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Hereditary orders

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Hereditary Orders.

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1. Class groups.

Throughout this article let R denote a Dedekind domain with quotient field K , and let $\mathcal{A}$ be an R -order in a separable K -algebra A . We shall give another approach to some results of Jacobinski [2] on hereditary orders. As general references for the background material needed for this article, the reader is referred to [3]-[5].

A (left) $\mathcal{A}$ -lattice is a left $\mathcal{A}$ -module M which is finitely generated and torsionfree as R -module. For each maximal ideal P of R , let R_P , M_P (and so on) denote P -adic completions. Two $\mathcal{A}$ -lattices M , N are in the same *genus* (notation: $M \sim N$) if for each P , $M_P \cong N_P$ as $\mathcal{A}_P$ -lattices. If $M \sim \mathcal{A}$, we call M *locally free* (of rank 1). Two $\mathcal{A}$ -lattices X , Y are *stably isomorphic* if $X + \mathcal{A}^{(k)} \cong Y + \mathcal{A}^{(k)}$ for some k . Let $[X]$ denote the stable isomorphism class of X .

The R -order $\mathcal{A}$ is *hereditary* if every left ideal of $\mathcal{A}$ is $\mathcal{A}$ -projective. We state without proof (see references):

(1) THEOREM. The following are equivalent:

- i) $\mathcal{A}$ is hereditary.
- ii) Every left $\mathcal{A}$ -lattice is projective.
- iii) Every right $\mathcal{A}$ -lattice is projective.

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The *locally free class group* $\text{Cl } A$ is defined to be the abelian group generated by $\{[M]: M \vee A\}$, with addition given by

$$[M] + [M'] = [M''] \quad \text{whenever } M + M' \cong A + M'.$$

Thus $[M] = 0$ in $\text{Cl } A$ if and only if M is stably isomorphic to A .

If A' is a maximal R -order in A , then the class group $\text{Cl } A'$ can be computed explicitly, especially when K is an algebraic number field, or when K is a function field and A satisfies the Eichler condition relative to R . In these cases, $\text{Cl } A'$ turns out to be a ray class group of the center of A' . Given any order A , we may find a maximal order A' in A with $A \subset A'$. It is obviously desirable to relate the two class groups $\text{Cl } A$ and $\text{Cl } A'$.

Whether or not K is a global field, each inclusion of orders $A \subset A'$ gives rise to a well defined homomorphism

$$\beta: \text{Cl } A \rightarrow \text{Cl } A', \quad \beta[X] = [A' \otimes_A X] \quad \text{for } [X] \in \text{Cl } A.$$

We now prove

(2) **THEOREM.** Let A be a hereditary order, and let A' be any R -order in A containing A . Then $\beta: \text{Cl } A \cong \text{Cl } A'$ is an isomorphism.

PROOF. (We do not assume that A' is a maximal order, nor that K is a global field.) Given any $[X'] \in \text{Cl } A'$, we may assume without loss of generality that X' is a A' -lattice in A . Let $u(A)$ denote the group of units of A . Since $X' \vee A'$, for each P we may write

$$X'_P = A'_P x_P \quad \text{for some } x_P \in u(A_P).$$

Furthermore, $X'_P = A'_P$ a.e. («almost everywhere»), so we may choose $x_P = 1$ a.e. Now set

$$X = A \cap \left\{ \bigcap_P A_P x_P \right\}.$$

Then X is a A -lattice in A such that $X_P = A_P x_P$ for all P , whence $[X] \in \text{Cl } A$. We have $A'X = X'$, since this holds locally at each P . Then $\beta[X] = [X']$, since $A' \otimes_A X \cong A'X$. This proves that β is an epimorphism.

Now let $S = \{P_1, \dots, P_n\}$ be the set of all P at which $A_P \neq A'_P$.

If $X \vee A$, then by Roiter's Lemma there is a A -exact sequence

$$(3) \quad 0 \rightarrow X \rightarrow A \rightarrow T \rightarrow 0,$$

where T is a left A -module such that $T_P = 0$ for all $P \in S$. Now A' is a right A -lattice, hence is A -projective by Theorem 1, since A is hereditary. Therefore the functor $A' \otimes_A \cdot$ is exact. Applying it to the sequence (3), we obtain a A' -exact sequence

$$(4) \quad 0 \rightarrow A'X \rightarrow A' \rightarrow A' \otimes_A T \rightarrow 0,$$

where we have identified $A' \otimes_A X$ with $A'X$.

We claim next that $A' \otimes_A T \cong T$ as left A -modules. Indeed, both are R -torsion modules, so we need only show that

$$A'_P \otimes_{A_P} T_P \cong T_P \quad \text{for all } P.$$

This is clear when $P \in S$, while for $P \notin S$ it follows from the fact that $A'_P = A_P$. This shows that $A' \otimes_A T \cong T$, so (4) may be rewritten as

$$(5) \quad 0 \rightarrow A'X \rightarrow A' \rightarrow T \rightarrow 0.$$

Comparing (3) and (5) and using Schanuel's Lemma, we obtain

$$(6) \quad X + A' \cong A + A'X \quad \text{as } A\text{-lattices.}$$

Now let $[X] \in \ker \beta$, so $[A'X] = 0$ in $\text{Cl } A'$. Thus $A'X$ is stably isomorphic to A' , so there exists a free A' -lattice F' such that

$$A'F + F' \cong A' + F'.$$

It follows from (6) that

$$(7) \quad X + (A' + F') \cong A + (A' + F').$$

Since $A' + F'$ is a A -lattice, and hence is A -projective, there exists a A -lattice L such that $A' + F' + L$ is a free A -lattice. Adding L to both sides of (7), it follows that X is stably isomorphic to A as

$\mathcal{A}$ -lattices. Therefore $[X] = 0$ in $\text{Cl } \mathcal{A}$. This completes the proof that β is an isomorphism.

For later use, we quote without proof

- (8) **THEOREM** (Jacobinski [1]). Let K be a global field, and let $\mathcal{A}$ be an arbitrary order. If $\mathcal{A}$ satisfies the Eichler condition relative to R , then $[X] = [Y]$ in $\text{Cl } \mathcal{A}$ if and only if $X \cong Y$.

2. Endomorphism rings.

For any $\mathcal{A}$ -lattice M , let $\text{End}_{\mathcal{A}} M$ denote its $\mathcal{A}$ -endomorphism ring. It is well known that if $\mathcal{A}'$ is a maximal R -order in $\mathcal{A}$, and M' is any $\mathcal{A}'$ -lattice, then $\text{End}_{\mathcal{A}'} M'$ is also a maximal order. We now establish a corresponding result for hereditary orders.

- (9) **THEOREM**. Let $\mathcal{A}$ be a hereditary order, and let M be any non-zero $\mathcal{A}$ -lattice. Then $\text{End}_{\mathcal{A}} M$ is a hereditary R -order in $\text{End}_{\mathcal{A}} KM$.

PROOF. It is clear that $\text{End}_{\mathcal{A}} M$ is an R -order in $\text{End}_{\mathcal{A}} KM$, any we must show that it is hereditary. Since M is $\mathcal{A}$ -projective, we may choose a $\mathcal{A}$ -lattice M' such that $M \oplus M' \cong \mathcal{A}^{(n)}$ for some n . Set $\Gamma = \text{End}_{\mathcal{A}} \mathcal{A}^{(n)}$; the projection map $e: \mathcal{A}^{(n)} \rightarrow M$ is then an idempotent in the ring Γ . We have

$$\text{End}_{\mathcal{A}} M \cong e\Gamma e.$$

Now $\mathcal{A}^{(n)}$ is a progenerator for the category of left $\mathcal{A}$ -modules, and hence Γ is Morita equivalent to $\mathcal{A}$. But Morita equivalence preserves the property of being hereditary, since by Theorem 1, an order $\mathcal{A}$ is hereditary if and only if every submodule of a finitely generated projective $\mathcal{A}$ -module is $\mathcal{A}$ -projective. It follows that Γ is hereditary, and it remains to prove that also $e\Gamma e$ is hereditary.

Let L be any left ideal in $e\Gamma e$; then $L \subset e\Gamma e \subset \Gamma$, and we can form the left ideal ΓL in Γ . Now L is finitely generated as $e\Gamma e$ -module, so there exists a left $e\Gamma e$ -epimorphism (for some r)

$$\varphi: (e\Gamma e)^{(r)} \rightarrow L, \quad \varphi(\alpha_1, \dots, \alpha_r) = \sum_{i=1}^r \alpha_i l_i, \quad \alpha_i \in e\Gamma e.$$

We may extend φ to a left Γ -epimorphism

$$\varphi': \Gamma^{(r)} \rightarrow \Gamma L, \quad \varphi'(\gamma_1, \dots, \gamma_r) = \sum_{i=1}^r \gamma_i l_i, \quad \gamma_i \in \Gamma.$$

But ΓL is Γ -projective since Γ is hereditary, and hence there exists a Γ -homomorphism $\psi': \Gamma L \rightarrow \Gamma^{(r)}$ such that $\varphi' \psi' = 1$ on ΓL . We may write

$$\psi'(y) = (\psi_1(y), \dots, \psi_r(y)), \quad y \in \Gamma L.$$

Then

$$y = \varphi' \psi'(y) = \sum_{i=1}^r \psi_i(y) l_i, \quad y \in \Gamma L.$$

Also we have

$$(10) \quad \psi_i(\gamma y) = \gamma \psi_i(y), \quad \gamma \in \Gamma, y \in \Gamma L,$$

so in particular

$$(11) \quad \psi_i(x) = e \psi_i(x), \quad x \in L.$$

Now define $\psi: L \rightarrow (e\Gamma e)^{(r)}$ by setting

$$\psi(x) = (\psi_1(x) e, \dots, \psi_r(x) e), \quad x \in L.$$

Each $\psi_i(x) e \in e\Gamma e$ by (11), and ψ is a left $e\Gamma e$ -homomorphism by (10). We have

$$\varphi \psi(x) = \sum_{i=1}^r \psi_i(x) e \cdot l_i = \sum_{i=1}^r \psi_i(x) l_i = x, \quad x \in L.$$

This shows that ψ splits the epimorphism φ , and proves that L is $e\Gamma e$ -projective. Hence $e\Gamma e$ is hereditary, and the theorem is established.

3. Restricted genus.

Two Λ -lattices M, N are in the same *restricted genus* if $M \vee N$ and $\Lambda' M \cong \Lambda' N$ for some maximal order Λ' containing Λ . We shall apply our preceding results to prove the following result of Jacobinski [2]:

- (12) **THEOREM.** Let A be a hereditary R -order in the separable K -algebra A , where K is a global field and where $\text{End}_A KM$ satisfies the Eichler condition relative to R . If M, N are A -lattices in the same restricted genus, then $M \cong N$.

PROOF. Let A' be a maximal order containing A , such that $A' M \cong A' N$. Let us set

$$\Gamma = \text{End}_A M, \quad \Gamma' = \text{End}_{A'} A' M, \quad B = \text{End}_A KM.$$

Then $\Gamma \subset \Gamma' \subset B$, and Γ' is a maximal order in the separable K -algebra B . The order Γ is hereditary, by Theorem 9.

Replacing N by αN for some nonzero $\alpha \in R$, we may assume hereafter that $N \subset M$. We shall view M as a left A -, right Γ -bimodule. Set

$$J = \text{Hom}_A(M, N).$$

Then J is a left ideal in Γ . We claim that $J \nmid \Gamma$, and that $N = MJ$.

For each P , there is an isomorphism $M_P \cong N_P$, and hence we may write

$$N_P = M_P u_P \quad \text{for some } u_P \in u(B_P).$$

Then

$$J_P = \text{Hom}_{A_P}(M_P, N_P) = \Gamma_P u_P,$$

which shows that $J \nmid \Gamma$. This also proves that $N = MJ$, since the equality holds locally at all P .

Now we observe that

$$\Gamma' J = \text{Hom}_{A'}(A' M, A' N),$$

since this holds locally at each P . But $A' N = A' M \cdot y$ for some $y \in u(B)$, since $A' M \cong A' N$. Therefore $\Gamma' J = \Gamma' y$, which shows that the element $[J] \in \text{Cl } \Gamma$ maps onto the zero element of $\text{Cl } \Gamma'$. Hence $[J] = 0$ by Theorem 2, and so J is stably isomorphic to Γ . Therefore $J = \Gamma z$ for some $z \in u(B)$ by Theorem 8, whence

$$N = MJ = M \cdot \Gamma z = Mz \cong M,$$

as desired. This completes the proof of the theorem.

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