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Abelian groups with many automorphisms


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1. - The results.

Given a group $G$, its automorphism group $\text{Aut } G$ acts as a group of permutations on the set of all subgroups of $G$. Problem 91 of Fuchs [2] proposes to determine the equivalence classes (orbits) of subgroups of $G$ determined by this action of $\text{Aut } G$. In this note we characterize the class of all abelian groups $G$ for which these equivalence classes are precisely the isomorphism classes of subgroups of $G$. These are the $T_1$-groups in the sense of Polimeni [4]: $G$ is a $T_1$-group if, given two subgroups $H$ and $K$ of $G$ which are isomorphic, there exists $\alpha$ in $\text{Aut } G$ such that $\alpha(H) = K$. This definition is weaker than the definition of a $T'_1$-group: $G$ is a $T'_1$-group if every isomorphism between any two of its subgroups can be extended to an automorphism of $G$. We shall prove the following theorem which shows among other things that every abelian $T_1$-group actually is a $T'_1$-group.

(1.1) THEOREM. For an abelian group $G$ the following statements are equivalent.

(i) $\text{Aut } G$ operates transitively on the isomorphism classes of subgroups of $G$ (i.e. $G$ is $T_1$).

(ii) Either $G$ is a torsion group all of whose $p$-components are homo-cocyclic of finite rank, or $G$ is a divisible group whose torsion-free rank and $p$-ranks are all finite.

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(iii) $G$ is a characteristic subgroup of a divisible group whose torsion-free rank and $p$-ranks are all finite.

(iv) Every isomorphism between any two subgroups of $G$ is induced by some automorphism of $G$ (i.e., $G$ is $T'$).

Let $\mathcal{A}$ be the class of all abelian groups $G$ such that every automorphism of any subgroup of $G$ can be extended to an automorphism of $G$. Clearly every abelian $T'_1$-group (and hence, in view of (1.1), every abelian $T_1$-group) is contained in $\mathcal{A}$. In [3] Mishina has shown that an abelian torsion group belongs to $\mathcal{A}$ if and only if all of its $p$-components are homo-cocyclic. This result together with (1.1) implies the following theorem.

(1.2) Theorem. Let $G$ be an abelian torsion group all of whose $p$-ranks are finite. Then the following conditions are equivalent.

(i) $\text{Aut } G$ operates transitively on the isomorphism classes of subgroups of $G$.

(ii) All $p$-components of $G$ are homo-cocyclic.

(iii) $G$ is a characteristic subgroup of a divisible torsion group.

(iv) Every automorphism of any subgroup of $G$ can be extended to an automorphism of $G$.

(v) Every isomorphism between any two subgroups of $G$ can be extended to an automorphism of $G$.

2. – The proof.

All groups, except groups of automorphisms, are assumed to be abelian and the notation and terminology will be that of Fuchs [1, 2]. We will make use of the following auxiliary results. Some proofs are left to the reader.

(2.1) Lemma. Characteristic subgroups of $T_1$-groups are $T_1$-groups.

(2.2) Lemma. Characteristic subgroups of $T'_1$-groups are $T'_1$-groups.

(2.3) Lemma. If $G$ is a $T_1$-group, then $G$ is not isomorphic to a proper subgroup of itself.

(2.4) Corollary. If $G$ is a $T_1$-group and $G = A \oplus \bigoplus_{i \in I} B_i$, where $0 \neq B_i \cong B_j$, for all $i, j$ in $I$, then $I$ is finite.
(2.5) Proposition. Let $G$ be a $T_1$-group and let $x, y \in G$ such that $o(x) = o(y)$. If $o(x) = \infty$ or $G$ is a $p$-group, then there exists $\alpha$ in $\text{Aut} \ G$ such that $\alpha(x) = y$.

Proof. Since $\langle x \rangle \cong \langle y \rangle$, there exists $\beta \in \text{Aut} \ G$ such that
\[ \langle y \rangle = \beta(\langle x \rangle) = \langle \beta(x) \rangle \]
and consequently $y$ generates $\langle \beta(x) \rangle$. If $o(x)$ is infinite, $y = k \cdot \beta(x)$, where $k = \pm 1$; if $G$ is a $p$-group, then $y = k \cdot \beta(x)$, for some integer $k$ relatively prime to $p$. In either case, the mapping $\gamma$ such that $g \mapsto kg$, for all $g \in G$, is an automorphism of $G$. Hence $\alpha = \gamma \circ \beta \in \text{Aut} \ G$ and
\[ \alpha(x) = (\gamma \circ \beta)(x) = \gamma(\beta(x)) = k \cdot \beta(x) = y \]
as desired.

(2.6) Corollary. Let $G$ be a $T_1$-group and let $x, y \in G$ such that $o(x) = o(y)$. If $o(x) = \infty$ or $G$ is a primary group, then, for all primes $p$, $h_p(x) = h_p(y)$.

A group is called homo-cocyclic if it is the direct sum of pairwise isomorphic cocyclic groups (cf. [1], p. 16).

(2.7) Proposition. If $G$ is a $p$-primary $T_1$-group, then $G$ is homo-cocyclic of finite rank.

Proof. Since $G[p]$ is a characteristic subgroup of $G$, (2.1) and (2.4) imply that $G[p]$ is finite and $G$ has finite rank $k$. By (2.6) any two non-zero elements in $G[p]$ have equal height in $G$. Hence $G = \bigoplus_{i=1}^{k} Z(p^n)$, for some $1 \leq n < \infty$, as claimed.

(2.8) Corollary. If $G$ is a torsion $T_1$-group, then every $p$-component of $G$ is homo-cocyclic of finite rank.

(2.9) Proposition. If $G$ is a non-torsion $T_1$-group, then $G$ is a divisible group whose torsion-free and $p$-ranks are all finite.

Proof. In view of (2.4) and the structure theorem for divisible groups (cf. [1], p. 104) it suffices to show that $G$ is divisible, which is equivalent to $h_p(x) = \infty$, for all $x \in G$, and all primes $p$. Let $x \in G$ and let $p$ be a prime. If $o(x) = \infty$, then, for all integers $n \geq 1$,
\[ o(x) = o(p^n x) \quad \text{and} \quad h_p(x) = h_p(p^n x) \geq n \]
by (2.6). Hence, \( h_p(x) = \infty \) whenever \( x \in G \) has infinite order. Suppose \( o(x) < \infty \). Since \( G \) is not torsion there exists \( y \in G \) such that \( o(y) = \infty \). Hence

\[
o(x + y) = \infty \quad \text{and} \quad h_p(x + y) = h_p(y) = \infty.
\]

By ([1], p. 6, exercise 11) \( h_p(x) \) cannot be finite and \( h_p(x) = \infty \) as claimed.

(2.10) **Proposition.** If \( D \) is a divisible group whose torsion-free rank and \( p \)-ranks are all finite, then \( D \) is a \( T_1 \)-group.

**Proof.** Let \( \varphi: A_1 \rightarrow A_2 \) be an isomorphism between the two subgroups \( A_1 \) and \( A_2 \) of \( D \). Let \( \overline{D}_i \) be the divisible hull of \( A_i \) in \( D \), \( i = 1, 2 \). Since \( \overline{D}_1 > A_2 \) is injective, there exists a homomorphism \( \sigma: \overline{D}_1 \rightarrow \overline{D}_2 \) such that \( \sigma|A_1 = \varphi \). The fact that \( \overline{D}_1 \) is an essential extension of \( A_1 \) and \( \varphi \) is monic implies \( \sigma \) is monic (cf. [1], p. 106 f). Also, \( \sigma \) is epic since \( \sigma(\overline{D}_1) \) is divisible and

\[
A_2 = \varphi(A_1) = \sigma(A_1) < \sigma(\overline{D}_1) < \overline{D}_2.
\]

Hence \( \sigma: \overline{D}_1 \rightarrow \overline{D}_2 \) is an isomorphism extending \( \varphi \). There exists \( D_1 \leq D \) such that

\[
\overline{D}_1 \oplus D_1 = D = \overline{D}_2 \oplus D_2.
\]

Since the torsion-free rank and all \( p \)-ranks of \( D \) are finite, \( \overline{D}_1 \simeq \overline{D}_2 \) implies \( D_1 \simeq D_2 \). Let \( \tau: D_1 \rightarrow D_2 \) be an isomorphism and define \( \alpha: D \rightarrow D \) by \( \alpha|\overline{D}_1 = \sigma \) and \( \alpha|D_i = \tau \). Then \( \alpha \in \text{Aut} \ D \) and \( \alpha|A_1 = \sigma|A_1 = \varphi \), completing the proof.

**Proof of (1.1).** The proof is cyclic. (ii) follows from (i) by (2.8) and (2.9). Assume the validity of (ii). Clearly, we may assume that \( G \) is torsion. If the \( p \)-component \( G_p \) of \( G \) is not divisible, then \( G_p = \bigoplus_{i=1}^{k} Z(p^n) \), for some integer \( n \) and \( G_p = D_p[p^n] \), where \( D_p = \bigoplus_{i=1}^{k} Z(p^n) \). Hence, for all \( p \), \( G_p \) is a characteristic subgroup of a divisible \( p \)-group \( D_p \) of finite rank, and \( G = \bigoplus_p G_p \) is characteristic in \( D = \bigoplus_p D_p \), as stated in (iii). Using (2.2) and (2.10), (iv) follows from (iii). The last implication is trivial.
REFERENCES


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