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An existence theorem for certain solutions of algebraic differential equations in sectors


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1. Introduction.

In this paper we consider first-order differential equations,

\[ \Omega(z, y, y') = \sum_{k,l \geq 0} f_{k,l}(z) y^k (y')^l = 0, \]

where \( \Omega \) is a polynomial in \( y \) and \( y' \), whose coefficients \( f_{k,l}(z) \) are complex functions, defined and analytic in a sectorial region which is approximately of the form,

\[ a < \arg(z - \beta \exp[i(a + b)/2]) < b, \]

(for some \( \beta > 0 \)), and where each non-zero \( f_{k,l}(z) \) has an asymptotic expansion in terms of real powers of \( z \), as \( z \to \infty \) over a filter base (denoted \( F(a, b) \)) which consists essentially of the sectors (2) as \( \beta \to +\infty \). (We are using here the stronger concepts of «asymptotically equivalent» (\( \sim \)) and «smaller rate of growth» (\( \ll \)) which were introduced by W. Strodt in [5; §13]. For the reader’s convenience, these concepts are reviewed in §2 below, and we point out that the class of equations treated here contains, as a special case, the class of equations having polynomial coefficients.) The equations (1) were

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among those equations which were treated in [2], [5] and [7], where
existence theorems were proved for solutions which are asymptotically
equivalent over $F(a, b)$ to logarithmic monomials (i.e. functions of
the form,

$$M(z) = Kz^{\alpha_{*}}(\log z)^{\alpha_{1}}(\log \log z)^{\alpha_{2}} \cdots (\log_{q} z)^{\alpha_{q}},$$

for real $\alpha_{j}$ and complex $K \neq 0$). If (1) has a solution which is $\sim M$
over $F(a, b)$, then $M$ must be a critical monomial [2; §§ 4, 5] of $\Omega$ (i.e.
$M$ is a point of instability of $\Omega$ in the sense that for some function
g $\sim M$, $\Omega(z, g, g')$ is not $\sim \Omega(z, M, M')$ over $F(a, b)$). An algorithm
for determining all critical monomials of $\Omega$ was developed in [2; §§ 21, 22]
and the powerful Strodt-Wright theorem [7; p. 221] states, in part,
that for every critical monomial $M$ of $\Omega$, there exists a solution $\sim M$
over a suitable $F(a_{1}, b_{1})$. (For the special class treated in this paper,
the algorithm shows that $\alpha_{1} = 0$ for $j > 2$).

In [1], existence theorems were proved for solutions of (1) which
are of larger rate of growth than all logarithmic monomials over $F(a, b)$
and also for solutions which are of smaller rate of growth than all
monomials. (These solutions are of the form $\exp \left[ \int W \right]$, where $W$ is $\sim$
to a logarithmic monomial of the form $Kz^{\alpha_{*}}$).

In [3], a converse result was proved which shows that for the class
of equations treated here, any solution $h(z)$ of (1), which is meromorphic
in an element of $F(a, b)$ and which is $\ll$ comparable $\ll$ with all loga-
rithmic monomials $M$ over $F(a, b)$ (in the sense that for any $M$, one
of the relations $h \ll M$, $M \ll h$ or $h \sim cM$ for some $c \neq 0$, is valid over
$F(a, b)$), must either be asymptotically equivalent to a logarithmic
monomial or of the form $\exp \int W$, where $W/Kz^{\alpha_{*}} \to 1$ over $F(a, b)$, for
real $\alpha_{0}$ and complex $K \neq 0$.

Of course, equations (1) can possess solutions which violate the
above comparability condition such as solutions which are $\sim$ to func-
tions of the form $cz^{\alpha}$, where $\alpha$ is a nonreal complex number and $c \neq 0$.
(It is easy to see that for such $\alpha$, $z^{\alpha}$ and $z^{\text{Re}(\alpha)}$ are not comparable over
any $F(a, b)$). Thusfar, no general existence theorem for such solu-
tions of nonlinear equations (1) has been proved, and in this paper,
we prove such a result. Because of the noncomparability property
of such functions $cz^{\alpha}$ with logarithmic monomials, the concept of
a point of instability of $\Omega$ cannot in general be used as the starting
point in the search for such solutions because as is pointed out in
$p.$ p. 253], it is possible for a function $cz^{\alpha}$, with $\alpha$ nonreal, to be a
point of instability of $\Omega$, and yet there be no solution of (1) which
is \( \sim c \zeta^x \) over any \( F(a, b) \). However, in the case of those \( \Omega \) which are homogeneous as polynomials in \( y \) and \( y' \), it is true that for nonreal \( \alpha \) the equation \( \Omega = 0 \) possesses a solution \( \sim c \zeta^x \) over \( F(a, b) \), if \( c \zeta^x \) is a point of instability of \( \Omega \). (See §§ 4, 6 below). It is this fact which is the basis for our method for nonhomogeneous \( \Omega \). Roughly speaking, we show that under certain conditions if \( c \zeta^x \) (with \( \alpha \) nonreal) is a point of instability of a homogeneous part of \( \Omega \), and if this homogeneous part is in some sense «dominant» for \( z \Re(a) \), then the whole equation \( \Omega = 0 \) possesses a solution which is \( \sim c \zeta^x \) over \( F(a, b) \). We remark that it is very easy to determine those functions \( c \zeta^x \) which are points of instability of a homogeneous part of \( \Omega \) (see § 6), and that in any given example, it is very easy to test whether the hypothesis is satisfied for the particular \( c \zeta^x \) involved.

The proof of the main result consists of using the exact solution \( \sim c \zeta^x \) of the homogeneous part of the equation involved (see § 4), to eventually transform the whole equation \( \Omega = 0 \) into a quasi-linear equation of the type treated by Strodt in [5; § 117], where a method of linear successive approximations was successful.

2. Preliminaries.

(a) [5; § 94]. Let \(-\pi < a < b < \pi\). For each nonnegative real-valued function \( g \) on \((0, (b - a)/2)\), let \( V(g) \) be the union \( \{ \delta \in (0, (b - a)/2) \mid a + \delta < \arg (z - h(\delta)) < b - \delta \} \) where \( h(\delta) = g(\delta) \exp [i(a + b)/2] \). The set of all \( V(g) \) (for all choices of \( g \)) is denoted \( F(a, b) \), and is a filter base of simply-connected regions which converges to \( \infty \) by [5; §§ 93, 95]. By \( \log z \), we will mean the principal branch of the logarithm in \( |\arg z| < \pi \). By induction, it is easy to see that the function \( \log(\log(z)) \) is defined and analytic in some element of \( F(a, b) \). If \( \alpha \) is a complex number, then as usual, \( z^\alpha \) will denote \( \exp[\alpha \log z] \).

(b) [5; §§ 13, 17]. If \( f \) is analytic in an element of \( F(a, b) \), then \( f \to 0 \) over \( F(a, b) \) means that for any \( \varepsilon > 0 \), there is an element of \( F(a, b) \) on which \( |f(z)| < \varepsilon \). The statement \( f \ll 1 \) over \( F(a, b) \) means that in addition to \( f \to 0 \), for all positive integers \( j \) and \( k \) we have \( \theta_k f \to 0 \), where \( \theta_k f = (z \log z \ldots \log_{k-1} z)f' \), and where \( \theta_k^j \) is the \( j \)-th iterate of the operator \( \theta_k \). Then \( f \ll g, f \gg g, f \sim g \) and \( f \approx g \) over \( F(a, b) \) mean respectively, \( f/g \ll 1, g/f \ll 1, f - g \ll g \) and finally \( f \sim cg \)
for some constant $c \neq 0$. The crucial property [5; § 28] of the relation $\ll$ is that if $f \ll 1$ over $F(a, b)$, then $\theta_j f \ll 1$ over $F(a, b)$ for all $j > 0$. From this, it easily follows that if $f \sim cz^\alpha$, where $c$ and $\alpha$ are nonzero complex numbers, then $f' \sim \alpha cz^{\alpha-1}$. It is easily verified that for any complex number $\alpha$ and any $\varepsilon > 0$, we have $z^{\text{Re}(\alpha)-\varepsilon} \ll z^\alpha \ll z^{\text{Re}(\alpha)+\varepsilon}$ over $F(a, b)$.

(c) [6; p. 244]. A logarithmic field of rank zero over $F(a, b)$ is a set $L$ of functions, each defined and meromorphic in an element of $F(a, b)$, with the following properties: (i) $L$ is a field (where, as usual, we identify two elements of $L$ if they agree on an element of $F(a, b)$); (ii) $L$ contains all functions of the form $cz^{\beta}$, for real $\beta$ and complex $c \neq 0$, and (iii) for every element $f$ in $L$ except zero, there exists a function $cz^{\beta}$, with $\beta$ real and $c \neq 0$, such that $f \sim cz^{\beta}$ over $F(a, b)$. (The set of all rational combinations of the functions $cz^{\beta}$ is the simplest example of such a field, and this field clearly contains the field of rational functions.)

(d) let the equation (1) have coefficients in a logarithmic field $L$ of rank zero. We say $\Omega$ is nontrivial if some coefficient is not identically zero. If $f_{kj} \neq 0$, then

$$f_{kj}(z) = c_{kj} z^{\alpha_{kj}} (1 + E_{kj}),$$

where $a_{kj}$ is real, $c_{kj}$ is a nonzero complex number, and $E_{kj}$ is an element of $L$ which is $\ll 1$ over $F(a, b)$. We denote by $\delta_0(f_{kj})$ the number $a_{kj}$. (If $f_{kj} \equiv 0$, we set $\delta_0(f_{kj}) = -\infty$). Let $A = (k + j; f_{kj} \neq 0)$. If $q \in A$, we denote by $\Omega^{(q)}$ the homogeneous part of $\Omega$ of degree $q$ in the indeterminates $y$ and $y'$. For a real number $\beta$, we set,

$$\Omega^{(q)*, \beta} = \beta q + \max \{ \delta_0(f_{kj}) - j; k + j = q \}.$$

Finally [2; § 17], the critical equation of $\Omega^{(q)}$ is the equation $G_q(x) = 0$, where

$$G_q(x) = \sum \{ c_{kj} x^j; (k, j) \in J_q \},$$

where

$$J_q = \{(k, j); k + j = q \text{ and } \delta_0(f_{kj}) - j = \Omega^{(q)*, 0}\}.$$
3. We now state our main result.

The proof will be given in § 5.

THEOREM. Let \( \Omega(z, y, y') = \sum f_{k,j}(z) y^k(y')^j \) be a nontrivial polynomial in \( y \) and \( y' \) whose coefficients belong to a logarithmic field of rank zero over \( F(a, b) \). Let \( A = \{k + j : f_{k,j} \neq 0\} \), and for \( q \in A \), let \( G_q(x) = 0 \) be the critical equation of \( \Omega^{(q)} \). Let \( \alpha \) be a nonzero complex number for which there exists an element \( p \) in \( A \) such that,

(a) \( \alpha \) is a simple root of the equation \( G_p(x) = 0 \), and

(b) \( \Omega^{(q)}[\star, \Re \alpha] < \Omega^{(q)}[\star, \Re \alpha] \) for all \( q \in A - \{p\} \).

Assume further that if \( A \neq \{p\} \), then there exists an element \( t \) in \( A - \{p\} \) such that,

(c) \( G_t(\alpha) \neq 0 \), and

(d) \( \Omega^{(q)}[\star, \Re \alpha] < \Omega^{(q)}[\star, \Re \alpha] \) for all \( q \in A - \{p, t\} \).

Then, for any complex number \( c \neq 0 \), the equation \( \Omega = 0 \), possesses a solution \( y_c \), which is analytic in an element of \( F(a, b) \) and satisfies \( y_c \sim cz^{\alpha} \) over \( F(a, b) \).

4. Lemma.

Let \( \Omega(z, y, y') = \sum f_{k,j}(z) y^k(y')^j \) be a nontrivial polynomial in \( y \) and \( y' \) whose coefficients belong to a logarithmic field of rank zero over \( F(a, b) \), and assume that \( \Omega \) is homogeneous as a polynomial in \( y \) and \( y' \) (i.e. for some \( p \), \( \Omega = \Omega^{(p)} \)). Then, if \( \alpha \) is a nonzero complex root of the critical equation \( G_p(x) = 0 \) of \( \Omega \), and \( c \) is a nonzero complex number, then the equation \( \Omega = 0 \) possesses a solution \( g \), which is analytic in an element of \( F(a, b) \), and satisfies \( g \sim cz^{\alpha} \) over \( F(a, b) \).

PROOF. Since \( \Omega = \Omega^{(p)} \), if we divide the equation \( \Omega = 0 \) by \( y^p \), and set \( v = y'/y \), we obtain

\[
H(v) = \sum_{k+j=p} f_{k,j}(z) v^j = 0 .
\]
Let $\beta = \Omega^*[\ast, 0]$, let $J_\sigma$ be as in (7) and let $\sigma$ be a complex number. Using the representation (4) for the coefficients, and noting that $a_{kj} - j = \beta$ if $(k, j) \in J_\sigma$, while $a_{kj} - j < \beta$ if $(k, j) \notin J_\sigma$, it follows easily from (6) and (8) that,

\begin{equation}
H(\sigma z^{-1}) = z^\theta (G_\sigma(\sigma) + E) \quad \text{where } E \ll 1 \quad \text{over } F(a, b).
\end{equation}

Let $\sigma_0$ be a complex number which is not a root of $G_\sigma$. Then since $G_\sigma(\sigma) = 0$, we have,

\begin{equation}
H(\sigma z^{-1}) \ll z^\theta \approx H((\sigma_0/\sigma) \alpha z^{-1}).
\end{equation}

Thus by \cite[\S 5(a)]{2}, $\alpha z^{-1}$ is a point of instability of $H$, so in the terminology of \cite[\S 5]{6}, the instability multiplicity of $\alpha z^{-1}$ for $H$ is at least 1, that is,

\begin{equation}
\text{inst}(\alpha z^{-1}, H) \geq 1.
\end{equation}

(This shows immediately that $H$ is of degree $\geq 1$). Since $H$ has coefficients in a logarithmic field of rank zero, it follows from \cite[Theorem II, p. 244]{6} (by applying this result to, in the terminology of \cite[p. 246]{6}, the logarithmic quadruple $(F, E_0 \ast (F), R, S_0)$, where $F = F(a, b)$ and $R$ is the set of real numbers), that there exists a logarithmic field of rank zero over $F(a, b)$ in which $H$ factors completely into linear factors. Hence there exist distinct functions $\varphi_1, \ldots, \varphi_q$ all lying in a logarithmic field of rank zero over $F(a, b)$, and all nonidentically zero, such that for some $(k, j)$ we have

\begin{equation}
H(\varphi_j) \ll \varphi_j \approx \alpha z^{-1}.\quad \text{(by (12))}.
\end{equation}

Since $\varphi_j$ lies in a logarithmic field of rank zero, we have,

\begin{equation}
\varphi_j \approx \alpha z^{-1}(1 + w_0),
\end{equation}

where $m_1, \ldots, m_q$ are positive integers. Now if $q = 0$ or none of the functions $\varphi_1, \ldots, \varphi_q$ are $\sim \alpha z^{-1}$, then by building up $H$ one factor at a time, it would follow from repeated applications of \cite[\S\S 24, 25]{6}, that $\text{inst}(\alpha z^{-1}, H) = 0$, contradicting (11). Thus $q > 1$ and for some $j$, $1 < j < q$, we have,

\begin{equation}
\varphi_j \sim \alpha z^{-1} \quad \text{and} \quad H(\varphi_j) \equiv 0 \quad \text{(by (12))}.
\end{equation}

Since $\varphi_j$ lies in a logarithmic field of rank zero, we have,
where $\omega_0 \ll 1$ and $\omega_0$ also lies in the field. If $\omega_0 = 0$, then $H(xz^{-1}) \equiv 0$, so clearly $cz^x$ would be a solution of $\Omega = 0$ and the proof would be complete. If $\omega_0 \neq 0$, then by property (iii) for a field of rank zero, (and noting that $\omega_0 \ll 1$), we have, over $F(a, b)$,

(15) \[ w_0 \sim Kz^\lambda, \quad \text{for some } \lambda < 0 \text{ and } K \neq 0. \]

Recalling that the elements of $F(a, b)$ are simply-connected, and choosing a point $r$ in the domain of $w_0$, let

(16) \[ U(z) = \int z^{\lambda-1} w_0(\zeta) d\zeta. \]

In view of (15), it follows from [4; Lemma $\zeta$ (b), p. 272], that there is a complex number $K_1$ such that

(17) \[ K_1 + U \sim (\alpha K/\lambda) z^\lambda \quad \text{over } F(a, b). \]

Now set,

(18) \[ U_1 = K_1 + U \quad \text{and} \quad W = \exp U_1. \]

Then if we set $g = cz^x W$, it follows easily from (14) that $g'/g = \varphi_i$, and hence since $H(\varphi_i) \equiv 0$ (by (13)), we see that $g$ is a solution of $\Omega = 0$. Hence the proof will be complete if we show $g \sim cz^x$, or equivalently that $W \sim 1$ over $F(a, b)$. To this end, let $V = W - 1$. Since $\lambda < 0$, clearly $U_1 \to 0$ and hence $W \to 1$ over $F(a, b)$. Thus $V \to 0$ over $F(a, b)$. Now let $k$ be a positive integer, and refer to the operators $\theta_k^j$ in §2(b). Clearly $\theta_k^j V = W \theta_k^j U_1$, since $W' = WU'_1$. By induction on $j$, it is easy to see that for $j > 1$ we have,

(19) \[ \theta_k^j V = WQ_j(\theta_k^1 U_1, \theta_k^2 U_1, \ldots, \theta_k^j U_1), \]

where $Q_j(u_1, \ldots, u_j)$ is a polynomial in $u_1, \ldots, u_j$, with constant coefficients and $Q_j(0, \ldots, 0) = 0$. Since $\lambda < 0$, it follows from (17) that $U_1 \ll 1$. Thus $\theta_k^j U_1 \to 0$ over $F(a, b)$ for all positive $k$ and $j$. Since $W \to 1$, it follows from (19) that $\theta_k^j V \to 0$ for all positive $k$ and $j$ and hence $V \ll 1$ over $F(a, b)$. Thus $W \sim 1$ and the proof of the lemma is now complete.
5. Proof of the main result (§ 3).

We can assume at the outset that \( A \neq \{p\} \), for if \( A = \{p\} \), then \( \Omega = \Omega^{(p)} \) and the result follows from the previous lemma.

Let the nonzero coefficients of \( \Omega \) have the representation (4) and let \( G_q \) and \( J_q \) be as in (6) and (7). By hypothesis, we have,

\[
G_p(x) = 0 \quad \text{and} \quad G'_p(x) \neq 0 .
\]

(Note that if \( p = 0 \), \( G_p(x) \) would have no roots.) For convenience, let

\[
b_q = \Omega^{(p)}[\ast, 0] \quad \text{for} \quad q \in A ,
\]

and let \( c \) be any nonzero constant. By the previous lemma, there exists an analytic function \( g \) in an element of \( F(a, b) \) such that over \( F(a, b) \),

\[
g \sim cz^\alpha \quad \text{and} \quad \Omega^{(p)}(z, g(z), g'(z)) = 0 .
\]

Since \( \alpha \neq 0 \), it follows from § 2(b) that we have the representations,

\[
g = cz^\alpha(1 + E_1) \quad \text{and} \quad g' = \alpha cz^{\alpha-1}(1 + E_2) \quad \text{where} \quad E_i \ll 1 .
\]

We now consider the equation,

\[
\Omega(z, y, y')/g^p z^b = 0 .
\]

Under the change of dependent variable \( y = gw \), (24) is transformed into the equation,

\[
A(z, w, w') = \sum d_{nm} w^n w'^m = 0 ,
\]

where for each \( q > 0 \), and \( m < q \), we have

\[
d_{q-m,m} = \sum_{j=m}^{q} T_{q, m, j} ,
\]
where
\begin{equation}
T_{q_m j} = g^{-p-z^{-p}} f_{q-j, i} \left( \frac{j}{m} \right) g^{z^{-j+m}(g')^{j-m}}.
\end{equation}

From the second relation in (22), it follows that
\begin{equation}
d_{\varphi 0} = 0.
\end{equation}

We now consider $d_{p-1,1}$. In view of the representations (4) and (23), and the definitions of $b_p$, $J_p$ and $G_p(x)$, it follows from a straightforward calculation of (26) for the case $q = p$ and $m = 1$, that over $F(a, b)$,
\begin{equation}
d_{p-1,1} \sim \lambda z, \quad \text{where } \lambda = G'_p(\alpha) \neq 0 \text{ by (20)}.
\end{equation}

From the representations (4) and (23), we see that if $f_{p-j, i} \neq 0$, then $T_{q_m j} \approx z_{q-p-j, j-b_p+m}$. By definition of $b_p$, it follows that for each $j$, either $T_{q_m j}$ is $\approx z^m$ or is $\ll z^m$. Thus by (26), it easily follows that for each $m$,
\begin{equation}
d_{q-m, m} \text{ is either } \approx z^m \text{ or is } \ll z^m.
\end{equation}

We now consider $d_{q-m, m}$ where $q \in A - \{p\}$. From (4) and (23), we see that if $f_{q-j, i} \neq 0$, then
\begin{equation}
T_{q_m j} \approx z^{\beta(q, 0) + m},
\end{equation}
where,
\begin{equation}
\beta(q, j) = \alpha q + a_{q-j, i} - j - (\alpha p + b_p).
\end{equation}

Now from (b) of the hypothesis, there exists $\delta > 0$ such that $Q^q[*, \Re \alpha] < Q^q[*, \Re \alpha] - 2\delta$ for all $q \in A - \{p\}$. Hence,
\[\Re (\beta(q, j)) < -2\delta,\]
and thus from (31) (and § 2(b)), we have $T_{q_m j} \ll z^{m-\delta}$. Thus from (26),
\begin{equation}
d_{q-m, m} \ll z^{m-\delta} \quad \text{for } q \in A - \{p\}.
\end{equation}

Of course if $q \notin A$, then each $f_{q-j, i} = 0$ so clearly each $d_{q-m, m} = 0$ by (26).
Hence, with (33) we have,

$$d_{q-m-m} \ll z^{m-\delta} \quad \text{if } q \neq p .$$

Under the change of dependent variable $w = 1 + v$, equation (25) is transformed into the equation,

$$\Phi(z, v, v') = \sum D_{k,j} v^j(v')^j = 0 ,$$

where

$$D_{k,j} = \sum_{n \geq k} \binom{n}{k} d_{n,j} .$$

From (28) and (34), we see that,

$$D_{k,0} \ll z^{-\delta} \quad \text{over } F(a, b) \text{ for } k > 0 .$$

From (29) and (34), we see that since $\delta > 0$,

$$D_{0,1} \sim \lambda z \quad \text{over } F(a, b) ,$$

while for $k > 1$,

$$D_{k,1} \text{ is either } \approx z \text{ or } \ll z .$$

Finally, from (30) and (34), we see that for each $k$, $j$,

$$D_{k,j} \text{ is either } \approx z^j \text{ or } \ll z^j .$$

We now investigate the coefficient $D_{00}$ more closely. Clearly, $D_{00} = \Omega(z, g, g')/g^p z^{b_p}$: In view of (22) and the definition of the set $A$, we have

$$D_{00} = g^{-p} z^{-b_p} \sum \{ \Omega^{(q)}(z, g, g') : q \in A - \{p\} \} .$$

In view of the representations (4) and (23), and the definitions of $b_z$, $J_z$ and $G_q(x)$, it follows from a straightforward calculation, that for $q \in A$, we have

$$\Omega^{(q)}(z, g, g') = c^z z^{a_q+b_q} (G_q(x) + E_q) \quad \text{where } E_q \ll 1 .$$
Now by hypotheses (e) and (d), there exist $t \in A - \{p\}$ and $\delta_1 > 0$ such that $G_t(x) \neq 0$, and

$$\Omega^{(e)}[\ast, \Re x] < \Omega^{(d)}[\ast, \Re x] - 2\delta_1 \quad \text{for} \quad q \in A - \{p, t\}.$$  

Thus from (42),

$$\Omega^{(d)}(z, g, g') \sim e^t \lambda_1 z^{x+t+b} \quad \text{where} \quad \lambda_1 = G_t(x) \neq 0.$$  

Furthermore, since $\Re(xq + b_t - (xt + b_t)) < -2\delta_1$ by (43) for $q \in A - \{p, t\}$, it follows from §2(b) that $z^{x+t+b} \ll z^{-\delta} \ll 1$, and hence by (42) and (44), we have that for each $q \in A - \{p, t\}$, the function $\Omega^{(d)}(z, g, g')$ is $\ll$ the function $\Omega^{(d)}(z, g, g')$ over $F(a, b)$.

Thus from (41), (44) and (23), we have,

$$D_{00} \sim K_1 z^\beta \quad \text{over} \quad F(a, b),$$

where

$$\beta = xt + b_t - (xp + b_p) \quad \text{and} \quad K_1 = e^{t-p} \lambda_1.$$  

By hypothesis (b), we have,

$$\Re(\beta) < 0.$$  

In view of (45) and (38), we have,

$$D_{00}/D_{01} \sim (K_1/\lambda) z^{\beta-1}.$$  

We consider the equation,

$$h' = -D_{00}/D_{01}. $$

Under the change of dependent variable,

$$h = -(K_1/\lambda \beta) z^\beta(1 + \varphi),$$

followed by multiplication by $(\lambda/K_1) z^{1-\beta}$, we have from (48) that equation (49) is transformed into an equation of the form,

$$\varphi + (z/\beta) \varphi' = U, \quad \text{where} \quad U \ll 1 \quad \text{over} \quad F(a, b).$$
In view of (47), it follows from [4; Lemma 6(b), p. 271], that equation (51) possesses a solution \( q_0 \), analytic in an element of \( F(a, b) \), and satisfying \( q_0 \ll 1 \) over \( F(a, b) \). Thus by (50), the equation (49) possesses a solution \( h_0 \) such that

\[
(52) \quad h_0 \sim -(K_1/\lambda \beta) z^\beta \quad \text{over } F(a, b).
\]

Under the change of dependent variable \( v = h_0 + h_0 u \), equation (35) is transformed into the equation,

\[
(53) \quad \Psi(z, u, u') = \sum t_{mn} u^m(u')^n = 0,
\]

where the coefficients are given by the formula,

\[
(54) \quad t_{mn} = (h_0)^n \sum_{k \geq m} \binom{k}{m} \Gamma_{kn},
\]

where

\[
(55) \quad \Gamma_{kn} = \sum_{i=0}^{k} \binom{k+i}{n} D_{i,k+n-i}(h_0)^i (h_0')^{k-i}.
\]

Since \( h_0 \) solves equation (49), it follows from (48) and (52) that for \( 0 \ll i \ll k \),

\[
(56) \quad h_0'(h_0')^{k-i} \approx z^{k\beta-k+i} \quad \text{over } F(a, b).
\]

It thus follows from (40) and (55) that for each \( k, n \),

\[
(57) \quad \Gamma_{kn} \text{ is either } \approx z^{k\beta+n} \text{ or } \ll z^{k\beta+n} \text{ over } F(a, b).
\]

Now \( \Gamma_{00} = D_{00} \) and \( \Gamma_{10} = D_{01} h_0' + D_{10} h_0 \). Since \( h_0 \) solves (49), it follows from (54) that

\[
(58) \quad t_{00} = D_{10} h_0 + \sum_{k \geq 2} \Gamma_{k0}.
\]

In view of (37) and (52), \( D_{10} h_0 / z^\beta \ll 1 \). Furthermore, if \( k \geq 2 \), then since \( \text{Re}((k-1)\beta) < 0 \) (by (47)), it follows from (57) and § 2(b) that \( \Gamma_{k0} / z^\beta \ll 1 \). Hence from (58),

\[
(59) \quad t_{00} / z^\beta \ll 1 \quad \text{over } F(a, b).
\]
An existence theorem for certain solutions etc. 79

Now $\Gamma_{o1} = D_{o1}$ so by (38),

$$\Gamma_{o1} \sim \lambda z.$$  

For $k \geq 1$, $\text{Re}(k\beta + 1) < 1$ by (47), so by (57) and §2(b), $\Gamma_{o1} \ll z$ for $k \geq 1$. Thus by (54) and (60), clearly, $t_{o1} \sim h_0 \lambda z$. Thus by (52),

$$t_{o1}/z^\theta \sim (-K_1/\beta)z \quad \text{over } F(a, b).$$

We now consider $t_{10} = \sum_{k \geq 1} k\Gamma_{o1}$. Now $\Gamma_{10} = D_{o1} h_0 + D_{10} h_0$. Since $h_0$ solves (49), it follows easily from (37), (38), (48) and (52), that, $D_{10} h_0 \ll D_{o1} h_0 \sim -K_1 z^\theta$. Thus,

$$\Gamma_{10} \sim -K_1 z^\theta.$$  

Now for $k \geq 2$, $\text{Re}((k-1)\beta) < 0$ (by (47)), so clearly from (57) and §2(b), we have

$$\Gamma_{k0} \ll z^\theta \quad \text{for } k \geq 2.$$  

Thus with (62), we have $t_{10} \sim -K_1 z^\theta$, and hence

$$t_{10}/z^\theta \sim -K_1 \quad \text{over } F(a, b).$$

From (63), it immediately follows that over $F(a, b)$,

$$t_{m0}/z^\theta \ll 1 \quad \text{for } m \geq 2.$$  

Finally, we consider the ratio $t_{mn}/t_{o1}$ for $n \geq 1$ and $m + n \geq 2$. We rewrite (54) in the form,

$$t_{mn}/t_{o1} = \sum_{k \geq m} \binom{k}{m} (h_0)^n (\Gamma_{kn}/t_{o1}).$$

In view of (52), (57) and (61), the term in the summation corresponding to $k$ is either $\approx$ or $\ll z^{\beta(k+n-1)+n-1}$. But $k \geq m$, so $k + n - 1 \geq m + n - 1 \geq 1$. Thus by (47), $\text{Re}(\beta(k + n - 1)) < 0$, and hence it follows from §2(b) that each term in the sum on the right of (66) is $\ll z^{n-1}$,

$$t_{mn}/t_{o1} \ll z^{n-1} \quad \text{over } F(a, b) \text{ if } n \geq 1 \text{ and } m + n \geq 2.$$
If follows from the asymptotic relations (59), (61), (64), (65) and (67) that the polynomial \( \Psi(z, u, u')/(-K_1 z^\delta) \) is normal over \( F(a, b) \) in the sense of [5; § 83]). In view of (47), it follows from [5; §§ 117, 118] that the equation,

\[
\Psi(z, u, u')/(-K_1 z^\delta) = 0 ,
\]

possesses a unique solution \( u_0 \), which is analytic in an element of \( F(a, b) \) and satisfies,

\[
u_0 \ll 1 \text{ over } F(a, b).
\]

From (35), it follows that \( w_0 = 1 + h_0 + h_0 u_0 \) satisfies equation (25). Thus the function \( y_0 = g(1 + h_0 + h_0 u_0) \) is analytic in an element of \( F(a, b) \) and is a solution of the original equation \( \Omega(z, y, y') = 0 \). In view of (69) and the fact that \( h_0 \ll 1 \) over \( F(a, b) \) by (47) and (52) (and § 2(b)), we see that \( y_0 \sim g \) over \( F(a, b) \) and hence by (22), \( y_0 \sim cz^\alpha \). This concludes the proof of the theorem.

6. Remark.

We point out here that for the differential polynomials \( \Omega \), and the functions \( cz^\alpha \) which are treated in the main result, it was shown in the course of the proof that \( cz^\alpha \) is a point of instability of a homogeneous \( \Omega \), if and only if \( \alpha \) is a root of the critical equation of \( \Omega \). (If \( \Omega = \Omega^{(\alpha)} \) and if \( \alpha \) is not a root of the critical equation \( G_\alpha(x) = 0 \), then from the calculation in (42) it follows that for any function \( g \sim cz^\alpha \), we have that \( \Omega(z, g, g') \sim c^\alpha G_\alpha(x) x^{\alpha+q} \), so clearly \( cz^\alpha \) is not a point of instability of \( \Omega \). Conversely, if \( \alpha \) is a root of \( G_\alpha(x) = 0 \), then by § 4, the equation \( \Omega = 0 \) possesses a solution \( \sim cz^\alpha \) and hence clearly \( cz^\alpha \) is a point of instability of \( \Omega \).)

REFERENCES


