^{a+1} \varphi(x^0 - a) \, dx^0 = 0 ,
$$
and analogously
\[ \delta \int_{\mathbb{R}^4} \sqrt{-g} \, dC_4 = 0. \]

Furthermore in the same way we prove that
\[ \delta \int_{\mathbb{R}^4} R \sqrt{-g} \, dC_4 = 0 = \delta \int_{\mathbb{R}^4} \sqrt{-g} \, dC_4. \]

Hence
\[ \delta \int_{\mathbb{R}^4} (R + 16\pi \rho_4 \phi) \sqrt{-g} \, dC_4 = 0 = \delta \int_{\mathbb{R}^4} (R + 16\pi \rho_4 \phi) \sqrt{-g} \, dC_4. \]

Furthermore
\[ \int_{\mathbb{R}^4} (R + 16\pi \rho_4 \phi) \sqrt{-g} \, dC_4 = \]
\[ = \int_{\mathbb{R}^4} dx^a (R + 16\pi \rho_4 \phi) \sqrt{-g} \, dC_4 = 2a \int_{\mathbb{R}^4} (R + 16\pi \rho_4 \phi) \sqrt{-g} \, dC_4. \]

From (15) and (16) we have
\[ \delta \int_{\mathbb{R}^4} (R + 16\pi \rho_4 \phi) \sqrt{-g} \, dC_4 = 2a \delta \int_{\mathbb{R}^4} (R + 16\pi \rho_4 \phi) \sqrt{-g} \, dC_4 \]
for the variation \( \delta g_{ab} \) and \( \delta g_{ab} \) above.

We also have
\[ \int_{\mathbb{R}^4} (A^{ab} + 8\pi \rho_4 \psi \xi^a \xi^b) \delta g_{ab} \sqrt{-g} \, dC_4 = \]
\[ = \int_{\mathbb{R}^4} (A^{ab} + 8\pi \rho_4 \psi \xi^a \xi^b) \phi (x^a - a) \delta g_{ab} \sqrt{-g} \, dC_4 = \]
\[ = \int_{\mathbb{R}^4} (A^{ab} + 8\pi \rho_4 \psi \xi^a \xi^b) \delta g_{ab} \sqrt{-g} \, dC_4 \int_{\mathbb{R}^4} \phi (x^a - a) \, dx^a = 0. \]

and analogously
\[ \int_{\mathbb{R}^4} (A^{ab} + 8\pi \rho_4 \psi \xi^a \xi^b) \delta g_{ab} \sqrt{-g} \, dC_4 = 0. \]
Furthermore

\[ \int_{C_0^3} (A^\alpha{}_{\beta} + 8\pi \rho e^{-4\varphi} U^\alpha{}_{\beta}) \delta g_{\alpha\beta} \sqrt{-g} \, dC_3 = \]

\[ = \int_{C_0^3} (A^\alpha{}_{\beta} + 8\pi \rho e^{-4\varphi} U^\alpha{}_{\beta}) \delta g_{\alpha\beta} \sqrt{-g} \, dC_3 \int_{a}^{b} \, dx^0 = \]

\[ = 2a \int_{C_0^3} (A^\alpha{}_{\beta} + 8\pi \rho e^{-4\varphi} U^\alpha{}_{\beta}) \delta g_{\alpha\beta} \sqrt{-g} \, dC_3. \]

From (14), (17), (18), (19), (20) we deduce

\[ \frac{\delta}{\delta \varphi} \int_{C_0^3} (R + 16\pi \rho e^{-4\varphi} \rho) \sqrt{-g} \, dC_3 = -\int_{C_0^3} (A^\alpha{}_{\beta} + 8\pi \rho e^{-4\varphi} U^\alpha{}_{\beta}) \delta g_{\alpha\beta} \sqrt{-g} \, dC_3 \]

for the variations \( \delta g_{\alpha\beta} \) specified above.

From (21) it follows that the variational condition \( \delta J = 0 \) is equivalent to the validity, in \( C_0 \), of the gravitational equations for \( C: A^\alpha{}_{\beta} + 8\pi \rho e^{-4\varphi} U^\alpha{}_{\beta} = 0 (x, \beta = 0, 1, 2, 3) \). If we remember that \( \rho, R, A^\alpha{}_{\beta}, U^\alpha{}_{\beta} \) do not depend on \( x^0 \), we can conclude that the variational condition \( \delta J = 0 \) is equivalent to the validity of the gravitational equations for \( C \) in the whole world tube \( W_C \). This proves the theorem.

REFERENCES


Manoscritto pervenuto in Redazione l’8 aprile 1976.