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A selection theorem

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1. – Introduction.

A well known theorem of Michael states that a lower semi-continuous multi-valued mapping, from a metric space into the non-empty closed and convex subsets of a Banach space, admits a continuous selection. It is also known that, when the multi-valued mapping is instead upper semi-continuous, in general we have only measurable selections.

This paper considers a compact convex valued mapping $F$ of two variables, $t$ and $x$, that is separately upper semi-continuous in $t$ for every fixed $x$ and lower semi-continuous in $x$ for every fixed $t$, and proves the existence of a selection $f(t, x)$, separately measurable in $t$ and continuous in $x$. As a consequence, an existence theorem for solutions of a multi-valued differential equation is presented.

2. – Notations and basic definitions.

In what follows $R$ are the reals, $X$ a separable metric space and $Z$ a Banach space. We shall denote by $K(Z)$ the set of non-empty compact and convex subsets of $Z$. $B[A, \varepsilon]$ is an open ball of radius $\varepsilon > 0$ about the set $A$, $\overline{A}$ is the closure of $A$. We shall use the symbol $d(\cdot, \cdot)$ both for the metric in $X$ and for the metric inherited from the norm in $Z$. Also $d(a, B)$ is the distance from the point $a$ to the set $B$, while (*) Indirizzo dell'A. : Istituto Matematico, Università, Via Belzoni 7, 35100 Padova.
\[ \delta^*(A, B) = \sup \{ d(a, B) : a \in A \} \] and \( D \) is the Hausdorff distance, i.e. \( D(A, B) = \sup \{ \delta^*(A, B), \delta^*(B, A) \} \). A mapping \( F \) from a subset \( I \) of the reals into the nonempty compact subsets of \( Z \) is called upper semicontinuous (u.s.c.) if \( \forall t^0 \in I, \forall \varepsilon > 0, \exists \delta > 0: |t - t^0| < \delta \Rightarrow F(t) \subset B[F(t^0), \varepsilon] \). A mapping \( F: X \to K(Z) \) is called lower semi-continuous (l.s.c.) if \( \forall x^0 \in X, \forall \varepsilon > 0, \exists \delta > 0: d(x, x^0) < \delta \Rightarrow F(x^0) \subset B[F(x), \varepsilon] \).

3. Main results.

**Lemma.** Let \( E \subset \mathbb{R} \) be compact; let \( X \) be a separable metric space, \( Z \) a Banach space. Let \( \Phi: E \times X \to K(Z) \) be upper semi-continuous in \( t \in E \) for every \( x \in X \) and lower semi-continuous in \( x \) for every \( t \in E \). Then for every \( \varepsilon > 0 \) there exist \( E_\varepsilon \), a compact subset of \( E \), with \( \mu(E \setminus E_\varepsilon) < \varepsilon \) and a single-valued continuous function \( f_\varepsilon: E_\varepsilon \times X \to Z \) such that for \( (t, x) \in E_\varepsilon \times X \),

\[ d(f_\varepsilon(t, x), \Phi(t, x)) < \varepsilon. \]

**Proof.** Let \( D = \{ x_j \} \) be a countable dense subset of \( X \). Set \( \Delta = \text{diam} (E) \). For every \( j \) set

\[ \delta_j(t) = \sup \{ \delta : 0 < \delta < \Delta : \exists y \in \Phi(t, x_j) : d(x, x_j) < \delta \Rightarrow d(y, \Phi(t, x)) < \varepsilon/2 \} \]

Since \( \Phi \) is l.s.c. in \( x \) for every \( t \), the set inside parenthesis is nonempty. The following \( a) \) and \( b) \) are the two main reasons for the above definition

\( a) \) The real valued functions \( \delta_j(t) \) are semi-continuous. Fix \( j \) and \( t^0 \). We wish to prove that

\[ \lim \delta_j(t) \leq \delta_j(t^0). \]

Assume this is false; then there exist \( \{ t_n \}, t_n \to t^0 \) and a positive \( \xi : \delta_j(t_n) > \delta_j(t^0) + \xi \). By the very definition of \( \delta_j \), for every \( n \) there exists \( y_n \in \Phi(t_n, x_j) \) such that \( d(x, x_j) < \delta_j(t^0) + \xi/2 \) implies \( d(y_n, \Phi(t_n, x_j)) < \varepsilon/2 \). Since \( \Phi(\cdot, x_j) \) is u.s.c. at \( t^0 \), \( d(y_n, \Phi(t^0, x_j)) \to 0 \). Then from the compactness of \( \Phi(t^0, x_j) \) it follows easily that there exists a subsequence converging to some \( y^0 \in \Phi(t^0, x_j) \). Now fix any \( x \) such that \( d(x, x_j) < \delta_j(t^0) + \xi/2 \). Then

\[ d(y^0, \Phi(t^0, x)) < d(y^0, y_n) + d(y_n, \Phi(t_n, x)) + \delta^*(\Phi(t_n, x), \Phi(t^0, x)). \]
Since \( d(y_0, y_n) \to 0, \delta^*(\Phi(t_n, x), \Phi(t_0, x)) \to 0 \) and \( d(y_n, \Phi(t_n, x)) < \varepsilon \), it follows that \( d(y_0, \Phi(t_0, x)) < \varepsilon/2 \).

Therefore \( \delta_j(t^o) + \xi/2 < \delta_j(t^o) \), a contradiction. This proves our claim on \( \delta_j(\cdot) \).

The functions \( \delta_j(\cdot) \), being semi-continuous, are measurable. Applying Lusin's Theorem we infer the existence of a compact \( E_1 \subset E \) with \( \mu(E \setminus E_1) < \varepsilon/2 \) such that on \( E_1 \) each \( \delta_j(\cdot) \) is continuous.

b) For every \( t \in E_1 \), \( V_{j,t} = \{ x : d(x, x_j) < \delta_j(t)/2 \} \). Then \( \{ V_{j,t} \} \) is a covering of \( X \) (for each fixed \( t \)).

It is enough to show that if \( \{ x_j \} \) converges to \( \hat{x} \), then \( \lim \delta_j(t) > 0 \).

Consider \( \hat{x} \): since \( \Phi(t, \cdot) \) is u.s.c., there exists \( \Lambda > 0 : d(x, \hat{x}) < \Lambda \) implies \( \Phi(t, \hat{x}) \subset B[\Phi(t, x), \varepsilon/4] \). We claim then: \( x_j \) sufficiently close to \( \hat{x} \) implies \( \delta_j(t) > \Lambda/2 \). In fact let \( d(x_j, \hat{x}) < \Lambda/2 \); let \( x' \in B[x_j, \Lambda/2] \), so that \( d(x', \hat{x}) < \Lambda \). Take any \( y \in \Phi(t, \hat{x}) \): there exists \( y_j \in \Phi(t, x_j) : d(y, y_j) < \varepsilon/4 \). Hence

\[
d'(y_j, \Phi(t, x')) < d(y_j, y) + d(y, \Phi(t, x')) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.
\]

This proves that \( \delta_j(t) > \Lambda/2 \) and our point b).

Consider now the mappings \( \Psi_j : E \to 2^e \) defined by

\[
\Psi_j(t) = \{ y \in \Phi(t, x_j) : d(x, x_j) < \delta_j(t) \Rightarrow d(y, \Phi(t, x)) < \varepsilon/2 \}.
\]

By the definition of \( \delta_j \), \( \Psi_j(t) \) is non-empty. Our next claim is that the restriction of \( \Psi_j \) to \( E_1 \) is u.s.c. We shall prove first that it has closed graph. Assume this is not true: there exist \( t^o \) and \( \{ t_n \}, t_n \to t^o \), points \( y_n \) and \( y^o \), with \( y_n \in \Psi_j(t_n) \) and \( y_n \to y^o \) such that \( y^o \notin \Psi_j(t^o) \), i.e. there exist \( \xi > 0 \) and \( \hat{x} : d(\hat{x}, x_j) < \delta_j(t^o) + \xi \) but

\[
d(y^o, \Phi(t^o, \hat{x})) > \varepsilon/2.
\]

By the continuity of \( \delta_j \), \( n \) large implies \( \delta_j(t_n) > d(\hat{x}, x_j) \), hence

\[
d(y^o, \Phi(t^o, \hat{x})) < d(y^o, y_n) + d(y_n, \Phi(t_n, \hat{x})) + \delta^*(\Phi(t_n, \hat{x}), \Phi(t^o, \hat{x})).
\]

Since \( y_n \in \Psi_j(t_n) \), \( d(y^o, \Phi(t^o, \hat{x})) < \varepsilon/2 \) or

\[
y^o \in \Psi_j(t^o).
\]
A contradiction, so $\Psi_j$ has closed graph. We have in addition, that $\Phi(\cdot, x_j)$ is u.s.c. and that its images are compact sets. This implies that $\Phi(E_1, x_j)$ is compact. Finally, $\Psi_j$, a closed mapping whose range is contained in a compact set, is u.s.c.

Drop an open set of measure at most $\varepsilon/2$ so that on its complement $E_\varepsilon \subset E_1$ (we have $\mu(E \setminus E_\varepsilon) < \varepsilon$) each $\Psi_j(\cdot)$ is continuous. Then for every $j$, for every $\tau \in E_\varepsilon$, there exist $\eta(j, \tau) > 0$, and $\gamma(j, \tau)$: $0 < \eta(j, \tau) < \gamma(j, \tau)$: $|\tau - t| < \gamma(j, \tau)$ implies $D(\Psi_j(t), \Psi_j(\tau)) < \varepsilon/2$ and $|\tau - t| < \eta(j, \tau)$ implies $\delta_j(t) > \frac{1}{2} \delta_j(\tau)$.

Consider the collection $\{O(j, \tau)\}$,

$$O(j, \tau) = \{(t, x) : |t - \tau| < \eta(j, \tau) \text{ and } x \in V_{i, \tau}\}.$$  

It is an open covering of the paracompact $E_\varepsilon \times S$. Let $\{V(j, \tau)\}$ be a (precise) locally finite refinement, $\{p_{i, \tau}\}$ a partition of unity subordinate to $V(j, \tau)$; choose $y_{i, \tau} \in \Psi_j(\tau)$ and set

$$f_\varepsilon(t, x) = \sum p_{i, \tau}(t, x) y_{i, \tau}.$$  

We claim that the above $f_\varepsilon$ has the required properties.

In fact, fix $(t, x) \in E_\varepsilon \times S$. Let $j, \tau$ be such that $p_{i, \tau}(t, x) > 0$. Hence $(t, x) \in O(j, \tau)$, i.e.

i) $|t - \tau| < \eta(j, \tau)$ and ii) $|x - x_j| < \frac{1}{2} \delta_j(\tau)$.

From point i), there exists $\hat{y} \in \Psi_j(t)$: $d(\hat{y}, y_{i, \tau}) < \varepsilon/2$. Moreover $|t - \tau| < \eta(j, \tau)$ implies $\frac{1}{2} \delta_j(\tau) < \delta_j(t)$. Hence from ii) and the definition of $\Psi_j(t)$, we have

$$d(y_{i, \tau}, \Phi(t, x)) < d(y_{i, \tau}, \hat{y}) + d(\hat{y}, \Phi(t, x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$  

The convexity of $\Phi(t, x)$ implies that the same relation holds for $f_\varepsilon(t, x)$, a convex combination of $y_{i, \tau}$'s. Q.E.D.

**Theorem 1.** Let $I \subset \mathbb{R}$ be compact, $X$ a separable metric space $F: I \times X \to K(Z)$ be u.s.c. for every fixed $x \in X$, l.s.c. for every fixed $t \in I$. Then there exists a mapping $f: I \times X \to Z$ such that

i) for every $(t, x) \in I \times X$, $f(t, x) \in F(t, x)$,

ii) for every $x \in X$, $f(\cdot, x): I \to Z$ is measurable,

iii) for every $t \in I$, $f(t, \cdot): X \to Z$ is continuous.
PROOF. Let $\varepsilon_n > 0: \sum \varepsilon_n < \mu(I)$. We claim first: there exist compact $E_n \subseteq I$ with $\mu(I \setminus E_n) < \varepsilon_n$ and continuous $f_n: E_n \times X \to Z$ such that

$$d(f_n(t, x), F(t, x)) < \varepsilon_n, \quad t \in E_n, \ n = 1, \ldots,$$

$$d(f_n(t, x), f_{n-1}(t, x)) < \varepsilon_{n-1}, \quad t \in E_{n-1} \cap E_n, \ n = 2, \ldots.$$

For $n = 1$ set in the preceding Lemma $\varepsilon = \varepsilon_1$, $E = I$, $\Phi = F$ and call $f_1$ the $f_\varepsilon$ obtained.

Assume we have constructed $E_n, f_n$ up to $n = N - 1$. Consider $I \setminus E_{n-1}$. It is an open set; there exist a compact subset of $I \setminus E_{n-1}$, with $\mu((I \setminus E_{n-1}) \setminus C_{N-1}) < \varepsilon_N/3$. In the Lemma set $E = C_{N-1}$, $F = \Phi$, $\varepsilon = \varepsilon_N/3$ to yield:

- a compact subset $K_N$ of $C_{N-1}$, with $\mu(C_{N-1} \setminus K_N) < \varepsilon_N/3$ and
- a function $f_1(t, x): K_N \times X \to Z$ such that

$$d(f_1(t, x), F(t, x)) < \varepsilon_N/3 < \varepsilon_N.$$

Consider now the set $E_{N-1} \times X$ and the mapping $\Phi: E_{N-1} \times X \to K(Z)$ defined by

$$\Phi(t, x) = F(t, x) \cap B[f_{N-1}(t, x), \varepsilon_{N-1}].$$

By our induction assumption, $\Phi(t, x)$ is non-empty. Moreover it is compact and convex. In addition it is u.s.c. in $t \in E_{N-1}$ for every fixed $x \in X$ (its graph is the intersection of two closed graphs and the range is contained in a compact set) and l.s.c. in $x$ for every fixed $t$ [1].

Applying the Lemma to $\Phi$, $E_{N-1}$ and $\varepsilon_N$, we infer the existence of a compact $K_N \subseteq E_{N+1}$, $\mu(E_{N-1} \setminus K_N) < \varepsilon_N$ and a $f^2: K_N \times X \to Z$ such that

$$d(f^2(t, x), \Phi(t, x)) < \varepsilon_N.$$

Hence for $f^2$ both

$$d(f^2(t, x), f_{N-1}(t, x)) < \varepsilon_{N-1}$$

and

$$d(f^2(t, x), F(t, x)) < \varepsilon_N \quad \text{hold}.$$
Set $E_N = K^1_N \cup K^2_N$ and define $f_N: E_N \times X \to Z$ by

$$f_N(t, x) = \begin{cases} f^1(t, x), & t \in K^1_N, \\ f^2(t, x), & t \in K^2_N. \end{cases}$$

We have that $\mu(I \setminus E_N) = \mu((E_{N-1} \setminus K^2_N) \cup ((I \setminus E_{N-1}) \setminus K^1_N)) \leq \varepsilon_N/3 + 2\varepsilon_N/3 = \varepsilon_N$, and the claim is proved.

Now set

$$A_N = \bigcup_{n=1}^{\infty} (I \setminus E_n).$$

Then $A_N \subset A_{N-1}$ and $\mu(\cap A_N) = \lim \mu(A_N) = 0$. Fix $t \notin \cap A_N$. Then $\{f_N(t, \cdot)\}$ is a Cauchy sequence of continuous functions and converges uniformly to a $q(t, x)$, continuous in $x$. Fix $x$. Then for every $t \notin \cap A_N$, $q(t, x)$ is the pointwise limit of $f_N(t, x)$, hence measurable. For $t \in \cap A_N$, let $\phi(t, \cdot)$ be any continuous selection from $F(t, \cdot)$ [1].

The function

$$f(t, x) = \begin{cases} q(t, x), & t \in I \setminus \cap A_N, \\ \phi(t, x), & t \in \cap A_N, \end{cases}$$

has the required properties. Q.E.D.

From Theorem 1 the following Theorem 2 can easily be proved:

**Theorem 2.** Let $Z$ be a finite dimensional space, $Q$ an open subset of $R \times Z$, $F: Q \to K(Z)$ be u.s.c. in $t$ for every fixed $x$ and l.s.c. in $x$ for every fixed $t$, $t$ and $x$ in $Q$. Moreover assume that the range of $F$ is contained in some compact subset of $Z$. Let $(t^0, x^0) \in Q$. Then the Cauchy problem

$$x' = F(t, x), \quad x(t^0) = x^0$$

admits at least one solution.

Also, applying a result of Scorza Dragoni [2] to the function $f$ of Theorem 1, the following Corollary can be derived:

**Corollary.** Let $I \subset R$ be compact, $X$ a separable metric space, $F: I \times X \to K(Z)$ be u.s.c. for every fixed $x \in X$, l.s.c. for every fixed $t \in I$. Then for every $\varepsilon > 0$ there exist $K_\varepsilon$, a compact subset of $I$ and a continuous $f_\varepsilon: K_\varepsilon \times X \to Z$ that is a selection from $F$. 
REFERENCES


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