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An Asymptotic Result for Weak Differential Inequalities.

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Introduction.

In this paper we present in weak form a result concerning differential inequalities which, for strong solutions, was proved in [1], [3], [4]. The functions here involved need not be differentiable and they are not supposed to belong to the domain of the given unbounded operator.

§ 1. – Let be H a Hilbert space and A, a linear closed operator, $\mathfrak{D}(A) \subset H \to H$, with dense domain in H.

Let be A^* the adjoint operator to A, defined on the dense set

$$\mathfrak{D}(A^*) = \{h \in H, (Ak, h) = (k, h^*)\} \qquad \forall k \in \mathfrak{D}(A),$$

through formula $A^*h = h^*$, so that $(Ah, k) = (h, A^*k)$, $\forall h \in \mathfrak{D}(A)$ and $k \in \mathfrak{D}(A^*)$.

Define now a natural class of vector-valued test-functions $K_{A^*}(0, \infty)$, consisting of continuously differentiable functions $0 < t < \infty \to H$, $\phi(t)$, having compact support in the open interval $(0, \infty)$, such that $\phi(t) \in \mathfrak{D}(A^*)$, $\forall t \in (0, \infty)$ and $(A^*\phi)(t)$ is continuous, $0 < t < \infty \to H$.

Our aim is to demonstrate the following:

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THEOREM. *Let us assume that $u(t)$ and $f(t)$ are strongly continuous functions, $0 \leqslant t < \infty \to H$, related through the integral identity:*

$$\int_0^\infty \big(u(t), \varphi'(t) + (A^*\varphi)(t)\big)\,dt = -\int_0^\infty \big(f(t), \varphi(t)\big)\,dt \tag{1.1}$$

for any $\varphi(t) \in K_{A^*}(0, \infty)$.

Assume also that on a sequence of vertical lines in the complex plane: $\operatorname{Re}\lambda = \sigma_n \to -\infty$ as $n \to \infty$, the resolvent operator

$$(\sigma_n + i\tau - A)^{-1} \in \mathfrak{L}(H; H) \qquad \text{for } n = 1, 2, \ldots, -\infty < \tau < \infty\,,$$

and verifies an estimate

$$\|(\sigma_n + i\tau - A)^{-1}\| \leqslant M\,, \qquad n = 1, 2, \ldots, -\infty < \tau < \infty\,.$$

In these conditions, if $\|f(t)\| \leqslant \phi(t)\|u(t)\|$, $0 \leqslant t < \infty$, where $\phi(t) \leqslant c < 1/M$, $0 \leqslant t < \infty$ and if $\sup\limits_{t \geqslant 0} \exp[-at]\,\|u(t)\| < \infty$ for any real number a, it follows that $u(t) = \theta$, $\forall t \geqslant 0$.

We shall start with following

LEMMA. *Let be $\zeta(t)$ a scalar-valued continuously differentiable function defined as:*

$$\zeta(t) = 0 \quad \text{for } t \leqslant 0\,, \qquad \zeta(t) = 1 \quad \text{for } t \geqslant t_0 > 0\,,$$

increasing between $t = 0$ and $t = t_0$.

Let be $v(t) = \exp[at]\zeta(t)u(t)$ for $t \geqslant 0$, $v(t) = \theta$ for $t < 0$, where a is a real number. Then the integral identity:

$$\int_{-\infty}^{+\infty} \big(v(t), \psi'(t) + (A^*\psi)(t)\big)\,dt = \tag{1.2}$$

$$= -\int_{-\infty}^{+\infty} \big(\exp[at]\zeta(t)f(t) + \exp[at]\zeta'(t)u(t) + av(t), \psi(t)\big)\,dt$$

is verified, $\forall \psi \in K_{A^*}(-\infty, \infty)$.

Here, $K_{A^*}(-\infty,\infty)$ has a similar definition to $K_{A^*}(0,\infty)$; precisely, it consists of continuously differentiable functions $-\infty<t<\infty\to H$ $\psi(t)$, having compact support, the range being in $\mathfrak{D}(A^*)$, and $(A^*\psi)(t)$ being H-continuous function.

Proof of Lemma. We have

$$(1.3)\qquad \int_{-\infty}^{+\infty}\big(v(t),\,\psi'(t)\big)\,dt=\int_0^\infty(\exp[at]\zeta(t)u(t),\,\psi'(t))\,dt=$$
$$=\int_0^\infty\big(u(t),\,\exp[at]\zeta(t)\psi'(t)\big)\,dt\,.$$

Write now the identity

$$(1.4)\qquad \exp[at]\zeta(t)\psi'(t)=\big(\exp[at]\zeta(t)\psi(t)\big)'-\big(\exp[at]\zeta(t)\big)'\psi(t)\,;$$

hence we get

$$(1.5)\qquad \int_{-\infty}^{+\infty}\big(v(t),\,\psi'(t)\big)\,dt=$$
$$=\int_0^\infty\Big(u(t),\,\big(\exp[at]\zeta(t)\psi(t)\big)'\Big)\,dt-\int_0^\infty\Big(u(t),\,\big(\exp[at]\zeta(t)\big)'\psi(t)\Big)\,dt\,.$$

Denote: $\phi(t)=\exp[at]\zeta(t)\psi(t)$ and take also a scalar-valued function $\nu_\varepsilon(t)$, depending on parameter $\varepsilon>0$, such that:

$$\nu_\varepsilon(t)=0\quad\text{for } 0\leqslant t\leqslant\varepsilon\,,\qquad \nu_\varepsilon(t)=1\quad\text{for } t\geqslant 2\varepsilon\,,$$
$$\nu_\varepsilon(t)\in C^1[0,\infty)\,,\qquad |\nu'_\varepsilon(t)|\leqslant\frac{C}{\varepsilon}\,,\qquad 0\leqslant t<\infty\,.$$

Now, it is easy to see that $\nu_\varepsilon(t)\phi(t)$ belongs to $K_{A^*}(0,\infty)$.

Consequently, using (1.1), we obtain

$$\int_0^\infty\big(u(t),\,(\nu_\varepsilon\phi)'(t)\big)\,dt=$$
$$=-\int_0^\infty\big(u(t),\,A^*(\nu_\varepsilon\phi)(t)\big)\,dt-\int_0^\infty\big(f(t),\,(\nu_\varepsilon\phi)(t)\big)\,dt\,,\qquad\forall\varepsilon>0\,.$$

We get now:

$$\int_0^\infty (u(t),\, (\nu_\varepsilon \phi')(t))\, dt = \int_\varepsilon^{2\varepsilon} (u(t),\, \nu_\varepsilon(t)\phi'(t))\, dt + \int_{2\varepsilon}^\infty (u(t),\, \phi'(t))\, dt\,.$$

Actually we can estimate

$$\left|\int_\varepsilon^{2\varepsilon} (u(t),\, \nu_\varepsilon(t)\,\phi'(t))\, dt\right| \leqslant \varepsilon \sup_{\varepsilon\leqslant t\leqslant 2\varepsilon} \|u(t)\|\,\|\phi'(t)\| C \quad \to 0 \text{as } \varepsilon \to 0\,.$$

Obviously it is:

$$\lim_{\varepsilon\to 0} \int_{2\varepsilon}^\infty (u(t),\, \phi'(t))\, dt = \int_0^\infty (u(t),\, \phi'(t))\, dt\,.$$

Consider also

$$\int_0^\infty (u(t), \nu'_\varepsilon(t)\,\phi(t))\, dt = \int_\varepsilon^{2\varepsilon} (u(t),\, \nu'_\varepsilon(t)\phi(t))\, dt\,.$$

Here we can estimate

$$\left|\int_\varepsilon^{2\varepsilon} (u(t),\, \nu'_\varepsilon(t)\,\phi(t))\, dt\right| < \frac{C}{\varepsilon}\,\varepsilon \sup_{\varepsilon\leqslant t\leqslant 2\varepsilon} \|u(t)\|\,\|\phi(t)\|\,.$$

which $\to 0$ as $\varepsilon \to 0$ because $\phi(0) = \zeta(0)\,\psi(0) = \theta$ and $u(t)$, $\phi(t)$ are continuous, $u(t)$ is bounded near $t = 0$ and $\phi(t) \to \theta$ as $t{\downarrow}0$.

Summing up these results we get

$$\lim_{\varepsilon\downarrow 0} \int_0^\infty (u(t),\, (\nu_\varepsilon \phi)'(t))\, dt = \int_0^\infty (u(t),\, \phi'(t))\, dt\,.$$

Consider also the expression

$$-\int_0^\infty (u(t),\, A^*(\nu_\varepsilon\phi)(t))\, dt = -\int_\varepsilon^{2\varepsilon} (u(t),\, (A^*\phi)(t))\, \nu_\varepsilon(t)\, dt - \int_{2\varepsilon}^\infty (u(t),\, (A^*\phi)(t))\, dt$$

which tends obviously as $\varepsilon\downarrow 0$ to $-\int_0^\infty (u(t), (A^*\phi)(t))\,dt$. Similarly we have

$$-\int_0^\infty (f(t), (v_\varepsilon \phi)(t))\,dt \to -\int_0^\infty (f(t), \phi(t))\,dt \qquad \text{as } \varepsilon\downarrow 0\,.$$

Summing up again we obtain the equality

$$\int_0^\infty (u(t), \phi'(t))\,dt = -\int_0^\infty (u(t), (A^*\phi)(t))\,dt - \int_0^\infty (f(t), \phi(t))\,dt$$

and remembering definition of $\phi(t)$, we have

$$\int_0^\infty \Big(u(t), (\exp[at]\zeta(t)\psi(t))'\Big)\,dt =$$

$$= -\int_0^\infty (u(t), \exp[at]\zeta(t)(A^*\psi)(t))\,dt - \int_0^\infty (f(t), \exp[at]\zeta(t)\psi(t))\,dt$$

or also, turning back to (1.5), we obtain

$$\int_{-\infty}^{+\infty} (v(t), \psi'(t))\,dt = -\int_0^\infty (u(t), \exp[at]\zeta(t)(A^*\psi)(t))\,dt -$$

$$-\int_0^\infty (f(t), \exp[at]\zeta(t)\psi(t))\,dt - \int_0^\infty (u(t), (\exp[at]\zeta(t))'\psi(t))\,dt =$$

$$= -\int_{-\infty}^\infty (v(t), (A^*\psi)(t))\,dt - \int_{-\infty}^\infty (\exp[at]\zeta(t) f(t), \psi(t))\,dt -$$

$$-\int_{-\infty}^\infty (\exp[at]\zeta'(t)u(t), \psi(t))\,dt - a\int_{-\infty}^\infty (v(t), \psi(t))\,dt\,.$$

Which is what we had to prove.

If we denote

$$h(t) = \exp[at]\zeta(t)f(t) + \exp[at]\zeta'(t)u(t) + av(t),$$

we see that $h(t)$ is continuous, $-\infty < t < \infty \to H$.

§ 2. – Let us consider now a sequence of scalar-valued, non-negative C^1-functions, $\{\alpha_n(t)\}_{n=1}^{\infty}$, vanishing for $|t| > 1/n$ and such that $\int_{-\infty}^{+\infty}\alpha_n(\tau)\,d\tau = 1$, $n = 1, 2, \ldots$. Consider the convolution

$$(2.1) \qquad (v * \alpha_n)(t) = \int_{-\infty}^{\infty} v(\tau)\alpha_n(t-\tau)\,d\tau = \int_{|t-\tau|\leqslant 1/n} v(\tau)\alpha_n(t-\tau)\,d\tau$$

where v was defined in Lemma.

As usual, $(v * \alpha_n)(t)$ is well-defined for $-\infty < t < +\infty$ and is H-continuously differentiable there.

As was proved in our papers [5], [6], from (1.2) we can deduce that $(v * \alpha_n)(t) \in \mathfrak{D}(A)$, $\forall t \in (-\infty, \infty)$ and that

$$(2.2) \qquad (v*\alpha_n)'(t) = A(v * \alpha_n)(t) + (h * \alpha_n)(t).$$

We have now the following

Proposition 1. *It is*

$$\int_{-\infty}^{+\infty}\|(v * \alpha_n)'(t)\|\,dt < \infty, \qquad \int_{-\infty}^{+\infty}\|(h * \alpha_n)(t)\|\,dt < \infty, \qquad n = 1, 2, \ldots.$$

Remark that

$$(v * \alpha_n)'(t) = \int_{-\infty}^{+\infty} v(\tau)\alpha_n'(t-\tau)\,d\tau = \int_{t-1/n}^{t+1/n} v(\tau)\alpha_n'(t-\tau)\,d\tau.$$

Now, $v(t)$ is estimated:

$$\|v(t)\| = \exp[at]\zeta(t)\|u(t)\|, \qquad t \geqslant 0, \quad \|v(t)\| = 0, \quad t \leqslant 0.$$

On the other hand, our main hypothesis on $u(t)$ implies that $\forall$ real α, $\exists N_\alpha > 0$, such that $\|u(t)\| \leqslant N_\alpha \exp[\alpha t]$, $t \geqslant 0$.

Take then $\alpha + a = \beta < 0$, and get $\|v(t)\| \leqslant N_\alpha \exp[\beta t]$ for $t \geqslant 0$, hence $\forall t \in R^1$ too. Hence

$$\|(v * \alpha_n)'(t)\| \leqslant \int_{t-1/n}^{t+1/n} \|v(\tau)\| \cdot |\alpha_n'(t-\tau)|\, d\tau = \int_{-1/n}^{1/n} \|v(t-\sigma)\| \cdot |\alpha_n'(\sigma)|\, d\sigma \leqslant$$

$$\leqslant N_\alpha \int_{-1/n}^{1/n} \exp[\beta(t-\sigma)] |\alpha_n'(\sigma)|\, d\sigma = N_\alpha \exp[\beta t] \int_{-1/n}^{1/n} \exp[|\beta|\sigma] |\alpha_n'(\sigma)|\, d\sigma \leqslant$$

$$\leqslant N_\alpha \exp[\beta t] \exp\left[\frac{|\beta|}{n}\right] \int_{-1/n}^{1/n} |\alpha_n'(\sigma)|\, d\sigma = C_n \exp[\beta t]\,, \qquad \forall t \in R^1.$$

Furthermore, for $t < -1/n$, $(v * \alpha_n)'(t) = 0$ because $v(\tau) = 0$ for $\tau \leqslant 0$. This proves integrability of $(v * \alpha_n)'(t)$ on real axis.

Consider now $h(t)$ which was defined as the sum

$$\exp[at]\, \zeta(t) f(t) + \exp[at] \zeta'(t) u(t) + av(t)\,.$$

It follows: $h = \theta$ for $t \leqslant 0$ and

$$\|h(t)\| \leqslant \exp[at]\, \|f(t)\| + \exp[at]\, |\zeta'(t)|\, \|u(t)\| + |a|\, \|\exp[at] \zeta(t) u(t)\| \qquad \text{for } t \geqslant 0\,.$$

Actually

$$\|f(t)\| \leqslant \phi(t) \|u(t)\| \leqslant c \|u(t)\| \leqslant c N_\alpha \exp[\alpha t]\,, \qquad t \geqslant 0\,.$$

Also, $\zeta'(t) = 0$ for $t \geqslant t_0$, and $\|av(t)\| \leqslant |a| N_\alpha \exp[\beta t]$, $t \geqslant 0$, $\alpha = \beta - a$. Consequently $\exp[at] \|f(t)\| \leqslant c N_\alpha \exp[\beta t]$, $t \geqslant 0$, and $\exists c_1 > 0$ such that

$$\exp[at]\, |\zeta'(t)|\, \|u(t)\| \leqslant c_1 \exp[\beta t]\,, \qquad t \geqslant 0$$

(in fact $\exp[(a-\beta)t]\, |\zeta'(t)|\, \|u(t)\| > 0$ only on $0 \leqslant t \leqslant t_0$; it is a continuous function there, take c_1 its supremum).

It is consequently

$$\|h(t)\| \leqslant cN_\alpha \exp[\beta t] + c_1 \exp[\beta t] + |a| N_\alpha \exp[\beta t] = C_2(\alpha, a) \exp[\beta t]\,,$$
$$t \geqslant 0\,,\ h = \theta\,,\ t \leqslant 0\,.$$

It we take again the convolution

$$(h * \alpha_n)(t) = \int\limits_{|t-\tau| \leqslant 1/n} h(\tau)\alpha_n(t-\tau)\, d\tau\,,$$

it is identically null for $t < -1/n$, and is otherwise estimated by

$$\|(h * \alpha_n)(t)\| \leqslant \int\limits_{-1/n}^{1/n} \|h(t-\sigma)\| \alpha_n(\sigma)\, d\sigma \leqslant C_2 \int\limits_{-1/n}^{1/n} \exp[\beta(t-\sigma)]\, \alpha_n(\sigma)\, d\sigma \leqslant$$
$$\leqslant C_2 \exp[\beta t] \exp\left[\frac{|\beta|}{n}\right] = c_{3,n} \exp[\beta t]\,, \qquad \text{for all real } t\,.$$

This proves our Proposition 1.

PROPOSITION 2. *The functions* $(v * \alpha_n)(t)$ *and* $A(v * \alpha_n)(t)$ *are norm-integrable on the real axis.*

The second term is obviously integrable by Prop. 1. The proof for the first term is similar to the one given above.

Let us multiply now equality (2.2) by $1/\sqrt{2\pi} \exp[-i\tau t]$, where $-\infty < \tau < \infty$, $i = \sqrt{-1}$ and then integrate on R^1; we obtain

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \exp[-i\tau t](v * \alpha_n)'(t)\, dt =$$
$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \exp[-i\tau t] A(v * \alpha_n)(t)\, dt + \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \exp[-i\tau t](h * \alpha_n)(t)\, dt\,.$$

If we effectuate partial integration in the left-hand integral-say on intervals $(-r, r)$, and use vanishing of $(v * \alpha_n)(t)$ for $t < -1/n$ and exponential decay for $t \to +\infty$, we obtain

$$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \exp[-i\tau t](v * \alpha_n)'(t)\, dt = i\tau \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \exp[-i\tau t](v * \alpha_n)(t)\, dt\,.$$

On the other hand, as $\exp[-i\tau t](v * \alpha_n)(t)$ is integrable on R^1 and

$$A\big(\exp[-i\tau t](v * \alpha_n)(t)\big) = \exp[-i\tau t]A(v * \alpha_n)(t)$$

has the same property we get, as well-known [2], that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-i\tau t](v * \alpha_n)(t)\,dt \in \mathfrak{D}(A)$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-i\tau t]A(v * \alpha_n)(t)\,dt = A\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-i\tau t](v * \alpha_n)(t)\,dt\,.$$

Hence, using a standard notation, we get

$$i\tau(v * \alpha_n)\hat{}\,(\tau) = A(v * \alpha_n)\hat{}\,(\tau) + (h * \alpha_n)\hat{}\,(\tau)\,.$$

Write now $h(t)$ in the form

$$h(t) = g(t) + av(t)\,, \qquad g(t) = \exp[at]\zeta(t)f(t) + \exp[at]\zeta'(t)u(t)\,.$$

Then $(h * \alpha_n)(t) = (g * \alpha_n)(t) + a(v * \alpha_n)(t)$ and consequently

$$(h * \alpha_n)\hat{}\,(\tau) = (g * \alpha_n)\hat{}\,(\tau) + a(v * \alpha_n)\hat{}\,(\tau)$$

and also

$$(i\tau - a - A)(v * \alpha_n)\hat{}\,(\tau) = (g * \alpha_n)\hat{}\,(\tau)\,,$$

where $-\infty < \tau < \infty$ and a is an arbitrary real number.

Take in particular $a = -\sigma_n$; then $(i\tau - a - A) = (\sigma_n + i\tau - A)$ which has a bounded inverse $\forall n = 1, 2, \ldots$. We get

$$(v * \alpha_n)\hat{}\,(\tau) = (\sigma_n + i\tau - A)^{-1}(g * \alpha_n)\hat{}\,(\tau)\,, \quad -\infty < \tau < \infty\,, \quad n = 1, 2, \ldots\,.$$

§ 3. – Remember now well-known relation

$$(v * \alpha_n)\hat{}\,(\tau) = \hat{v}(\tau)\hat{\alpha}_n(\tau)\,, \qquad (g * \alpha_n)\hat{}\,(\tau) = \hat{g}(\tau)\hat{\alpha}_n(\tau)\,.$$

Consider also a particular sequence $\{\alpha_n\}_1^\infty$, constructed as follows: Take one function $0 \leqslant \alpha(t) \in C^1(-\infty, \infty)$, $=0$ for $|t| \geqslant 1$, with $\int_{-\infty}^{\infty} \alpha(t) \cdot dt = 1$. Then put $\alpha_n(t) = n\alpha(nt)$ and see that it verifies the required above conditions: also it is $\hat{\alpha}_n(\tau) = \hat{\alpha}(\tau/n)$ as easily seen. Hence, from section 2, we get now:

$$\hat{v}(\tau)\,\hat{\alpha}\left(\frac{\tau}{n}\right) = (\sigma_n + i\tau - A)^{-1}\hat{g}(\tau)\,\hat{\alpha}\left(\frac{\tau}{n}\right), \qquad -\infty < \tau < \infty,\; n = 1, 2, \ldots$$

and

$$\left\| \hat{v}(\tau)\,\hat{\alpha}\left(\frac{\tau}{n}\right)\right\| \leqslant M\|\hat{g}(\tau)\|\left|\hat{\alpha}\left(\frac{\tau}{n}\right)\right|, \qquad -\infty < \tau < \infty,\; n = 1, 2, \ldots .$$

If we let here $n \to \infty$, we get $\hat{\alpha}(\tau/n) \to \hat{\alpha}(0) = 1$, $\forall \tau \in (-\infty, \infty)$ hence $\|\hat{v}(\tau)\| \leqslant M\|\hat{g}(\tau)\|$, $-\infty < \tau < \infty$ and by Plancherel's equality we get

$$\int_{-\infty}^{\infty} \|v(t)\|^2\,dt \leqslant M^2 \int_{-\infty}^{\infty} \|g(t)\|^2\,dt .$$

The final steps are the standard ones (see [4]); we give here the full details, for sake of completeness.

Because of definition of $v(t)$ we get, for $a = -\sigma_n$, $n = 1, 2, \ldots$

$$\int_{-\infty}^{\infty} \|v(t)\|^2\,dt = \int_0^{\infty} \exp[2at]\zeta^2(t)\|u(t)\|^2\,dt \leqslant M^2 \int_{-\infty}^{\infty} \|g(t)\|^2\,dt = M^2 \int_0^{\infty} \|g(t)\|^2\,dt$$

because $g = \theta$ for $t < 0$; also, as $\zeta(t) = 1$ for $t \geqslant t_0$, we deduce

$$\int_{t_0}^{\infty} \exp[2at]\,\|u(t)\|^2\,dt \leqslant M^2 \int_0^{\infty} \|g(t)\|^2\,dt .$$

Actually, $g(t) = \exp[at]f(t)$ for $t \geqslant t_0$; for $0 \leqslant t \leqslant t_0$, $g(t)$ is estimated by:

$$\|g(t)\| \leqslant \exp[at]\|f(t)\| + C_1 \exp[at]\|u(t)\| \leqslant \exp[at](\phi(t) + C_1)\,\|u(t)\| <$$
$$< \exp[at]\left(\frac{1}{M} + C_1\right)\|u(t)\|, \qquad 0 \leqslant t \leqslant t_0 .$$

For $t \geqslant t_0$, $\|g(t)\| \leqslant \exp[at]\,\|f(t)\| \leqslant c\|u(t)\|\exp[at]$ where $c < 1/M$.

We obtain:

$$\int_{t_0}^{\infty} \exp[2at]\|u(t)\|^2\,dt \leqslant M^2\left(\int_0^{t_0} \exp[2at]\left(\frac{1}{M}+C_1\right)^2\|u(t)\|^2\,dt\right) +$$

$$+ M^2\left(\int_{t_0}^{\infty} c^2\|u(t)\|^2\exp[2at]\,dt\right) \text{hence} \left(\int_{t_0}^{\infty}\exp[2at]\|u(t)\|^2\,dt\right)(1-M^2c^2) \leqslant$$

$$\leqslant M^2\left(\frac{1}{M}+C_1\right)^2\int_0^{t_0}\exp[2at]\,\|u(t)\|^2\,dt\,.$$

Here $a = -\sigma_n$, $n = 1, 2, \ldots$.

As $\sigma_n \to -\infty$, a is > 0 for $n \geqslant \bar{n}$; hence $\forall \delta > 0$ and $n \geqslant \bar{n}$

$$\int_{t_0+\delta}^{\infty}\exp[2at]\|u(t)\|^2\,dt \leqslant \frac{M^2(1/M+C_1)^2}{1-M^2c^2}\exp[2at_0]\int_0^{t_0}\|u(t)\|^2\,dt$$

and also

$$\exp[2a(t_0+\delta)]\int_{t_0+\delta}^{\infty}\|u(t)\|^2\,dt \leqslant \frac{M^2(1/M+C_1)^2}{1-M^2c^2}\exp[2at_0]\int_0^{t_0}\|u(t)\|^2\,dt$$

which implies

$$\int_{t_0+\delta}^{\infty}\|u(t)\|^2\,dt \leqslant \exp[-2a\delta]\,\frac{M^2(1/M+C_1)^2}{1-M^2c^2}\int_0^{t_0}\|u(t)\|^2\,dt\,.$$

As $n \to \infty$, $\exp[-2a\delta] = \exp[2\sigma_n\delta] \to 0$, $\forall \delta > 0$; we get hence forth

$$\int_{t_0+\delta}^{\infty}\|u(t)\|^2\,dt = 0 \qquad \forall\delta > 0\,,\ \text{so}\ u = \theta\ \text{on}\ (t_0, \infty)\,.$$

As t_0 is arbitrary > 0, $u = \theta$ on $[0, \infty)$ too.

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