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A Selection Theorem.

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1. – Introduction.

A well known theorem of Michael states that a lower semi-continuous multi-valued mapping, from a metric space into the non-empty closed and convex subsets of a Banach space, admits a continuous selection. It is also known that, when the multi-valued mapping is instead upper semi-continuous, in general we have only measurable selections.

This paper considers a compact convex valued mapping F of two variables, t and x, that is separately upper semi-continuous in t for every fixed x and lower semi-continuous in x for every fixed t, and proves the existence of a selection $f(t, x)$, separately measurable in t and continuous in x. As a consequence, an existence theorem for solutions of a multi-valued differential equation is presented.

2. – Notations and basic definitions.

In what follows $\boldsymbol{R}$ are the reals, X a separable metric space and Z a Banach space. We shall denote by $K(Z)$ the set of non-empty compact and convex subsets of Z. $B[A, \varepsilon]$ is an open ball of radius $\varepsilon > 0$ about the set A, $\bar{A}$ is the closure of A. We shall use the symbol $d(\cdot, \cdot)$ both for the metric in X and for the metric inherited from the norm in Z. Also $d(a, B)$ is the distance from the point a to the set B, while

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$\delta^*(A, B) = \sup\{d(a, B)\colon a \in A\}$ and D is the Hausdorff distance, i.e. $D(A, B) = \sup\{\delta^*(A, B), \delta^*(B, A)\}$. A mapping F from a subset I of the reals into the nonempty compact subsets of Z is called upper semicontinuous (u.s.c.) if $\forall t^0 \in I$, $\forall \varepsilon > 0$, $\exists \delta > 0\colon |t - t^0| < \delta \Rightarrow F(t) \subset \subset B[F(t_0), \varepsilon]$. A mapping $F\colon X \to K(Z)$ is called lower semi-continuous (l.s.c.) if $\forall x^0 \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0\colon d(x, x^0) < \delta \Rightarrow F(x^0) \subset B[F(x), \varepsilon]$.

3. – Main results.

LEMMA. Let $E \subset \boldsymbol{R}$ be compact; let X be a separable metric space, Z a Banach space. Let $\Phi\colon E \times X \to K(Z)$ be upper semi-continuous in $t \in E$ for every $x \in X$ and lower semi-continuous in x for every $t \in E$. Then for every $\varepsilon > 0$ there exist E_ε, a compact subset of E, with $\mu(\mathrm{E} \setminus E_\varepsilon) \leqslant \varepsilon$ and a single-valued continuous function $f_\varepsilon\colon E_\varepsilon \times X \to Z$ such that for $(t, x) \in E_\varepsilon \times X$,

$$d\big(f_\varepsilon(t, x), \Phi(t, x)\big) \leqslant \varepsilon .$$

PROOF. Let $D = \{x_j\}$ be a countable dense subset of X. Set $\Delta = \operatorname{diam}(E)$. For every j set

$$\delta_j(t) = \sup\{\delta\colon 0 < \delta < \Delta\colon \exists y \in \Phi(t, x_j)\colon d(x, x_j) < \delta \Rightarrow d\big(y, \Phi(t, x)\big) \leqslant \varepsilon/2\}$$

Since Φ is l.s.c. in x for every t, the set inside parenthesis is nonempty. The following *a*) and *b*) are the two main reasons for the above definition

a) The real valued functions $\delta_j(t)$ are semi-continuous. Fix j and t^0. We wish to prove that

$$\overline{\lim}\, \delta_j(t) \leqslant \delta_j(t^0) .$$

Assume this is false; then there exist $\{t_n\}$, $t_n \to t^0$ and a positive $\xi\colon \delta_j(t_n) > \delta_j(t^0) + \xi$. By the very definition of δ_j, for every n there exists $y_n \in \Phi(t_n, x_j)$ such that $d(x, x_j) \leqslant \delta_j(t^0) + \xi/2$ implies $d\big(y_n, \Phi \cdot \cdot (t_n, x)\big) \leqslant \varepsilon/2$. Since $\Phi(\cdot, x_j)$ is u.s.c. at t^0, $d\big(y_n, \Phi(t^0, x_j)\big) \to 0$. Tehn from the compactness of $\Phi(t^0, x_j)$ it follows easily that there exists a subsequence converging to some $y^0 \in \Phi(t^0, x_j)$. Now fix any x such that $d(x, x_j) \leqslant \delta_j(t^0) + \xi/2$. Then

$$d\big(y^0, \Phi(t^0, x)\big) \leqslant d(y^0, y_n) + d\big(y_n, \Phi(t_n, x)\big) + \delta^*\big(\Phi(t_n, x), \Phi(t^0, x)\big) .$$

Since $d(y^0, y_n) \to 0$, $\delta^*(\Phi(t_n, x), \Phi(t^0, x)) \to 0$ and $d(y_n, \Phi(t_n, x)) \leqslant \varepsilon$, it follows that $d(y^0, \Phi(t^0, x)) \leqslant \varepsilon/2$.

Therefore $\delta_j(t^0) + \xi/2 \leqslant \delta_j(t^0)$, a contradiction. This proves our claim on $\delta_j(\cdot)$.

The functions $\delta_j(\cdot)$, being semi-continuous, are measurable. Applying Lusin's Theorem we infer the existence of a compact $E_1 \subset E$ with $\mu(E \setminus E_1) \leqslant \varepsilon/2$ such that on E_1 each $\delta_j(\cdot)$ is continuous.

b) For every $t \in E_1$, $V_{j,t} = \{x\colon d(x, x_j) < \delta_j(t)/2\}$. Then $\{V_{j,t}\}$ is a covering of X (for each fixed t).

It is enough to show that if $\{x_j\}$ converges to $\hat{x}$, then $\underline{\lim}\ \delta_j(t) > 0$. Consider $\hat{x}$: since $\Phi(t, \cdot)$ is u.s.c., there exists $\Delta > 0$: $d(x, \hat{x}) < \Delta$ implies $\Phi(t, \hat{x}) \subset B[\Phi(t, x), \varepsilon/4]$. We claim then: x_j sufficiently close to $\hat{x}$ implies $\delta_j(t) \geqslant \Delta/2$. In fact let $d(x_j, \hat{x}) < \Delta/2$; let $x' \in B[x_j, \Delta/2]$, so that $d(x', \hat{x}) < \Delta$. Take any $y \in \Phi(t, \hat{x})$: there exists $y_j' \in \Phi(t, x_j)$: $d(y, y_j') < \varepsilon/4$. Hence

$$d(y_j', \Phi(t, x')) \leqslant d(y_j', y) + d(y, \Phi(t, x')) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2 .$$

This proves that $\delta_j(t) \geqslant \Delta/2$ and our point *b*).

Consider now the mappings $\Psi_j\colon E \to 2^z$ defined by

$$\Psi_j(t) = \{y \in \Phi(t, x_j)\colon d(x, x_j) < \delta_j(t) \Rightarrow d(y, \Phi(t, x)) \leqslant \varepsilon/2\} .$$

By the definition of δ_j, $\Psi_j(t)$ is non-empty. Our next claim is that the restriction of Ψ_j to E_1 is u.s.c. We shall prove first that it has closed graph. Assume this is not true: there exist t^0 and $\{t_n\}$, $t_n \to t^0$, points y_n and y^0, with $y_n \in \Psi_j(t_n)$ and $y_n \to y^0$ such that $y^0 \notin \Psi_j(t^0)$, i.e. there exist $\xi > 0$ and $\hat{x}$: $d(\hat{x}, x_j) \leqslant \delta_j(t^0) - \xi$ but

$$d(y^0, \Phi(t^0, \hat{x})) > \varepsilon/2 .$$

By the continuity of δ_j, n large implies $\delta_j(t_n) > d(\hat{x}, x_j)$, hence

$$d(y^0, \Phi(t^0, \hat{x})) \leqslant d(y^0, y_n) + d(y_n, \Phi(t_n, \hat{x})) + \delta^*(\Phi(t_n, \hat{x}), \Phi(t^0, \hat{x})) .$$

Since $y_n \in \Psi_j(t_n)$, $d(y^0, \Phi(t^0, \hat{x})) \leqslant \varepsilon/2$ or

$$y^0 \in \Psi_j(t^0) .$$

A contradiction, so Ψ_j has closed graph. We have in addition, that $\Phi(\cdot, x_j)$ is u.s.c. and that its images are compact sets. This implies that $\Phi(E_1, x_j)$ is compact. Finally, Ψ_j, a closed mapping whose range is contained in a compact set, is u.s.c.

Drop an open set of measure at most $\varepsilon/2$ so that on its complement $E_\varepsilon \subset E_1$ (we have $\mu(E \setminus E_\varepsilon) < \varepsilon$) each $\Psi_j(\cdot)$ is continuous. Then for every j, for every $\tau \in E_\varepsilon$, there exist $\varrho(j, \tau) > 0$ and $\eta(j, \tau)$: $0 < \eta(j, \tau) \leqslant \varrho(j, \tau)$: $|\tau - t| < \varrho(j, \tau)$ implies $D(\Psi_j(t), \Psi_j(\tau)) < \varepsilon/2$ and $|\tau - t| < \eta(j, \tau)$ implies $\delta_j(t) > \frac{1}{2}\delta_j(\tau)$.

Consider the collection $\{O(j, \tau)\}$,

$$O(j, \tau) = \{(t, x) \colon |t - \tau| < \eta(j, \tau) \text{ and } x \in V_{j,\tau}\} .$$

It is an open covering of the paracompact $E_\varepsilon \times S$. Let $\{V(j, \tau)\}$ be a (precise) locally finite refinement, $\{p_{j,\tau}\}$ a partition of unity subordinate to $V(j, \tau)$; choose $y_{j,\tau} \in \Psi_j(\tau)$ and set

$$f_\varepsilon(t, x) = \sum p_{j,\tau}(t, x) y_{j,\tau} .$$

We claim that the above f_ε has the required properties.

In fact, fix $(t, x) \in E_\varepsilon \times S$. Let j, τ be such that $p_{j,\tau}(t, x) > 0$. Hence $(t, x) \in O(j, \tau)$, i.e.

$$\text{i)} \quad |t - \tau| < \eta(j, \tau) \qquad \text{and} \qquad \text{ii)} \quad |x - x_j| < \tfrac{1}{2}\delta_j(\tau) .$$

From point i), there exists $\hat{y} \in \Psi_j(t)$: $d(\hat{y}, y_{j,\tau}) < \varepsilon/2$. Moreover $|t - \tau| < \eta(j, \tau)$ implies $\frac{1}{2}\delta_j(\tau) < \delta_j(t)$. Hence from ii) and the definition of $\Psi_j(t)$, we have

$$d(y_{j,\tau}, \Phi(t, x)) \leqslant d(y_{j,\tau}, \hat{y}) + d(\hat{y}, \Phi(t, x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

The convexity of $\Phi(t, x)$ implies that the same relation holds for $f_\varepsilon(t, x)$, a convex combination of $y_{j,\tau}$'s. Q.E.D.

Theorem 1. Let $I \subset \boldsymbol{R}$ be compact, X a separable metric space $F \colon I \times X \to K(Z)$ be u.s.c. for every fixed $x \in X$, l.s.c. for every fixed $t \in I$. Then there exists a mapping $f \colon I \times X \to Z$ such that

i) for every $(t, x) \in I \times X$, $f(t, x) \in F(t, x)$,

ii) for every $x \in X$, $f(\cdot, x) \colon I \to Z$ is measurable,

iii) for every $t \in I$, $f(t, \cdot) \colon X \to Z$ is continuous.

Proof. Let $\varepsilon_n \downarrow 0: \sum \varepsilon_n < \mu(I)$. We claim first: there exist compact $E_n \subset I$ with $\mu(I \setminus E_n) \leqslant \varepsilon_n$ and continuous $f_n: E_n \times X \to Z$ such that

$$d(f_n(t, x), F(t, x)) \leqslant \varepsilon_n, \qquad t \in E_n, \ n = 1, \dots,$$

$$d(f_n(t, x), f_{n-1}(t, x)) \leqslant \varepsilon_{n-1}, \qquad t \in E_{n-1} \cap E_n, \ n = 2, \dots .$$

For $n = 1$ set in the preceding Lemma $\varepsilon = \varepsilon_1$, $E = I$, $\Phi = F$ and call f_1 the f_ε obtained.

Assume we have constructed E_n, f_n up to $n = N - 1$. Consider $I \setminus E_{N-1}$. It is an open set; there exist C_{N-1}, a compact subset of $I \setminus E_{N-1}$, with $\mu((I \setminus E_{N-1}) \setminus C_{N-1}) < \varepsilon_N/3$. In the Lemma set $E = C_{N-1}$, $F = \Phi$, $\varepsilon = \varepsilon_N/3$ to yield:

a compact subset K_N^1 of C_{N-1}, with $\mu(C_{N-1} \setminus K_N^1) \leqslant \varepsilon_N/3$ and

a function $f^1(t, x)$: $K_N^1 \times X \to Z$ such that

$$d(f^1(t, x), F(t, x)) \leqslant \varepsilon_N/3 < \varepsilon_N .$$

Consider now the set $E_{N-1} \times X$ and the mapping Φ: $E_{N-1} \times X \to K(Z)$ defined by

$$\Phi(t, x) = F(t, x) \cap \overline{B[f_{N-1}(t, x), \varepsilon_{N-1}]} .$$

By our induction assumption, $\Phi(t, x)$ is non-empty. Moreover it is compact and convex. In addition it is u.s.c. in $t \in E_{N-1}$ for every fixed $x \in X$ (its graph is the intersection of two closed graphs and the range is contained in a compact set) and l.s.c. in x for every fixed t [1].

Applying the Lemma to Φ, E_{N-1} and ε_N, we infer the existence of a compact $K_N^2 \subset E_{N+1}$, $\mu(E_{N-1} \setminus K_{N-1}^2) < \varepsilon_N$ and a f^2: $K_N^2 \times X \to Z$ such that

$$d(f^2(t, x), \Phi(t, x)) < \varepsilon_N .$$

Hence for f^2 both

$$d(f^2(t, x), f_{N-1}(t, x)) \leqslant \varepsilon_{N-1}$$

and

$$d(f^2(t, x), F(t, x)) < \varepsilon_N \qquad \text{hold} .$$

Set $E_N = K_N^1 \cup K_N^2$ and define $f_N\colon E_N \times X \to Z$ by

$$f_N(t, x) = \begin{cases} f^1(t, x)\,, & t \in K_N^1\,,\\ f^2(t, x)\,, & t \in K_N^2\,. \end{cases}$$

We have that $\mu(I \setminus E_N) = \mu\big((E_{N-1} \setminus K_N^2) \cup ((I \setminus E_{N-1}) \setminus K_N^1)\big) \leqslant \varepsilon_N/3 + 2\varepsilon_N/3 = \varepsilon_N$, and the claim is proved.

Now set

$$A_N = \bigcup_{n=N}^{\infty} (I \setminus E_N)\,.$$

Then $A_N \subset A_{N-1}$ and $\mu(\cap A_N) = \lim \mu(A_N) = 0$. Fix $t \notin \cap A_N$. Then $\{f_N(t, \cdot)\}$ is a Cauchy sequence of continuous functions and converges uniformly to a $\varphi(t, x)$, continuous in x. Fix x. Then for every $t \notin \cap A_N$, $\varphi(t, x)$ is the pointwise limit of $f_N(t, x)$, hence measurable. For $t \in \cap A_N$, let $\hat{\varphi}(t, \cdot)$ be any continuous selection from $F(t, \cdot)$ [1].

The function

$$f(t, x) = \begin{cases} \varphi(t, x)\,, & t \in I \setminus \cap A_N\,,\\ \hat{\varphi}(t, x)\,, & t \in \cap A_N\,, \end{cases}$$

has the required properties. Q.E.D.

From Theorem 1 the following Theorem 2 can easily be proved:

THEOREM 2. Let Z be a finite dimensional space, Ω an open subset of $R \times Z$, $F\colon \Omega \to K(Z)$ be u.s.c. in t for every fixed x and l.s.c. in x for every fixed t, t and x in Ω. Moreover assume that the range of F is contained in some compact subset of Z. Let $(t^0, x^0) \in \Omega$. Then the Cauchy problem

$$x' \in F(t, x)\,, \qquad x(t^0) = x^0$$

admits at least one solution.

Also, applying a result of Scorza Dragoni [2] to the function f of Theorem 1, the following Corollary can be derived:

COROLLARY. Let $I \subset \boldsymbol{R}$ be compact, X a separable metric space, $F\colon I \times X \to K(Z)$ be u.s.c. for every fixed $x \in X$, l.s.c. for every fixed $t \in I$. Then for every $\varepsilon > 0$ there exist K_ε, a compact subset of I and a continuous $f_\varepsilon\colon K_\varepsilon \times X \to Z$ that is a selection from F.

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