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Remarks on Holomorphic Vector Fields on Non-Compact Manifolds.

GIULIANA GIGANTE (*)

Introduction.

Let M be a Kähler manifold and let Z and ω be respectively a holomorphic vector field and a holomorphic linear differential form on M . If M is compact, then the function $\omega(Z)$ is constant on M .

This fact yields some useful informations on the structure of the Lie algebra $\mathfrak{h}(M)$ of holomorphic vector fields on M , on the Lie algebra $\mathfrak{i}(M)$ of infinitesimal isometries on M and on the vanishing of certain cohomology groups on M [3].

The purpose of this note is that of extending some of the above results to the non-compact case. If M is a complete Kähler manifold more specific hypothesis are required for $\omega(Z)$ to be constant. We discuss in § 2 the case where $\omega(Z)$ is square summable on M , and the Ricci curvature is positive outside a compact of M , thus extending to the non-compact case some results of K. Yano [7].

Section 3 contains some results concerning the relationship between the zero set of Z and the vanishing of some cohomology group of M . Recent results of A. Lichnerowicz [5] and A. Howard [2] are extended to the non compact case.

In § 1, we discuss briefly some problems concerning $\mathfrak{i}(M)$ in the case when M is a complete Riemannian manifold.

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1. In this section, M will be a paracompact, connected, oriented manifold of dimension n , endowed with a positive definite, complete riemannian metric g of class C^∞ . We shall denote by C^r (resp. $\mathcal{D}^r$) the space of real C^∞ r -forms (resp. C^∞ r -forms with compact support); $*$: $C^r \rightarrow C^{n-r}$ is the canonical real operator, associated with the riemannian metric such that $**\varphi = (-1)^{r(n-1)}\varphi$, for any $\varphi \in C^r$. Then, for $x \in M$ and $\varphi, \psi \in C^r$: $(\varphi \wedge * \psi)_x = A_x(\varphi, \psi) dm(x)$, where dm is the volume element defined by the riemannian metric and $A_x(\varphi, \psi)$ is the scalar product defined by the riemannian metric g at x . Let $\mathcal{L}_r^2$ be the Hilbert space, which is the completion of $\mathcal{D}^r$ with respect to the norm $\|\varphi\|^2 = (\varphi, \varphi) = \int_M \varphi \wedge * \varphi = \int_M A(\varphi, \varphi) dm$; d : $C^r \rightarrow C^{r+1}$ denotes the exterior differentiation operator and δ : $C^r \rightarrow C^{r-1}$ —defined by

$$\delta\varphi = (-1)^{r-1} * d * \varphi,$$

for any $\varphi \in C^r$ —is its formal adjoint.

In [6], it is shown that, if W'_r denotes the completion of $\mathcal{D}^r$ with respect to the norm $\eta(\varphi)^2 = \|\varphi\|^2 + \|d\varphi\|^2 + \|\delta\varphi\|^2$, then $W'_r = \{\varphi \in \mathcal{L}_r^2: d\varphi \in \mathcal{L}_{r+1}^2, \delta\varphi \in \mathcal{L}_{r-1}^2\}$. The Laplace-Beltrami ⁽¹⁾ operator Δ : $C^r \rightarrow C^r$, defined by $\Delta = d\delta + \delta d$, is essentially selfadjoint and its selfadjoint extension, denoted by $\mathbf{\Delta}$, has domain: $W''_r = \{\varphi \in W'_r: \delta d\varphi \in \mathcal{L}_r^2, d\delta\varphi \in \mathcal{L}_{r-1}^2\}$.

For $\varphi \in C^1$, we denote by $R\varphi$ the differential 1-form defined locally by $(R\varphi)_i dx^i = (R^j_i \varphi_j) dx^i$, where R_{ij} are the local components of the Ricci tensor. We denote by ∇ , the covariant derivation with respect to the riemannian connection defined by g . Assume that the Ricci tensor R satisfies the condition: (α) $A_x(R\varphi, \varphi) \geq 0$ for any $\varphi \in \mathcal{D}^1$, outside a compact K of M . Then the following facts hold, for any $\varphi \in W'_1$:

- (i) $\|\nabla\varphi\| < \infty$, where $\|\nabla\varphi\|^2 = \int_M A_x(\nabla\varphi, \nabla\varphi) dm(x)$
- (ii) $\|\nabla\varphi\|^2 + (R\varphi, \varphi) = \|d\varphi\|^2 + \|\delta\varphi\|^2$; moreover if $\varphi \in W''_1$

$$\|\nabla\varphi\|^2 + (R\varphi, \varphi) = (\mathbf{\Delta}\varphi, \varphi).$$
- (iii) if $\varphi \in \mathcal{L}_1^2$ and $\Delta\varphi = 0$, then $|\varphi|$ is bounded.

Assume that R satisfies the stronger condition: (β) there exists

⁽¹⁾ These and further results on the behavior of $\mathbf{\Delta}$ will be found in a forthcoming paper of the author.

$\gamma > 0$, such that: $A_x(R\varphi, \varphi) \geq \gamma A_x(\varphi, \varphi)$, for any $\varphi \in \mathcal{D}^1$, outside a compact set K of M . Then $\mathfrak{L}_1^2 = H_1 \oplus d(\delta W_1'') \oplus \delta(dW_1'')$, where $H_1 = \{\text{kernel } \Delta\}$ has a finite dimension.

For any vector field $X = \sum_i \zeta^i (\partial/\partial x^i)$ on M , ζ shall denote the 1-form $\sum_i \zeta_i dx^i$, corresponding to X , under the duality defined by the metric g , i.e. $\eta(X) = A(\zeta, \eta)$, for any $\eta \in C^1$.

THEOREM 1.1. If condition (α) holds and $\zeta, R\zeta$ are in $\mathfrak{L}_1^2$, then conditions 1) $\Delta\zeta = 2R\zeta$, 2) $\delta\zeta = 0$, imply that X is an infinitesimal isometry (i.e. X generates a local 1-parameter group of local isometries).

PROOF. Since $\zeta \in W_1''$ and $\Delta\zeta = 2R\zeta$, then $\|\nabla\zeta\| < \infty$ and $(\Delta\zeta, \zeta) = \|\nabla\zeta\|^2 + (R\zeta, \zeta)$; so

$$(a) \quad (R\zeta, \zeta) - \|\nabla\zeta\|^2 = 0.$$

Moreover, it follows from a straight-forward computation that

$$(b) \quad -\text{div}(A_X X + (\text{div } X) X) = \sum_{i,j} (R_{ij} \zeta^i \zeta^j + \nabla_j \zeta^i \cdot \nabla_i \zeta^j - \nabla_i \zeta^i \cdot \nabla_j \zeta^j),$$

where $A_X = L_X - \nabla_X$, and L_x is the Lie derivation with respect to X .

Let η be the 1-form corresponding to $A_X X$, then $\int_M |\eta| \cdot dm < \infty$ since $\eta^i = -\sum_j (\nabla_j \zeta^i) \zeta^j$, $\int_M |\delta\eta| dm < \infty$ by (b) since $\text{div } X = 0$. So from Gaffney Lemma (cf. e.g. [6], p. 51)

$$(c) \quad (R\zeta, \zeta) - \|\nabla\zeta\|^2 = \frac{1}{2} \int_M \text{trace}((A_X + {}^t A_X)^2) dm.$$

Since $\text{trace}((A_X + {}^t A_X)^2)$ is the square of the length of the symmetric tensor $A_X + {}^t A_X$, and $\frac{1}{2} \int_M \text{trace}((A_X + {}^t A_X)^2) dm = 0$ by (a) and (c), then $A_X + {}^t A_X = 0$, which is equivalent to say that X is an infinitesimal isometry ([3], pag. 43).

REMARK 1.1. It is well known [3], that if X is an infinitesimal isometry, then 1) $\Delta\zeta = 2R\zeta$ and 2) $\delta\zeta = 0$.

REMARK 2.1. K. Yano has shown (in [7]) that, if V is a compact oriented riemannian manifold, every infinitesimal affine transformation is an infinitesimal isometry; J. Hano (in [1]) gave the following exten-

sion: if V is complete, every infinitesimal affine transformation with bounded length is an infinitesimal isometry. From theorem 1.1, it follows that:

COROLLARY 2.1. If condition (α) is satisfied, every infinitesimal affine transformation such that ζ and $R\zeta$ are in $\mathcal{L}_1^2$, is an infinitesimal isometry.

PROOF. Let X be an infinitesimal affine transformation, then $\Delta\zeta = 2R\zeta$ and $\delta\zeta$ is a constant function ([3], pag. 44-45). Since $\zeta \in W_1''$, we have $\delta\zeta \in \mathcal{L}^2$. If $\text{Vol } M = \infty$, then $\delta\zeta = 0$. If $\text{Vol } M < \infty$, then $\delta\zeta \in \mathcal{L}^1$ and $\zeta \in \mathcal{L}^1$, so from Gaffney Lemma $\int_M \delta\zeta \, dm = 0$, which implies $\delta\zeta = 0$.

REMARK 3.1. If the Ricci tensor is negative definite, then M has no infinitesimal isometry such that $\zeta \in \mathcal{L}^2$ and $R\zeta \in \mathcal{L}^2$. Indeed $\Delta\zeta = 2R\zeta$ implies $\zeta \in W_1''$ and $0 < 2(R\zeta, \zeta) = (\Delta\zeta, \zeta) = \|\delta\zeta\|^2 > 0$. In particular, if M is an Einstein manifold with $c < 0$, M has no infinitesimal isometry in $\mathcal{L}^2$.

2. Throughout this and the following section M will always be a paracompact, connected, complex manifold of dimension n , endowed with a complete Kähler metric. We shall denote by $C^{p,q}$ (resp. $\mathcal{D}^{p,q}$) the space of $C^\infty(p, q)$ -forms (resp. $C^\infty(p, q)$ -forms with compact support); the canonical isomorphism $*$: $C^{p,q} \rightarrow C^{n-q, n-p}$, defined by the star operator of the Riemannian structure underlying the hermitian structure of M , allows us to introduce in $\mathcal{D}^{p,q}$ the scalar product $\int_M \varphi \wedge * \bar{\psi}$.

That enables us to define, as in the Riemannian case, the spaces $\mathcal{L}_{(p,q)}^2$ and $W'_{(p,q)}$. The complex Laplace operator $\square$: $C^{p,q} \rightarrow C^{p,q}$, defined by $\square = \theta\bar{\partial} + \bar{\partial}\theta$, where θ , is the formal adjoint of $\bar{\partial}$, is essentially selfadjoint, and the domain of its selfadjoint extension is $W''_{(p,q)} = \{\varphi \in W'_{(p,q)}: \bar{\partial}\theta\varphi \in \mathcal{L}_{(p,q)}^2, \theta\bar{\partial}\varphi \in \mathcal{L}_{(p,q)}^2\}$. Let $R_{\alpha\bar{\beta}}$ be the Ricci tensor of the Kähler metric $g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$, $R_{\alpha\bar{\beta}} = -(\partial^2 \log g)/(\partial z^\alpha \partial \bar{z}^\beta)$.

THEOREM 1.2. Let the riemannian structure of M satisfy condition (α) of §1. Let $Z = \sum \zeta^\alpha (\partial/\partial z^\alpha)$ be a complex vector field of type $(1, 0)$ and let $\zeta = \sum \zeta_\alpha dz^\alpha$ ($\zeta_\alpha = g_{\alpha\bar{\beta}} \zeta^\beta$) be the corresponding $(0, 1)$ -form. Then, if $\zeta \in W''_{(0,1)}$ and $(\square\zeta, \zeta) = (R''\zeta, \zeta)$, where $R''\zeta = \sum R_{\alpha\bar{\beta}} \cdot \zeta^\alpha d\bar{z}^\beta$, Z is holomorphic.

PROOF. Since $\|\nabla\zeta\| < \infty$ and $(\square\zeta, \zeta) = \|\nabla''\zeta\|^2 + (R''\zeta, \zeta)$ (∇'' is defined for any tensor field K by the properties $\nabla K = \nabla'K + \nabla''K$, $\nabla''_W K = 0$, $\nabla'_W K = 0$ for all vectors W of type $(1, 0)$), then $\nabla''\zeta = 0$, i.e. $\nabla''Z = 0$ which implies that Z is holomorphic ([3] pag. 93).

REMARK 1.2. It is a well known fact ([3] pag. 93) that, if Z is holomorphic, then $\square\zeta = R''\zeta$.

Let Z be, as before, a complex vector field on M . Then $Z = X - iJX$, where X is a vector field on the underlying differentiable manifold, and J defines the complex structure of M . Let ζ be the corresponding $(1, 0)$ form to Z .

THEOREM 2.2. Suppose that M satisfies the hypothesis of theorem 1.2. If $\zeta \in \mathcal{L}_{(0,1)}^2$ and $R''\zeta \in \mathcal{L}_{(0,1)}^2$, X is an infinitesimal isometry if and only if Z is holomorphic and $\operatorname{div} X = 0$.

PROOF. The 1-form corresponding to X is $\frac{1}{2}(\zeta + \bar{\zeta})$. By Remark 1.1 if X is an infinitesimal isometry, $\operatorname{div} X = 0$ and

$$\Delta(\zeta + \bar{\zeta}) = 2 \sum R_{ij}(\zeta + \bar{\zeta})^j dx^i,$$

then $\square\zeta = R''\zeta$, so $\zeta \in W_{(0,1)}''$ and $(\square\zeta, \zeta) = (R''\zeta, \zeta)$ and Z is holomorphic by theorem 1.2. If Z is holomorphic and $\operatorname{div} X = 0$, then by Remark 1.2 $\zeta \in W_{(0,1)}''$ and $\square\zeta = R''\zeta$; so $\frac{1}{2}(\zeta + \bar{\zeta}) \in W_1''$ and $\Delta(\zeta + \bar{\zeta}) = 2 \sum_{i,j} R_{ij}(\zeta + \bar{\zeta})^j dx^i$.

The conclusion follows from theorem 1.1.

THEOREM 3.2. If condition (β) of § 1 holds, and if Z is a holomorphic vector such that $\zeta \in \mathcal{L}_{(0,1)}^2$, then:

- i) if $\operatorname{vol} M = \infty$, there exists a unique $f \in \mathcal{L}_{(0,0)}^2$ such that $\zeta = \bar{\partial}f$.
- ii) if $\operatorname{vol} M < \infty$, there exists $f \in \mathcal{L}_{(0,0)}^2$, unique after the normalization $\int_M f \cdot dm = 0$, such that $\zeta = H\zeta + \bar{\partial}f$, where $\Delta H\zeta = 0$; moreover if zero Z , the zero set of Z , is $\neq \emptyset$, then $\zeta = \bar{\partial}f$.

PROOF. Since M is a Kähler manifold, condition (β) enables us to write $\zeta = H\zeta + \bar{\partial}(\theta\eta)$, $\theta\eta \in \mathcal{L}_{(0,0)}^2$ and $\Delta H\zeta = 0$, moreover $|H\zeta|$ is bounded (§ 1). Therefore, $\overline{H\zeta}(Z)$ is a constant, since it is a holomorphic function in $\mathcal{L}_{(0,0)}^2$. (This constant is zero if $\operatorname{vol} M = \infty$. It vanishes also

when $\text{vol } M < \infty$, provided that $\text{Zero } Z \neq \emptyset$). But $\overline{H\zeta}(Z)(z) = A(H\zeta, \zeta)(z) = |H\zeta|^2(z) + A(H\zeta, \bar{\partial}f)(z)$ then $\|H\zeta\|^2 = \int_M \overline{H\zeta}(Z) dm$.

As for the uniqueness of f , let g be in $\mathcal{L}_{(0,0)}^2$ such that $\zeta = H\zeta + \bar{\partial}g$, then $\bar{\partial}(f-g) = 0$ so $f-g$ is constant in $\mathcal{L}_{(0,0)}^2$.

THEOREM 4.2. If M is as in theorem 3.2 and $\text{vol } M < \infty$, then $\zeta = \bar{\partial}f$ if, and only if, $\alpha(Z) = 0$, for any holomorphic 1-form in $\mathcal{L}_{(1,0)}^2$.

PROOF. Since α is harmonic, then $|\alpha|$ is bounded; so $\alpha(Z)$ is a holomorphic function in $\mathcal{L}_{(0,0)}^2$. Then $\alpha(Z)(z) = k$ and

$$\int_M k dm = \int_M A(\bar{\alpha}, \bar{\partial}f)(z) dm = 0,$$

so $k = 0$. To prove the converse, just take $\alpha = H\bar{\zeta}$.

THEOREM 5.2. Under the hypothesis on M and Z of theorem 3.2 and with the above notation, X in an infinitesimal isometry if, and only if, the real part of f : $\text{Re } f$, is a constant.

PROOF. By theorem 2.2, we need only show that $\text{div } X = 0$ if, and only if, $\text{Re } f = \text{constant}$. Indeed $\delta(\zeta + \bar{\zeta}) = \delta(\bar{\partial}f + \partial\bar{f}) = \Delta(\text{Re } f)$; so $\text{div } X = 0 \Leftrightarrow \delta(\zeta + \bar{\zeta}) = 0 \Leftrightarrow \Delta(\text{Re } f) = 0 \Leftrightarrow \text{Re } f = \text{constant}$, since $\text{Re } f$ belongs to $\mathcal{L}_{(0,0)}^2$.

3. Some applications.

In this section, M will be as at the beginning of § 2. Let $h(M)$ and $i(M)$ be respectively the space of holomorphic vector fields and of infinitesimal isometries on M .

THEOREM 1.3. If M is a Kähler manifold and $R_{\bar{g}} = 0$ then $h(M) \cap \mathcal{L}^2$ coincides with $i(M) \cap \mathcal{L}^2$, and consists of parallel vector fields (for the Riemannian connection).

Moreover, if $\text{vol } M = \infty$, they vanish.

PROOF. If $Z \in h(M) \cap \mathcal{L}^2$, then $\square\zeta = 0$ and $\bar{\zeta}(Z)(z) = |\zeta|^2(z)$ is a harmonic function in $\mathcal{L}_{(0,0)}^2$, so it is constant. ($\zeta = 0$ if $\text{vol } M = \infty$)

and since $\Delta(|\zeta|^2) = 2A(\Delta\zeta, \zeta) - 2A(R\zeta, \zeta) - 2|\nabla\zeta|^2$, we have $\nabla\zeta = 0$. Moreover, since $\zeta = H\bar{\zeta}$, then $\delta(\zeta + \bar{\zeta}) = 0$, which implies $Z \in i(M) \cap \mathfrak{L}^2$.

THEOREM 2.3. Let M be a Kähler manifold, which satisfies the condition (α) , with $K = \Phi$. If Z is a non-trivial holomorphic vector field, such that the corresponding form ζ is in $\mathfrak{L}_{(0,1)}^2$, then any holomorphic n -form φ in $\mathfrak{L}_{(n,0)}^2$ vanishes if $\text{Zero } Z \neq \emptyset$.

PROOF. If φ is an n -holomorphic form in $\mathfrak{L}_{(n,0)}^2$, $|\varphi|$ must be constant, since $(\Delta\varphi, \varphi) = \|\nabla\varphi\|^2$. At this point, we can remark that if $\text{vol } M = \infty$, φ must vanish, without any further hypothesis on the holomorphic vector fields. If $\text{vol } M < \infty$, then $\zeta = H\bar{\zeta} + \lim_n \bar{\partial}f_n$ in $\mathfrak{L}_{(0,1)}^2$ where f_n has compact support, and $H\bar{\zeta} = 0$. Since condition (α) holds, $|H\bar{\zeta}|$ is bounded and $\overline{H\bar{\zeta}}(Z)$ is a holomorphic function which belongs to $\mathfrak{L}_{(0,0)}^2$. Then $\overline{H\bar{\zeta}}(Z)$ is a constant, equal to zero if $\text{Zero } (Z) \neq \emptyset$. So $\zeta = \lim_n \bar{\partial}f_n$; since $d\varphi = 0$ and the $(n-1)$ -holomorphic form $i_Z(\varphi)$ (where i_Z denotes the interior product with respect to Z) is in $\mathfrak{L}_{(n-1,0)}^2$; then $d \circ i_Z(\varphi) = 0$ and $L_Z(\varphi) = d \circ i_Z(\varphi) + i_Z \circ d(\varphi) = 0$. Moreover, since $i_Z(*\bar{\varphi}) = 0$ and $d(*\bar{\varphi}) = 0$: $L_Z(*\bar{\varphi}) = 0$. Hence $L_Z(\bar{f}_n|\varphi|^2 dm) = Z(\bar{f}_n)|\varphi|^2 dm$ for all n , and by Stokes's theorem:

$$0 = \int_M A(\bar{\partial}f_n, |\varphi|^2 \zeta) dm = |\varphi|^2 \int_M A(\bar{\partial}f_n, \zeta) dm = |\varphi|^2 \langle \bar{\partial}f_n, \zeta \rangle \rightarrow |\varphi|^2 \|\zeta\|^2.$$

REMARK 3.3. The condition (α) has been used in the above proof only to grant that $|H\bar{\zeta}|$ is bounded. So theorem 3.3 still holds if condition (α) on the Ricci curvature of M , is replaced by the condition that the holomorphic vector field Z with $\text{Zero}(Z) \neq \emptyset$ has a bounded length.

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