Rendiconti del Seminario Matematico della Università di Padova

William D. L. Appling

Upper and lower integral difference functionals, closest approximations, and integrability

Rendiconti del Seminario Matematico della Università di Padova, tome 52 (1974), p. 1-24

<http://www.numdam.org/item?id=RSMUP_1974__52__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 52 (1974)

Upper and Lower Integral Difference Functionals, Closest Approximations, and Integrability.

WILLIAM D. L. APPLING (*)

1. Introduction.

Suppose U is a set, $\boldsymbol{F}$ is a field of subsets of U, $\mathfrak{p}$ is the set of all functions from $\boldsymbol{F}$ into $\exp(\mathbb{R})$, $\mathfrak{p}_B$ is the set of all elements of $\mathfrak{p}$ with bounded range union, $\mathfrak{p}_{AB}$ is the set of all bounded finitely additive functions from $\boldsymbol{F}$ into $\mathbb{R}$ and $\mathfrak{p}_A^+$ is the set of all nonnegative-valued elements of $\mathfrak{p}_{AB}$.

Suppose that for each β in $\mathfrak{p}$, $\sum$-bounded on U with respect to the subdivision $\mathfrak{D}$ of U (section 2), L and G are functions from $\boldsymbol{F}$ into $\mathbb{R}$ such that if V is in $\boldsymbol{F}$, then $L(\beta)(V)$ and $G(\beta)(V)$ are defined, respectively, as the sup and inf of the set:

$\left\{\sum_{\mathfrak{E}} \beta(I)\colon\ \mathfrak{E}\right.$ a subdivision of V and a subset of a refinement of $\mathfrak{D}$ $b(I)$ in $\beta(I)$ for each I in $\left.\mathfrak{E}\right\}$.

We pause here to remark that in section 2 of this paper we shall either discuss the notions, such as the immediatly preceding one, and assertions that appear in this introduction, or refer the reader to certain previous papers for them. We let « $\mathfrak{E} \ll \mathfrak{D}$ » mean « $\mathfrak{E}$ is a refinement of $\mathfrak{D}$ ».

Certain refinement-sum inequalities imply that if β is in $\mathfrak{p}$ and is $\sum$-bounded on U with respect to a subdivision $\mathfrak{D}$ of U, then, for each V

(*) Indirizzo dell'A.: Dept. of Mathematics, North Texas State University, Denton, Txas 76203, U.S.A.

in F, each of the integrals (section 2)

$$\int_V L(\beta)(I) \quad \text{and} \quad \int_V G(\beta)(I)$$

exist, and

$$\int_V G(\beta)(I) \leqslant \int_V L(\beta)(I) \ ;$$

furthermore, if V is in F, then

$$\int_V \beta(I)$$

exists iff

$$\int_V L(\beta)(I) = \int_V G(\beta)(I) \, ,$$

in which case

$$\int_V L(\beta)(I) = \int_V \beta(I) = \int_V G(\beta)(I) \, .$$

In a previous paper [4] the author proved the following theorem:

THEOREM 1.A.1. If N is a positive integer and f is a function from $\mathbb{R}^N$ into $\mathbb{R}$, then the following two statements are equivalent:

1) If $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space and $\alpha_1, \ldots, \alpha_N$ is a sequence of functions from $\boldsymbol{F}$ into $\mathbb{R}$ (or, for that matter, into $\exp(\mathbb{R})$) with bounded range (or bounded range union) such that if $i = 1, \ldots, N$, then

$$\int_U \alpha_i(I) \, \mu(I)$$

exists, then

$$\int_U f(\alpha_1(I), \ldots, \alpha_N(I)) \, \mu(I)$$

exists, and

2) f is continuous.

Consider the following abstraction of statement 1) of the above theorem:

(*): Suppose that N is a positive integer and Y is a function from the set of all N-tuples of elements of $\mathfrak{p}_B$ into $\mathfrak{p}_B$ such that if η is in $\mathfrak{p}_{AB}$ and $\alpha_1, \ldots, \alpha_N$ is a $\mathfrak{p}_B$-N-tuple such that for $i = 1, \ldots, N$,

$$\int_U \alpha_i(I)\,\eta(I)$$

exists, then

$$\int_U Y(\alpha_1, \ldots, \alpha_N)(I)\,\eta(I)$$

exists (We note, and refer the reader to Theorem 2.A.1 of section 2 of this paper, that (*) is equivalent to a statement of (*) in which $\mathfrak{p}_{AB}$ is replaced by $\mathfrak{p}_A^+$.).

In view of the fact that a function, W, satisfying the conditions of (*) can be constructed by simply choosing, for each $\mathfrak{p}_B$-N-tuple $\alpha_1, \ldots, \alpha_N$, a continuous function $f_{\alpha_1,\ldots,\alpha_N}$ from $\mathbb{R}^N$ into $\mathbb{R}$ and letting $W(\alpha_1, \ldots, \alpha_N)$ be given by

$$W(\alpha_1, \ldots, \alpha_N)(I) = f_{\alpha_1,\ldots,\alpha_N}(\alpha_1(I), \ldots, \alpha_N(I))\,,$$

we shall confine our attention to a $\mathfrak{p}_B$-N-tuple $\alpha_1, \ldots, \alpha_N$, an element β of $\mathfrak{p}_B$ such that if η is in $\mathfrak{p}_{AB}$ and if $i = 1, \ldots, N$, then

$$\int_U \alpha_i(I)\,\eta(I)$$

exists, then

$$\int_U \beta(I)\,\eta(I)$$

exists, and deduce a continuity-type statement that involves a notion of « dominated equi-integrability ». Specifically, we show the following (section 6):

THEOREM 6.1. Suppose N is a positive integer, β is in $\mathfrak{p}_B$ and α_1, $\ldots, \alpha_N$ is a sequence of elements of $\mathfrak{p}_B$ such that if η is in $\mathfrak{p}_{AB}$ and each of the integrals for $i = 1, \ldots, N$

$$\int_U \alpha_i(I)\,\eta(I)$$

exists, then

$$\int_U \beta(I)\,\eta(I)$$

exists. Then, if $0 < c$ and μ is in $\mathfrak{p}_A^+$, then there is $d > 0$ such that if $\varkappa$ is in $\mathfrak{p}_{AB}$, for all V in F,

$$\int_V |\varkappa(I)| \leqslant \mu(V)\,,$$

and for $i = 1, \ldots, N$,

$$\int_U [L(\alpha_i\varkappa)(I) - G(\alpha_i\varkappa)(I)] < d\,,$$

then

$$\int_U [L(\beta\varkappa)(I) - G(\beta\varkappa)(I)] < c\,.$$

It is trivial that the conclusion of the above theorem implies the hypothesis, so we forbear stating the theorem as a characterization theorem. We note (see Theorem 2.A.1 of section 2) that the hypothesis of Theorem 6.1 is equivalent to a statement of the hypothesis in which $\mathfrak{p}_{AB}$ is replaced by $\mathfrak{p}_A^+$.

One of the principal facts what we shall use in proving Theorem 6.1 is the fact (see section 3) that if γ is in $\mathfrak{p}_B$, then the set $\mathfrak{J}_\gamma$ of all η in $\mathfrak{p}_{AB}$ such that

$$\int_U \gamma(I)\,\eta(I)$$

exists, is a C-set in accordance with the following definition:

DEFINITION. The statement that M is a C-set means that $M \subseteq \mathfrak{p}_{AB}$ such that:

1) If η is in M, $\varkappa$ is in $\mathfrak{p}_{AB}$ and $\int|\eta| - \int|\varkappa|$ is in $\mathfrak{p}_A^+$, then $\varkappa$ is in M, and

2) if μ is in $\mathfrak{p}_A^+$ and ν is a function from F into $\mathbb{R}$ given by

$$\nu(V) = \sup\{\varkappa(V) : \varkappa \text{ in } M \cap \mathfrak{p}_A^+,\ \mu - \varkappa \text{ in } \mathfrak{p}_A^+\},$$

then ν is in $M \cap \mathfrak{p}_A^+$.

In [5] the following « nearest point theorem » is shown:

THEOREM 1.A.2. Suppose M is a C-set and for each η in $\mathfrak{p}_{AB}$, $\tau(\eta)$ and $\lambda(\eta)$ are functions from $\boldsymbol{F}$ into $\mathbb{R}$ given, respectively, by

$$\tau(\eta)(I) = \begin{cases} 1 & \text{if } 0 \leqslant \eta(I)\,, \\ -1 & \text{if } \eta(I) < 0\,, \end{cases}$$

$$\lambda_M(\eta)(I) = \sup\left\{\mu(I) : \mu \text{ in } M \cap \mathfrak{p}_A^+,\ \int |\eta| - \mu \text{ in } \mathfrak{p}_A^+\right\}.$$

Then there is a transformation $\boldsymbol{a}_M$ from $\mathfrak{p}_{AB}$ into M such that if η is in $\mathfrak{p}_{AB}$ and V is in F, then

$$\boldsymbol{a}_M(\eta)(V) = \int_V \tau(\eta)(I)\,\lambda_M(\eta)(I)\,.$$

Furthermore, if η is in $\mathfrak{p}_{AB}$, $\varkappa$ is in M and

$$\varkappa \neq \boldsymbol{a}_M(\in)\,,$$

then

$$\int_U |\eta(I) - \boldsymbol{a}_M(\eta)(I)| < \int_U |\eta(I) - \varkappa(I)|\,.$$

Throughout this paper, for each C-set M, we shall let λ_M and $\boldsymbol{a}_M$ denote the λ_M and $\boldsymbol{a}_M$ associated with M by Theorem 1.A.1. Note that if M is a C-set, and η is in $\mathfrak{p}_{AB}$, then

$$\boldsymbol{a}_M\Big(\int|\eta|\Big) = \lambda_M\Big(\int|\eta|\Big) = \lambda_M(\eta)\,.$$

As stated above, and proved in section 3, for each γ in $\mathfrak{p}_B$, $\mathfrak{J}_\gamma$ is a C-set. Another fact that we shall use in proving Theorem 6.1, is the following theorem which establishes a « dominated convergence » relationship between the upper and lower integral difference functional and $\boldsymbol{a}_{\mathfrak{J}_\gamma}$.

THEOREM 5.1. If γ is in p_B, $\varkappa$ is in $\mathfrak{p}_A^+$ and $0 < c$, then there is $d > 0$ such that if η is in $\mathfrak{p}_{AB}$, $\varkappa - \int|\eta|$ is in $\mathfrak{p}_A^+$ and

$$\int[L(\gamma\eta)(I) - G(\gamma\eta)(I)] < d\,,$$

then

$$\int|\eta(I) - a_{\mathfrak{J}_\gamma}(\eta)(I)| < c\,.$$

2. Preliminary theorems and definitions.

Throughout this paper all integrals will be limits for (finite) refinements of (finite) subdivisions, of the appropriate sums. We refer the reader to [2] and [3] for more detailed remarks about the above mentioned notions as well as for various refiniment-sum inequalities and the integral existence essertions that follow from them, such as, for example, the existence of the « upper » and « lower » integrals mentioned in the introduction. The reader is also referred to [3] for a statement of Kolmogoroff's [7] differential equivalence theorem as well as various integral existence and integral equivalence assertions that follow from it. If, in a given argument in one of the subsequent sections of this paper, the existence of an integral or the equivalence of an integral to an integral is an easy consequence of the above mentioned material, we shall feel free to merely write the integral or make the equivalence assertion and leave the proof to the reader.

We end this section with an extension of a previous interval function theorem of the author [1]. The argument for this extension carries over from the interval function case with only minor modifications and we therefore omit it.

THEOREM 2.A.1. If γ is in $\mathfrak{p}_B$ and η is in $\mathfrak{p}_{AB}$, then

$$\int_U \gamma(I)\,\eta(I)$$

exists iff

$$\int_U \gamma(I) \int_I |\eta(J)|$$

exists.

3. Functional equations, inequalities and continuity properties of upper and lower integral difference functionals.

We begin by stating two easily proved lemmas. Suppose each of α and β is in $\mathfrak{p}$ and is $\sum$-bounded on U with respect to $\mathfrak{D} \ll \{U\}$.

LEMMA 3.1. If V is in $\boldsymbol{F}$, then

$$\int_V G(\alpha)(I) + \int_V G(\beta)(I) \leqslant \int_V G(\alpha+\beta)(I) \leqslant \int_V L(\alpha+\beta)(I) \leqslant \int_V L(\alpha)(I) + \int_V L(\beta)(I) ,$$

so that

$$0 \leqslant \int_V [L(\alpha+\beta)(I) - G(\alpha+\beta)(I)] \leqslant \int_V [L(\alpha)(I) - G(\alpha)(I)] + \int_V [L(\beta)(I) - G(\beta)(I)] .$$

LEMMA 3.2. If s in is $\mathbb{R}$, then $s\alpha$ is $\sum$-bounded on U with respect to $\mathfrak{D}$. Suppose V is in $\boldsymbol{F}$. If $0 \leqslant s$, then

$$\int_V L(s\alpha)(I) = s\int_V L(\alpha)(I) \quad \text{and} \quad \int_V G(s\alpha)(I) = s\int_V G(\alpha)(I) ;$$

and if $s < 0$, then

$$\int_V L(s\alpha)(I) = s\int_V G(\alpha)(I) \quad \text{and} \quad \int_V G(s\alpha)(I) = s\int_V L(\alpha)(I) ,$$

so that, in either case,

$$\int_V [L(s\alpha)(I) - G(s\alpha)(I)] = |s|\int_V [L(\alpha)(I) - G(\alpha)(I)] .$$

THEOREM 3.1. Suppose β is in $\mathfrak{p}_B$, η is in $\mathfrak{p}_{AB}$, K is in $\mathbb{R}$, $|x| \leqslant K$ for all x in the range union of β, and V is in $\boldsymbol{F}$. Then

$$\left|\int_V L(\beta\eta)(I)\right| \leqslant K\int_V |\eta(I)| \quad \text{and} \quad \left|\int_V G(\beta\eta)(I)\right| \leqslant K\int_V |\eta(I)| ,$$

so that

$$\int_V [L(\beta\eta)(I) - G(\beta\eta)(I)] \leqslant 2K \int_V |\eta(I)| \, .$$

PROOF. Suppose P is either L or G. It is easy to see that if I is in $\boldsymbol{F}$, then

$$|P(\beta\eta)(I)| \leqslant K \int_I |\eta(J)| \, ,$$

so that if $\mathfrak{D} \ll \{V\}$, then

$$\left| \sum_{\mathfrak{D}} P(\beta\eta)(I) \right| \leqslant \sum_{\mathfrak{D}} |P(\beta\eta)(I)| \leqslant K \sum_{\mathfrak{D}} \int_I |\eta(J)| = K \int_V |\eta(J)| \, .$$

Therefore

$$\left| \int_V P(\beta\eta)(I) \right| \leqslant K \int_V |\eta(I)| \, ,$$

so that

$$\int_V [L(\beta\eta)(I) - G(\beta\eta)(I)] \leqslant \left| \int_V L(\beta\eta)(I) \right| + \left| \int_V G(\beta\eta)(I) \right| \leqslant 2K \int_V |\eta(I)| \, .$$

THEOREM 3.2. Suppose β is in $\mathfrak{p}_B$, each of η and $\varkappa$ is in $\mathfrak{p}_{AB}$, $|x| \leqslant K$ for all x in the range union of β and V is in $\boldsymbol{F}$. Then

$$\left| \int_V [L(\beta\eta)(I) - G(\beta\eta)(I)] - \int_V [L(\beta\varkappa)(I) - G(\beta\varkappa)(I)] \right| \leqslant 2K \int_V |\eta(I) - \varkappa(I)| \, .$$

PROOF. By Lemma 3.1, we see that

$$\int_V [L(\beta\eta)(I) - G(\beta\eta)(I)] - \int_V [L(\beta\varkappa)(I) - G(\beta\varkappa)(I)] \leqslant$$
$$\leqslant \int_V [L(\beta(\eta - \varkappa))(I) - G(\beta(\eta - \varkappa))(I)]$$

and

$$\int_V [L(\beta\varkappa)(I) - G(\beta\varkappa)(I)] - \int_V [L(\beta\eta)(I) - G(\beta\eta)(I)] \leqslant$$
$$\leqslant \int_V [L(\beta(\varkappa - \eta))(I) - G(\beta(\varkappa - \eta))(I)] \, .$$

By Theorem 3.1, the right expression of each of the above inequalities does not exceed

$$2K\int_V |\eta(I)-\varkappa(I)|\,.$$

Since the left expression of the inequality of the statement of Theorem 3.2 is the left expression of one of the above inequalities, the theorem follows.

We prove a homogeneity theorem that we shall use in section 5. We first state a lemma which is a fairly easy consequence of the Bochner-Radon-Nikodym Theorem. A proof of this lemma can be found in [6].

LEMMA 3.A.1. If α is in $\mathfrak{p}_B$, μ is in $\mathfrak{p}_A^+$ and $\int_U \alpha(I)\mu(I)$ exists, then

$$\int_U\Big[\int_V |\alpha(V)\mu(I)-\int_I \alpha(J)\mu(J)|\Big]=0\,,$$

i.e., if $0<c$, then there is $\mathfrak{D}\ll\{U\}$ such that if $\mathfrak{E}\ll\mathfrak{D}$ and for each V in $\mathfrak{E}$, $a(V)$ is in $\alpha(V)$, then

$$\sum_{\mathfrak{E}}\int_V\Big|\alpha(V)\mu(I)-\int_I \alpha(J)\mu(J)\Big|<c\,.$$

THEOREM 3.3. Suppose T is a transformation from $\mathfrak{p}_{AB}$ into $\mathfrak{p}_{AB}$ and K is a number such that if each of η and $\varkappa$ is in $\mathfrak{p}_{AB}$ and V is in $\boldsymbol{F}$, then

$$|T(\eta)(V)-T(\varkappa)(V)|\leqslant K\int_V |\eta(I)-\varkappa(I)|\,.$$

Suppose α is in $\mathfrak{p}_B$, λ is in $\mathfrak{p}_{AB}$ and $\int_U \alpha(I)\lambda(I)$ exists. Then

$$\int_U\Big|T\Big(\int\alpha\lambda\Big)(I)-T(\alpha(I)\lambda)(I)\Big|=0\,,$$

i.e., if $0<c$, then there is $\mathfrak{D}\ll\{U\}$ such that if $\mathfrak{E}\ll\mathfrak{D}$ and a is a

function from $\mathfrak{E}$ such that $a(I)$ is in $\alpha(I)$ for all I in $\mathfrak{E}$, then

$$\sum_{\mathfrak{E}} \left| T\left(\int \alpha\lambda\right)(I) - T(\alpha(I)\,\lambda)(I) \right| < c\,.$$

Proof. Let μ denote the element of $\mathfrak{p}_A^+$ given by

$$\mu(I) = \int_I |\lambda(J)|\,.$$

By Theorem 2.A.1,

$$\int_U \alpha(I)\,\mu(I)$$

exists, so that, by Lemma 3.A.1,

$$\int_U \left[\int_V \left| \alpha(V)\,\mu(I) - \int_I \alpha(J)\,\mu(J) \right| \right] = 0\,.$$

Now, suppose $0 < c$. There is $\mathfrak{D} \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}$ and for each V in $\mathfrak{E}$, $a(V)$ is in $\alpha(V)$, then

$$\sum_{\mathfrak{E}} \int_V \left| a(V)\mu(I) - \int_I \alpha(J)\mu(J) \right| < c/(K+1)\,,$$

so that

$$\sum_{\mathfrak{E}} \left| T\left(\int \alpha\lambda\right)(V) - T(\alpha(V)\,\lambda)(V) \right| \leqslant \sum_{\mathfrak{E}} K \int_V \left| \int_I \alpha(J)\,\lambda(J) - a(V)\,\lambda(I) \right| =$$

$$= K \sum_{\mathfrak{E}} \int_V |\alpha(I)\,\lambda(I) - a(V)\,\lambda(I)| = K \sum_{\mathfrak{E}} \int_V |\alpha(I) - a(V)|\,|\lambda(I)| =$$

$$= K \sum_{\mathfrak{E}} \int_V |\alpha(I) - a(V)| \int_I |\lambda(J)| = K \sum_{\mathfrak{E}} \int_V |\alpha(I) - a(V)|\,\mu(I) =$$

$$= K \sum_{\mathfrak{E}} \int_V |\alpha(I)\mu(I) - a(V)\mu(I)| = K \sum_{\mathfrak{E}} \int_V \left| \int_I \alpha(J)\mu(J) - a(V)\mu(I) \leqslant \right.$$

$$\leqslant Kc/(K+1) < c\,.$$

Therefore

$$\int_U \left| T\left(\int \alpha\lambda \right)(I) - T(\alpha(I)\,\lambda)(I) \right| = 0\,.$$

Now suppose β is in $\mathfrak{p}_B$ and for each $\varkappa$ in $\mathfrak{p}_{AB}$, $Z(\varkappa)$ is the element of $\mathfrak{p}_{AB}$ given by

$$Z(\varkappa)(V) = \int_V [L(\beta\varkappa)(I) - G(\beta\varkappa)(I)]\,.$$

We now use Theorems 3.2 and 3.3 to show a homogeneity assertion about Z that we shall use in proving Theorem 5.1.

THEOREM 3.4. If α is in $\mathfrak{p}_B$, $\varkappa$ is in $\mathfrak{p}_{AB}$, and $\int_U \alpha(I)\varkappa(I)$ exists, then for each V in $\boldsymbol{F}$,

$$\int_V |\alpha(I)|Z(\varkappa)(I)$$

exists and is $Z\left(\int \alpha\varkappa\right)(V)$.

PROOF. By Theorem 3.2, for each η and ι in $\mathfrak{p}_{AB}$ and V in $\boldsymbol{F}$, and $K \geqslant |x|$ for all x in β's range union,

$$|Z(\eta)(V) - Z(\iota)(V)| \leqslant 2K \int_V |\eta(I) - \iota(I)|\,,$$

so that by Theorem 3.3,

$$\int_U \left| Z\left(\int \alpha\varkappa \right)(I) - Z(\alpha(I)\,\varkappa)(I) \right| = 0\,,$$

which clearly implies that if V is in $\boldsymbol{F}$, then

$$\int_V Z(\alpha(I)\,\varkappa)(I)$$

exists and is

$$Z\left(\int \alpha\varkappa \right)(V)\ ;$$

but by Lemma 3.1, for each I in $\boldsymbol{F}$ and x in $\mathbb{R}$.

$$Z(x\varkappa)(I) = |x|Z(\varkappa)(I)\,,$$

so that

$$\int_V Z(\alpha(I)\varkappa)(I) = \int_V |\alpha(I)|Z(\varkappa)(I)\,,$$

so that

$$\int_V |\alpha(I)|Z(\varkappa)(I)$$

exists and is

$$Z\left(\int \alpha\varkappa\right)(V)\,.$$

We state two immediate corollaries of Theorem 3.4.

Corollary 3.5. If η is in $\mathfrak{p}_{AB}$ and V is in $\boldsymbol{F}$, then

$$Z(\eta)(V) = Z\left(\int|\eta|\right)(V)\,.$$

Proof. Let B be the function on $\boldsymbol{F}$ given by

$$B(I) = \begin{cases} 1 & \text{if } \eta(I) \geqslant 0\,, \\ -1 & \text{if } \eta(I) < 0\,, \end{cases}$$

For each I in $\boldsymbol{F}$,

$$B(I)\,\eta(I) = |\eta(I)|\,,$$

so that if V is in $\boldsymbol{F}$, then $\int_V B(I)\eta(I)$ exists and is $\int_V|\eta(I)|$. By Theorem 3.4, if V is in $\boldsymbol{F}$, then $\int_V|B(I)|Z(\eta)(I)$ exists and is $Z\left(\int B\eta\right)(V)$, which is $Z\left(\int|\eta|\right)(V)$, but

$$\int_V |B(I)|Z(\eta)(I) = \int_V 1\cdot Z(\eta)(I) = Z(\eta)(V)\,,$$

so that

$$Z\left(\int|\eta|\right)(V) = Z(\eta)(V)\,.$$

COROLLARY 3.6. If each of η and $\varkappa$ is in $\mathfrak{p}_{AB}$ and for all V in $\boldsymbol{F}$,

$$\int_V |\varkappa(I)| \leqslant \int_V |\eta(I)|,$$

then for all V in $\boldsymbol{F}$,

$$Z(\varkappa)(V) \leqslant Z(\eta)(V).$$

PROOF. Let $\eta^* = \int|\eta|$ and $\varkappa^* = \int|\varkappa|$. Clearly, if I is in $\boldsymbol{F}$, then

$$\varkappa^*(I)/\eta^*(I) \leqslant 1,$$

and if V is in $\boldsymbol{F}$, then

$$\varkappa^*(V) = \int_V [\varkappa^*(I)/\eta^*(I)]\,\eta^*(I).$$

Now, if V is in $\boldsymbol{F}$, then, by Theorem 3.4 and Corollary 3.5,

$$Z(\varkappa)(V) = Z(\varkappa^*)(V) = Z\Big(\int[\varkappa^*/\eta^*]\,\eta^*\Big)(V) = \int_V [\varkappa^*(I)/\eta^*(I)]\,Z(\eta^*)(I) \leqslant$$

$$\leqslant \int_V 1\cdot Z(\eta^*)(I) = Z(\eta^*)(V) = Z(\eta)(V).$$

4. A C-set discussion of integrability.

In this section we prove the assertion, made in the introduction, that if β is in $\mathfrak{p}_B$, then $\mathfrak{J}_\beta$ is a C-set.

THEOREM 4.1. If β is in $\mathfrak{p}_B$, then $\mathfrak{J}_\beta$ is a C-set.

PROOF. As in previous duscussions, for each η in $\mathfrak{p}_{AB}$, we shall let $Z(\eta)$ be the element of $\mathfrak{p}_{AB}$ given by

$$Z(\eta)(V) = \int_V [L(\beta\eta)(I) - G(\beta\eta)(I)].$$

Clearly

$$\mathfrak{J}_\beta = \{\eta : \eta \text{ in } \mathfrak{p}_{AB},\ Z(\eta)(U) = 0\}.$$

Now, if η is in $\mathfrak{J}_\beta$, $\varkappa$ is in $\mathfrak{p}_{AB}$ and

$$\int_V |\varkappa(I)| \leqslant \int_V |\eta(I)|$$

for all V in $\boldsymbol{F}$, then by Corollary 3.6,

$$0 \leqslant Z(\varkappa)(U) \leqslant Z(\eta)(U) = 0 ,$$

so that $\varkappa$ is in $\mathfrak{J}_\beta$; thus 1) of the definition of a C-set is satisfied for $\mathfrak{J}_\beta$.

If each of ζ and $\varkappa$ is in $\mathfrak{p}_A^+ \cap \mathfrak{J}_\beta$, then, by Corollary 3.6 and Lemma 3.1,

$$0 \leqslant Z\Big(\int \max\{\zeta, \varkappa\}\Big)(U) \leqslant Z(\zeta + \varkappa)(U) \leqslant Z(\zeta)(U) + Z(\varkappa)(U) = 0 + 0 = 0 ,$$

so that $\int \max\{\zeta, \varkappa\}$ is in $\mathfrak{J}_\beta$.

Now suppose η is in $\mathfrak{p}_A^+$ and λ is the function from $\boldsymbol{F}$ into $\mathbb{R}$ given by

$$\lambda(I) = \sup\{\pi(I) \colon \pi \text{ in } \mathfrak{p}_A^+ \cap \mathfrak{J}_\beta,\ \eta - \pi \text{ in } \mathfrak{p}_A^+\} .$$

Clearly λ is nonnegative-valued. We next show that λ is in $\mathfrak{p}_A^+$. Suppose V_1 and V_2 are mutually exclusive sets of $\boldsymbol{F}$ and $0 < c$. There are $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ in $\mathfrak{p}_A^+ \cap \mathfrak{J}_\beta$ such that each of $\eta - \varkappa_1$, $\eta - \varkappa_2$ and $\eta - \varkappa_3$ is in $\mathfrak{p}_A^+$ and

$$\lambda(V_1) - \varkappa_1(V_1) < c/4 , \qquad \lambda(V_2) - \varkappa_2(V_2) < c/4$$

and

$$\lambda(V_1 \cup V_2) - \varkappa_3(V_1 \cup V_2) < c/4 .$$

Let

$$\nu = \int \max\{\varkappa_1, \varkappa_2, \varkappa_3\} .$$

From the preceding paragraph ν is in $\mathfrak{J}_\beta$ and $\eta - \nu$ is in $\mathfrak{p}_A^+$, so that if V is in $\boldsymbol{F}$, then

$$\nu(V) \leqslant \lambda(V) .$$

Furthermore,

$$\lambda(V_1) - \nu(V_1) \leqslant \lambda(V_1) - \varkappa_1(V_1) < c/4 ,$$

$$\lambda(V_2) - \nu(V_2) \leqslant \lambda(V_2) - \varkappa_2(V_2) < c/4 ,$$

and

$$\lambda(V_1 \cup V_2) - \nu(V_1 \cup V_2) \leqslant \lambda(V_1 \cup V_2) - \varkappa_3(V_1 \cup V_2) < c/4 ,$$

so that

$$|\lambda(V_1) + \lambda(V_2) - \lambda(V_1 \cup V_2)| = |\lambda(V_1) - \nu(V_1) + \lambda(V_2) - \nu(V_2) - \lambda(V_1 \cup V_2) + \nu(V_1 \cup V_2)| \leqslant |\lambda(V_1) - \nu(V_1)| + |\lambda(V_2) - \nu(V_2)| + |\nu(V_1 \cup V_2) - \lambda(V_1 \cup V_2)| < 3c/4 < c .$$

Therefore

$$\lambda(V_1) + \lambda(V_2) = \lambda(V_1 \cup V_2) .$$

Therefore λ is in $\mathfrak{p}_A^+$.

Again, suppose $0 < c$. There is $K > 0$ such that $|x| \leqslant K$ for all x in the range union of β. There is π in $\mathfrak{J}_\beta \cap \mathfrak{p}_A^+$ such that $\eta - \pi$ is in $\mathfrak{p}_A^+$ and

$$\lambda(U) - \pi(U) < c/(2K+1) .$$

Clearly, now, $\lambda - \pi$ is in $\mathfrak{p}_A^+$. By Theorem 3.2,

$$|Z(\lambda)(U) - Z(\pi)(U)| \leqslant 2K \int_U |\lambda(I) - \pi(I)| = 2K[\lambda(U) - \pi(U)] \leqslant 2Kc/(2K+1) < c ,$$

so that

$$0 \leqslant Z(\lambda)(U) < c + Z(\pi)(U) = c + 0 = c .$$

Therefore

$$Z(\lambda)(U) = 0 ,$$

so that λ is in $\mathfrak{J}_\beta \cap \mathfrak{p}_A^+$. Therefore 2) of the definition is satisfied for $\mathfrak{J}_\beta$.

Therefore $\mathfrak{J}_\beta$ is a C-set.

In subsequent discussions in this paper, for each β in $\mathfrak{p}_B$, we shall let $\boldsymbol{a}_\beta$ and λ_β denote, respectively, $\boldsymbol{a}_M$ and λ_M, where $M = \mathfrak{J}_\beta$.

We now prove a theorem that we shall use in proving Theorem 6.1. We begin by stating a lemma (see [5]) that is a generalization of an assertion made in the proof of Theorem 4.1.

LEMMA 4.2. If M is a C-set and each of η and $\varkappa$ is in $M \cap \mathfrak{p}_A^+$, then so is $\int \max\{\eta, \varkappa\}$.

THEOREM 4.2. If G is a collection of C-sets, then $\bigcap_G X$ is a C-set, and if $\varkappa$ is in $\mathfrak{p}_A^+$ and V is in $\boldsymbol{F}$, then letting $M = \bigcap_G X$,

$$\boldsymbol{a}_M(\varkappa)(V) = \boldsymbol{b}(\varkappa)(V),$$

where

$$\boldsymbol{b}(\varkappa)(V) = \inf\left\{\int_V \min\{\boldsymbol{a}_{X_1}(\varkappa)(I), \ldots, \boldsymbol{a}_{X_n}(\varkappa)(I)\} : X_1, \ldots, X_n \text{ in } G\right\}.$$

PROOF. Clearly O belongs to every C-est, so that the elements of G have an element in common.

Now suppose η is in M, $\varkappa$ is in $\mathfrak{p}_{AB}$ and

$$\int_V |\varkappa(I)| \leqslant \int_V |\eta(I)|$$

for all V in $\boldsymbol{F}$. Since, for each X in G, η is in X, it immediately follows that $\varkappa$ is in X, so that $\varkappa$ is in M. Therefore 1) of the definition of a C-set is satisfied for M.

Now suppose μ is in $\mathfrak{p}_A^+$ and λ is the function from $\boldsymbol{F}$ into $\mathbb{R}$ given by

$$\lambda(V) = \sup\{\varkappa(V) : \varkappa \text{ in } M \cap \mathfrak{p}_A^+,\ \mu - \varkappa \text{ in } \mathfrak{p}_A^+\}.$$

Suppose X is in G. If V is in $\boldsymbol{F}$, then, since $M \subseteq X$,

$$\lambda(V) \leqslant \sup\{\zeta(V) : \zeta \text{ in } X \cap \mathfrak{p}_A^+,\ \mu - \zeta \text{ in } \mathfrak{p}_A^+\} = \boldsymbol{a}_X(\mu)(V).$$

Therefore, if λ is in $\mathfrak{p}_A^+$, then λ is in $X \cap \mathfrak{p}_A^+$. So suppose V_1 and V_2 are mutually exclusive sets of $\boldsymbol{F}$ and $0 < c$. There are $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ in $M \cap \mathfrak{p}_A^+$ such that for $i = 1, 2$, or 3, $\mu - \varkappa_1$ is in $\mathfrak{p}_A^+$,

$$0 \leqslant \lambda(V_1) - \varkappa_1(V_1) < c/3, \qquad 0 \leqslant \lambda(V_2) - \varkappa_2(V_2) < c/3,$$

and

$$0 \leqslant \lambda(V_1 \cup V_2) - \varkappa_3(V_1 \cup V_2) < c/3.$$

If Y is in G, then $\varkappa_1$, $\varkappa_2$ and $\varkappa_3$ are in $Y \cap \mathfrak{p}_A^+$, so that by Lemma 4.2, $\varkappa_4$, which shall denote

$$\int \max\{\varkappa_1, \varkappa_2, \varkappa_3\},$$

is in $Y \cap \mathfrak{p}_A^+$. Therefore $\varkappa_4$ is in $M \cap \mathfrak{p}_A^+$. Furthermore, $\mu - \varkappa_4$ is clearly in $\mathfrak{p}_A^+$, so that if V is in $\boldsymbol{F}$ then

$$\varkappa_4(V) \leqslant \lambda(V) .$$

Moreover, for $i = 1, 2$, or 3, $\varkappa_4 - \varkappa_i$ is in $\mathfrak{p}_A^+$. Therefore,

$$0 \leqslant \lambda(V_1) - \varkappa_4(V_1) < c/3 , \qquad 0 \leqslant \lambda(V_2) - \varkappa_4(V_2) < c/3 ,$$

and

$$0 \leqslant \lambda(V_1 \cup V_2) - \varkappa_4(V_1 \cup V_2) < c/3 ,$$

so that

$$\begin{aligned} &|\lambda(V_1) + \lambda(V_2) - \lambda(V_1 \cup V_2)| \leqslant |\lambda(V_1) - \varkappa_4(V_1)| + \\ &\quad + |\lambda(V_2) - \varkappa_4(V_2)| + |\varkappa_4(V_1) + \varkappa_4(V_2) - \lambda(V_1 \cup V_2)| = \\ &\quad = |\lambda(V_1) - \varkappa_4(V_1)| + |\lambda(V_2) - \varkappa_4(V_2)| + |\varkappa_4(V_1 \cup V_2) - \lambda(V_1 \cup V_2)| < \\ &\quad < 3c/3 = c . \end{aligned}$$

Therefore λ is in $\mathfrak{p}_A^+$, so that, from previous remarks, λ is in $X \cap \mathfrak{p}_A^+$.

Therefore λ is in $M \cap \mathfrak{p}_A^+$, so that 2) of the definition of a C-set is satisfied for M.

Therefore M is a C-set.

Now, suppose $\varkappa$ is in $\mathfrak{p}_A^+$. Clearly, for all X in G, $\boldsymbol{a}_X(\varkappa) - \boldsymbol{a}_M(\varkappa)$ is in $\mathfrak{p}_A^+$, so that if $X_1, \ldots, X_n$ are in G, then

$$\left[\int \min\{\boldsymbol{a}_{X_1}(\varkappa), \ldots, \boldsymbol{a}_{X_n}(\varkappa)\}\right] - \boldsymbol{a}_M(\varkappa)$$

is in $\mathfrak{p}_A^+$. Therefore, if V is in $\boldsymbol{F}$, then

$$\boldsymbol{a}_M(\varkappa)(V) \leqslant \boldsymbol{b}(\varkappa)(V) .$$

We show that $\boldsymbol{b}(\varkappa)$, which is clearly nonnegative-valued, is in $\mathfrak{p}_A^+$. Suppose V_1 and V_2 are mutually exclusive sets of $\boldsymbol{F}$ and $0 < c$. There are finite subcollections,

$$X_1, \ldots, X_p ; \qquad Y_1, \ldots, Y_q ; \qquad \text{and} \qquad Z_1, \ldots, Z_r$$

of G such that, letting β denote $\boldsymbol{b}(\varkappa)$,

$$0 \leqslant \left[\int_{V_1} \min\{\boldsymbol{a}_{X_1}(\varkappa)(I), \ldots, \boldsymbol{a}_{X_p}(\varkappa)(I)\}\right] - \beta(V_1) < c/3 ,$$

$$0 \leqslant \left[\int_{V_2} \min\{\boldsymbol{a}_{Y_1}(\varkappa)(I), \ldots, \boldsymbol{a}_{Y_q}(\varkappa)(I)\}\right] - \beta(V_2) < c/3 ,$$

and

$$0 \leqslant \left[\int_{V_1 \cup V_2} \min\{\boldsymbol{a}_{Z_1}(\eta)(I), \ldots, \boldsymbol{a}_{Z_r}(\eta)(I)\}\right] - \beta(V_1 \cup V_2) < c/3 ,$$

so that if $W_1, \ldots, W_t$ is the union of the above collections of X's, Y's and Z's and V^* is V_1, V_2 or $V_1 \cup V_2$, then

$$0 \leqslant \left[\int_{V^+} \min\{\boldsymbol{a}_{W_1}(\varkappa)(I), \ldots, \boldsymbol{a}_{W_t}(\varkappa)(I)\}\right] - \beta(V^*) < c/3 ,$$

so that, letting π denote $\int \min\{\boldsymbol{a}_{W_1}(\varkappa), \ldots, \boldsymbol{a}_{W_t}(\varkappa)\}$, we have that

$$|\beta(V_1) + \beta(V_2) - \beta(V_1 \cup V_2)| \leqslant |\beta(V_1) - \pi(V_2)| + |\beta(V_2) - \pi(V_2)| + \\ + |\pi(V_1) + \pi(V_2) - \beta(V_1 \cup V_2)| = |\beta(V_1) - \pi(V_1)| + |\beta(V_2) - \pi(V_2)| + \\ + |\pi(V_1 \cup V_2) - \beta(V_1 \cup V_2)| < 3c/3 = c .$$

Therefore β is in $\mathfrak{p}_A^+$. From this and the definition of β, it follows that $\varkappa - \beta$ is in $\mathfrak{p}_A^+$, and that β is in X for all X in G, so that β is in $M \cap \mathfrak{p}_A^+$. Therefore $\boldsymbol{a}_M(\varkappa) - \beta$ is in $\mathfrak{p}_A^+$, so that from the initial remarks of this paragraph it follows that

$$\boldsymbol{a}_M(\varkappa) = \boldsymbol{b}(\varkappa) .$$

5. A dominated convergence theorem.

In this section we prove Theorem 5.1, as stated in the introduction.

PROOF OF THEOREM 5.1. As in section 3, following the proof of Theorem 3.3, for each η in $\mathfrak{p}_{AB}$, let $Z(\eta)$ be the element of $\mathfrak{p}_{AB}$ given by

$$Z(\eta)(V) = \int_V [L(\beta\eta)(I) - G(\beta\eta)(I)] .$$

Now suppose, on the contrary, that there is $c > 0$ such that if $d > 0$, then there is η in $\mathfrak{p}_{AB}$ such that $\varkappa - \int|\eta|$ is in $\mathfrak{p}_A^+$ and

$$Z(\eta)(U) < d\,,$$

but

$$\int_U |\eta(I) - \boldsymbol{a}_\beta(\eta)(I)| \geqslant c\,,$$

so that by Corollary 3.5, if $\zeta = \int|\eta|$, then

$$Z(\zeta)(U) < d$$

and, with reference to Theorem 1.A.2,

$$\int_U |\zeta(I) - \boldsymbol{a}_\beta(\zeta)(I)| = \int_U |\eta(I) - \boldsymbol{a}_\beta(\eta)(I)| \geqslant c\,.$$

Let $\boldsymbol{a} = \boldsymbol{a}_\beta$. It follows that there is a sequence $\{\zeta_i\}_{i=1}^\infty$ of elements of $\mathfrak{p}_A^+$ such that if n is a positive integer, then $\varkappa - \zeta_n$ is in $\mathfrak{p}_A^+$,

$$Z(\zeta_n)(U) < 2^{-n}\,,$$

and

$$\int_U |\zeta_n(I) - \boldsymbol{a}(\zeta_n)(I)| \geqslant c\,.$$

For each positive integer n, let

$$v_n = \int \max\{\zeta_n, \boldsymbol{a}(\varkappa)\}\,,$$

so that, by Corollary 3.6 and Lemma 3.1,

$$\begin{aligned} Z(v_n)(U) &= \int_U [L(\beta v_n)(I) - G(\beta v_n)(I)] \leqslant \\ &\leqslant \int_U [L(\beta[\zeta_n + \boldsymbol{a}(\varkappa)])(I) - G(\beta[\zeta_n + \boldsymbol{a}(\varkappa)])(I)] \leqslant \\ &\leqslant \int_U [L(\beta\zeta_n)(I) - G(\beta\zeta_n)(I)] + \int_U [L(\beta\boldsymbol{a}(\varkappa))(I) - G(\beta\boldsymbol{a}(\varkappa))(I)] = \\ &= Z(\zeta_n)(U) + 0 < 2^{-n}. \end{aligned}$$

For each positive integer m, $\varkappa - v_m$ is in $\mathfrak{p}_A^+$. For each positive integer n and positive integer w, let

$$\varphi_n^{(w)} = \int \max\{v_n, \ldots, v_{n+w}\} \,;$$

clearly each of $\varkappa - \varphi_n^{(w)}$ and $\varphi_n^{(w)} - \boldsymbol{a}(\varkappa)$ is in $\mathfrak{p}_A^+$.

For each positive integer n, let μ_n be the function from $\boldsymbol{F}$ into $\mathbf{R}$ defined by

$$\mu_n(V) = \sup\{\varphi_n^{(w)}(V) : w \text{ a positive integer}\} .$$

It is easy to show that if n is a positive integer, then μ_n is in $\mathfrak{p}_A^+$, and we leave the proof to the reader. Furthermore, each of $\varkappa - \mu_n$ and $\mu_n - \boldsymbol{a}(\varkappa)$ is clearly in $\mathfrak{p}_A^+$ for every positive integer n.

Now, for each positive integer n and positive integer w, again by Corollary 3.6 and Lemma 3.1,

$$Z(\varphi_n^{(w)})(U) \leqslant Z\Big(\sum_{j=0}^{w} v_{n+j}\Big)(U) \leqslant \sum_{j=0}^{w} Z(v_{n+j})(U) < \sum_{j=0}^{w} 2^{-(n+j)} < 2^{-(n-1)} ,$$

and

$$Z(\mu_n)(U) \leqslant Z(\mu_n - \varphi_n^{(w)})(U) + Z(\varphi_n^{(w)})(U) < 2M[\mu_n(U) - \varphi_n^{(w)}(U)] + 2^{-(n-1)}, \text{ where } M = \sup\{|x| : x \text{ in } \beta\text{'s range union}\} .$$

Therefore, if n is a positive integer and $0 < s$, then there is a positive integer w such that

$$\mu_n(U) - \varphi_n^{(w)}(U) < s/[2(M+1)] ,$$

so that

$$Z(\mu_n)(U) \leqslant 2Ms/[2(M+1)] + 2^{-(n-1)} < s + 2^{-(n-1)} ,$$

so that

$$Z(\mu_n)(U) \leqslant 2^{-(n-1)} .$$

If n is a positive integer, then $\int \min\{\zeta_n, \boldsymbol{a}(\varkappa)\}$ is in $\mathfrak{J}_\beta$, so that by

Theorem 1.A.2,

$$c \leqslant \int_U |\zeta_n(I) - \boldsymbol{a}(\zeta_n)(I)| \leqslant \int_U |\zeta_n(I) - \int_I \min\{\zeta_n, \boldsymbol{a}(\varkappa)\}| =$$

$$= \int_U |\zeta_n(I) - \min\{\zeta_n(I), \boldsymbol{a}(\varkappa)(I)\}| = \int_U |\max\{\zeta_n(I), \boldsymbol{a}(\varkappa)(I)\} - \boldsymbol{a}(\varkappa)(I)| =$$

$$= v_n(U) - \boldsymbol{a}(\varkappa)(U) \leqslant \mu_n(U) - \boldsymbol{a}(\varkappa)(U) .$$

Let μ be the function from $\boldsymbol{F}$ into $\mathbb{R}$ defined by

$$\mu(V) = \inf\{\mu_n(V) : n \text{ a positive integer}\} .$$

Note that if n is a positive integer, then $\mu_n - \mu_{n+1}$ is in $\mathfrak{p}_A^+$; this and a few other elementary considerations imply that μ is in $\mathfrak{p}_A^+$, and we leave the details to the reader. Also, each of $\varkappa - \mu$ and $\mu - \boldsymbol{a}(\varkappa)$ is in $\mathfrak{p}_A^+$ and, most importantly, from the inequality of the last paragraph,

$$c \leqslant \mu(U) - \boldsymbol{a}(\varkappa)(U) ,$$

so that, since $\varkappa - \mu$ is in $\mathfrak{p}_A^+$, it follows that μ is not in $\mathfrak{Z}_\beta$, so that

$$Z(\mu)(U) > 0 ;$$

but, if n is a positive integer, then, since $\mu_n - \mu$ is in $\mathfrak{p}_A^+$, it follows from Corollary 3.6 that

$$Z(\mu)(U) \leqslant Z(\mu_n)(U) \leqslant 2^{-(n-1)} ;$$

this implies that

$$Z(\mu)(U) = 0 ,$$

a contradiction.

Therefore the theorem is true.

6. The inclusion and continuity theorem.

In this section we prove Theorem 6.1, as stated in the introduction.

Proof of Theorem 6.1. There is K in $\mathbb{R}$ such that $|x| \leqslant K$ for all x in the range union of β. For $i = 1, \ldots, n$, let $\boldsymbol{a}_i = \boldsymbol{a}_{\alpha_i}$, and $\lambda_i = \lambda_{\alpha_i}$

Let $\mathfrak{J}^* = \bigcap_{i=1}^{n} \mathfrak{J}_{\alpha_i}$. Clearly $\mathfrak{J}^* \subseteq \mathfrak{J}_\beta$. By Theorem 4.2, $\mathfrak{J}^*$ is a C-set. Let $\boldsymbol{a}^* = \boldsymbol{a}_{\mathfrak{J}^*}$. Suppose $0 < c'$ and μ is in $\mathfrak{p}_A^+$. Let $c = c'/(2K+1)$. It clearly follows from Theorem 5.1 that there is $d > 0$ such that if $i = 1, \ldots, n$, $\varkappa$ is in $\mathfrak{p}_{AB}$ and $\mu - \int |\varkappa|$ is in $\mathfrak{p}_A^+$ and

$$\int_U [L(\alpha_i \varkappa)(I) - G(\alpha_i \varkappa)(I)] < d ,$$

then

$$\int_U |\varkappa(I) - \boldsymbol{a}_i(\varkappa)(I)| < c/n .$$

Suppose $\varkappa$ is in $\mathfrak{p}_{AB}$, $\mu - \int |\varkappa|$ is in $\mathfrak{p}_A^+$ and for $i = 1, \ldots, n$

$$\int_U [L(\alpha_i \varkappa)(I) - G(\alpha_i \varkappa)(I)] < d .$$

Let $\varkappa^* = \int |\varkappa|$. By Corollary 3.5, for $i = 1, \ldots, n$,

$$\int_U [L(\alpha_i \varkappa^*)(I) - G(\alpha_i \varkappa^*)(I)] = \int_U [L(\alpha_i \varkappa)(I) - G(\alpha_i \varkappa)(I)] < d ,$$

so that, from the above conditions and Theorem 1.A.2,

$$c/n > \int_U |\varkappa(I) - \boldsymbol{a}_i(\varkappa)(I)| = \int_U |\varkappa^*[I) - \lambda_i(\varkappa^*)(I)| ,$$

which implies that

$$\int_U |\varkappa^*(I) - \int_I \min\{\lambda_1(\varkappa^*)(J), \ldots, \lambda_n(\varkappa^*)(J)\}| =$$

$$\int_U \left| \int_I \min\{\varkappa^*(J), \ldots, \varkappa^*(J)\} - \int_I \min\{\lambda_1(\varkappa^*)(J), \ldots, \lambda_n(\varkappa^*)(J)\} \right| =$$

$$\int_U |\min\{\varkappa^*(I), \ldots, \varkappa^*(I)\} - \min\{\lambda_1(\varkappa^*)(I), \ldots, \lambda_n(\varkappa^*)(I)\}| \leqslant$$

$$\int_U \sum_{w=1}^{n} |\varkappa^*(I) - \boldsymbol{a}_i(\varkappa^*)(I)| = \sum_{w=1}^{n} \int_U |\varkappa^*(I) - \boldsymbol{a}_i(\varkappa^*)(I)| < nc/n = c .$$

By Theorem 4.2, for each I in $\boldsymbol{F}$,

$$\boldsymbol{a}^*(\varkappa^*)(I) = \int_I \min\{\boldsymbol{a}_1(\varkappa^*)(J), \ldots, \boldsymbol{a}_n(\varkappa^*)(J)\},$$

so that

$$\int_U |\varkappa^*(I) - \boldsymbol{a}^*(\varkappa^*)(I)| < c .$$

Since $\mathfrak{J}^* \subseteq \mathfrak{J}_\beta$, it follows that

$$\int_U |\varkappa^*(I) - \boldsymbol{a}_\beta(\varkappa^*)(I)| \leqslant \int_U |\varkappa^*(I) - \boldsymbol{a}^*(\varkappa^*)(I)| < c ,$$

so that, by Theorem 3.2,

$$\int_U [L(\beta\varkappa)(I) - G(\beta\varkappa)(I)] = \left| \int_U [L(\beta\varkappa^*)(I) - G(\beta\varkappa^*)(I) - 0 \right| =$$

$$\left| \int_U [L(\beta\varkappa^*)(I) - G(\beta\varkappa^*)(I)] - \int_U [L(\beta\boldsymbol{a}_\beta(\varkappa^*)(I) - G(\beta\boldsymbol{a}_\beta(\varkappa^*)(I)] \right| \leqslant$$

$$2K\int |\varkappa^*(I) - \boldsymbol{a}_\beta(\varkappa^*)(I)| \leqslant 2Kc \leqslant 2Kc'/(2K+1) < c' ,$$

and the theorem follows.

REFERENCES

[1] W. D. L. Appling, *Interval functions and the Hellinger integral*, Duke Math. J., **29** (1962), 515-520.

[2] W. D. L. Appling, *Integral representations and refinement unboundedness*, Proc. Amer. Math. Soc., **19** (1968), 837-842.

[3] W. D. L. Appling, *Summability of real-valued set functions*, Riv. Mat. Parma, (2), **8** (1967), 77-100.

[4] W. D. L. Appling, *Continuity and set function summability*, Annali di Matematica pura e applicata, **87** (1970), 357-374.

[5] W. D. L. APPLING, *A generalization of absolute continuity and of an analogue of the Lebesgue decomposition theorem*, Riv. Mat. Parma, accepted for publication; to appear.

[6] W. D. L. APPLING, *Set functions, finite additivity and distribution functions*, Annali di Matematica pura e applicata, accepted for publication; to appear.

[7] A. KOLMOGOROFF, *Untersuchungen uber den Integralbegriff*, Math. Ann., **103** (1930), 654-696.

Manoscritto pervenuto in redazione il 1 settembre 1973.