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On an Evolution Equation in Banach Spaces.

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SUNTO - Vengono provati risultati sull'esistenza, unicità e regolarità delle soluzioni di un'equazione di evoluzione semilineare con condizioni ai limiti non lineari.

1. Introduction.

In this paper we shall prove existence, uniqueness and regularity theorems for an evolution equation in Banach spaces. The evolution equation is a semilinear one and, in addition, the boundary conditions imposed are non linear in nature. The existence and uniqueness results extend those obtained by the author in [12] and the regularity results seem to be new.

The equation under study is of the form

$$\begin{cases} \lambda u(t) + \dfrac{du(t)}{dt} + A(t)u(t) + F\big(t, u(t)\big) = v(t) & \lambda > 0\ ,\ 0 \leqslant t \leqslant T\ , \\ \varphi\big(u(0)\big) = u(T)\ , \end{cases}$$

where, for each t, the linear operator $-A(t)$ is the generator of a contraction semigroup, F is a non linear function and φ generates the non linear boundary conditions.

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In [12] we dealt with the case when $A(t)$ was independent of t. Here this requirement is relaxed and similar existence and uniqueness results are obtained for this more general situation. The solutions obtained are « generalized solutions » in general; however, under supplementary hypotheses, they are shown to be strict solutions; this yields regularity results.

2. Assumptions and results.

Let X be a Banach space and $B(X, X)$ the space of the linear bounded operators from X to X; let Y be the space $C(0, T; X)$ or $L^p(0, T; X)$, $p \geqslant 1$. We denote by $|\cdot|$ the norm on X, by $|\cdot|_{B(X,X)}$ the norm on $B(X, X)$ and by $\|\cdot\|$ the norm on Y. If g is a Lipschitz continuous function from X to X (from Y to Y) we denote by $|g|_L$ (by $\|g\|_L$) the Lipschitz norm of g, i.e.

$$|g|_L = \sup\left\{\frac{|g(x)-g(y)|}{|x-y|},\ x, y \in X,\ x \neq y\right\}.$$

We shall study the equation

$$\text{(E)}\qquad \begin{cases} \lambda u(t) + \dfrac{du(t)}{dt} + A(t)u(t) + F(t, u(t)) = v(t), \\ \qquad\qquad\qquad\qquad \lambda > 0,\ v(t) \in Y,\ 0 \leqslant t \leqslant T\,, \\ \varphi(u(0)) = u(T)\,, \end{cases}$$

where, for each t, $-A(t)$ is the generator of a linear contraction semigroup, F is a function from $D_F \subset [0, T] \times X$ to X and φ a (in general nonlinear) closed operator from $D_\varphi \subset X$ to X.

DEFINITION 1. *Let $v(t)$ belong to $C(0, T; X)$ $\big($to $L^p(0, T; X)\big)$. The function $u(t)$ from $[0, T]$ to X is a strict solution of equation (E) if it is continuous in $[0, T]$ with $\varphi(u(0)) = u(T)$, continuously differentiable in $[0, T]$ $\big($differentiable a.e. in $(0, T)$ and du/dt belongs to $L^p(0, T; X)\big)$, for each $t \in [0, T]$ $\big($a.e. in $(0, T)\big)$ $u(t)$ belongs to $D_{A(t)}$ and $(t, u(t))$ belongs to D_F, and (E) is satisfied for $t \in [0, T]$ $\big($a.e. in $(0, T)\big)$.*

DEFINITION 2. *Let L be an operator from $D_L \subset Y$ to Y. We call u a generalized solution of the equation $Lu = v$, $v \in Y$, if there exists a sequence $\{u_n\} \in D_L$ such that $u_n \to u$ and $Lu_n \to v$.*

We define the operators

$$\begin{cases} D_B = \left\{u \in Y;\ \dfrac{du}{dt} \in Y,\ A(t)u(t) \in Y,\ \varphi(u(0)) = u(T)\right\}, \\ (Bu)(t) = \dfrac{du(t)}{dt} + A(t)u(t)\,, \end{cases}$$

and

$$D_f = \{u \in Y;\ fu \in Y\}\,, \qquad (fu)(t) = F(t, u(t))\,.$$

Definition 3. *We call u a generalized solution of equation* (E) *it it is a generalized solution of the equation*

$$\lambda u + Bu + fu = v\,. \tag{1}$$

In [12] we gave sufficient conditions for the existence and uniqueness of a generalized solution of equation (E) in the case when $A(t)$ was independent of t. Here the assumptions on $A(t)$ are that

1) $-A(t)$ is for each t the infinitesimal generator of a contraction semigroup; the domain $D_{A(t)} = D$ of $A(t)$ is independent of t; there exists a unique *evolution operator*, $U(t, s)$, for the family $\{-A(t)\}$, i.e. a function from $\Gamma = \{(t, s) \in \mathbb{R}^2,\ 0 \leqslant s \leqslant t \leqslant T\}$ to $B(X, X)$ such that for each $x \in X$, $U(t, s)x$ is continuous from Γ to X; for each $x \in D$, $U(t, s)x \in D$ and

$$\frac{\partial U(t, s)x}{\partial t} = -A(t)\,U(t, s)\,x;$$

$U(t, t) = I$ and $|U(t, s)|_{B(X,X)} \leqslant 1$; every strict solution of equation

$$\frac{du}{dt} + A(t)u(t) = v(t)\,, \qquad v(t) \in Y\,,$$

can be expressed by $u(t) = U(t, 0)\,u(0) + \int_0^t U(t, s)\,v(s)\,ds$ (cfr. [5], [7], [8], [10], [11]).

Moreover, when we dealt with the space $C(0, T; X)$, we shall assume that

2) for each $x \in D$, $A(t)x$ is continuous from $[0, T]$ to X; $(\lambda + A(t))^{-1}$ is strongly continuous from $[0, T]$ to $B(X, X)$ for each $\lambda > 0$;

3) $A^{-1}(t)$ is, for each t in $[0, T]$ a bounded operator and if we set

$$k(t, s) = A(t)\,U(t, s)A^{-1}(s)\,,$$

then, for each $v(s) \in C(0, T; X)$, the function $(s, t) \to k(t, s)v(s)$ is integrable on Γ and, for each $s \in [0, T]$, the function $t \to k(t, s)v(s)$ is continuous on $(s, T]$ uniformly for s in $[0, T]$.

When we deal with the space $L^p(0, T; X)$ we shall assume, instead of 2) and 3), the following

2′) $A(t)$ and $(\lambda + A(t))^{-1}$, $\lambda > 0$, are strongly measurable on $(0, T)$; the function $t \to U(t, s)x$ is weakly absolutly continuous on $[s, T]$ for each $x \in D$ and $s \in [0, T]$.

3′) $A^{-1}(t)$, is, a.e. in $(0, T)$, a bounded linear operator and if we set $k(t, s) = A(t)\,U(t, s)A^{-1}(s)$ then $k(t, s)v(s) \in L^p(\Gamma; X)$ for each $v(s) \in L^p(0, T; X)$.

Under such hypotheses on $A(t)$, we shall prove the following existence and uniqueness results:

THEOREM 1. *Let the conditions* 1), 2) *and* 3) (*the conditions* 1), 2′) *and* 3′)) *be satisfied. For each* $\lambda > 0$ *let* $(\varphi - \exp[-\lambda T]\,U(T, 0))^{-1}$ *be a Lipschitz continuous function from* X *to* X *such that*

$$|(\varphi - \exp[-\lambda T]\,U(T, 0))^{-1}|_L \leq \frac{1}{1 - \exp[-\lambda T]} \tag{2}$$

and

$$(\varphi - \exp[-\lambda T]\,U(T, 0))^{-1} D \subset D\,. \tag{3}$$

Let f *be a accretive continuous operator from* Y *to* Y. *Then for each* $v(t) \in C(0, T; X)$ (*for each* $v(t) \in L^p(0, T; X)$) *there exists a unique generalized solution of equation* (E).

THEOREM 2. *Let the conditions* 1), 2) *and* 3) (*the conditions* 1), 2′) *and* 3′)) *and*

$$U(T, s)X \subset D \tag{4}$$

be satisfied. Let φ^{-1} *be a Lipschitz continuous function from* X *to* X *such that*

$$|\varphi^{-1}|_L \leqslant 1 \quad \text{and} \quad \varphi^{-1}(D) \subset D\,. \tag{5}$$

Let f *be a accretive continuous operator from* Y *to* Y. *Then, for each* $v(t) \in C(0, T; X)$ (*for each* $v(t) \in L^p(0, T; X)$), *there exists a unique generalized solution for equation* (E).

We shall prove also:

THEOREM 3. *Let the hypotheses of Theorem* 2 *be satisfied and let the equation*

$$\begin{cases} \lambda u(t) + \dfrac{du(t)}{dt} + A(t)u(t) = v(t)\,, \\ u(0) = 0\,, \end{cases} \tag{6}$$

have a strict solution for each $v(t) \in Y$ *when* $\lambda > 0$. *Then the equation* (E) *has a unique strict solution.*

From this theorem, using similar arguments to those used by Da Prato in [2], regularity results for the solutions of some partial differential equations are deduced.

3. Proofs of the theorems.

In order to prove Theorems 1 and 2, we show that, in such hypotheses, the closure, $\bar{B}$, of the graph of the operator B is m-accretive. As f is a continuous accretive operator from Y to Y, it follows from Barbu's result [1] that $\bar{B} + f$ is m-accretive, and hence equation (E) has a unique generalized solution for each $v(t) \in Y$.

We give the proofs only in the case Y is the space $C(0, T; X)$; the proofs in the case Y is the space $L^p(0, T; X)$ are analogous.

LEMMA 1. *Let* $v(t)$ *be a member of* Y *such that* $v(t) \in D$ *for each* $t \in [0, T]$ *and* $A(t)v(t) \in Y$. *If*

$$u(t) = \int_0^t U(t, s)v(s)\,ds\,,$$

then the functions $u(t)$, $A(t)u(t)$ *and* $du(t)/dt$ *belong to* Y.

PROOF. Since the map $(t, s) \to U(t, s)v(s)$ is continuous on Γ, $u(t) \in Y$. From condition 3), it follows that $A(t)\,U(t, s)A^{-1}(s)A(s)v(s)$ is integrable and the map $t \to \int_0^t A(t)\,U(t, s)A^{-1}(s)A(s)v(s)\,ds$ is continuous. As $A(t)$ is closed,

$$\int_0^t A(t)\,U(t, s)v(s)\,ds = A(t)\int_0^t U(t, s)v(s)\,ds\,.$$

Thus $A(t)u(t)$ and $du(t)/dt = v(t) + A(t)u(t)$ belong to Y.

LEMMA 2. *Let the conditions* 1), 2) *and* 3) *be satisfied. Let the map* $\varphi - U(T, 0)$ *be invertible and let* $(\varphi - U(T, 0))^{-1}$ *be continuous and*

$$(\varphi - U(T, 0))^{-1}D \subset D\,. \tag{7}$$

Then the equation

$$Bu = v\,, \qquad v \in Y\,, \tag{8}$$

has the unique generalized solution

$$u(t) = U(t, 0)(\varphi - U(T, 0))^{-1}\int_0^T U(T, s)v(s)\,ds + \int_0^t U(t, s)v(s)\,ds\,. \tag{9}$$

PROOF. Let u be a generalized solution of equation (8). Then there exists a sequence $\{u_n\} \in D_B$ such that $u_n \to u$ and $Bu_n \to v$.

Setting

$$v_n = u_n'(t) + A(t)u_n(t)\,,$$

we have

$$u_n = U(t, 0)u_n(0) + \int_0^t U(t, s)v_n(s)\,ds\,,$$

where

$$u_n(0) = (\varphi - U(T, 0))^{-1}\int_0^T U(T, s)v_n(s)\,ds\,.$$

Since $v_n \xrightarrow{Y} v$ and $\|U(t, s)\| \leq 1$, it follows that

$$\int_0^t U(t, s) v_n(s)\, ds \xrightarrow{X} \int_0^t U(t, s) v(s)\, ds$$

and, as $(\varphi - U(T, 0))^{-1}$ is continuous,

$$u_n(0) \xrightarrow{X} (\varphi - U(T, 0))^{-1} \int_0^T U(T, s) v(s)\, ds .$$

Therefore u is given by (9).

We shall prove now that function (9) is a generalized solution of equation (8). Set

$$u_n(t) = U(t, 0) u_{n,0} + \int_0^t U(t, s) n(n + A(s))^{-1} v(s)\, ds$$

where

$$u_{n,0} = (\varphi - U(T, 0))^{-1} \int_0^T U(T, s) n(n + A(s))^{-1} v(s)\, ds .$$

From Lemma 1 and condition 3), it follows easily that u_n belongs to D_B.

It is known that $n(n + A(t))^{-1} v(t) \xrightarrow{X} v(t)$. Hence from condition 1) we have also that $n(n + A(t))^{-1} v(t) \xrightarrow{Y} v(t)$. It follows that $u_n \xrightarrow{Y} u$ and

$$Bu_n = \frac{du_n(t)}{dt} + A(t) u_n(t) = n(n + A(t))^{-1} v(t) \xrightarrow{Y} v .$$

Proof of Theorem 1. It is enough to prove that $\bar{B}$ is a m-accretive graph. From Lemma 2, the equation

$$\lambda u + Bu = v , \qquad \lambda > 0 , \qquad v \in E$$

has the unique generalized solution

$$u(t) = \exp[-\lambda t]\, U(t, 0)(\varphi - \exp[-\lambda T]\, U(T, 0))^{-1} \int_0^T \exp[-\lambda(T - s)] \cdot$$

$$\cdot\, U(T, s) v(s)\, ds + \int_0^t \exp[-\lambda(t - s)]\, U(t, s) v(s)\, ds .$$

Then $(\lambda+\bar{B})^{-1}$, $\lambda>0$, is the graph of a function from Y to Y. Moreover, if v_1 and v_2 belong to Y, we have

$$|(\lambda+\bar{B})^{-1}v_1(t)-(\lambda+\bar{B})^{-1}v_2(t)|\leqslant\exp[-\lambda t]\frac{1}{1-\exp[-\lambda T]}\cdot$$
$$\cdot\int_0^T\exp[-\lambda(T-s)]|v_1(s)-v_2(s)|ds+$$
$$+\int_0^t\exp[-\lambda(t-s)]|v_1(s)-v_2(s)|ds\leqslant\frac{\exp[-\lambda t]}{\lambda}\|v_1-v_2\|+$$
$$+\frac{1-\exp[-\lambda t]}{\lambda}\|v_1-v_2\|=\frac{1}{\lambda}\|v_1-v_2\|,$$

and then $\|(\lambda+\bar{B})^{-1}v_1-(\lambda+\bar{B})^{-1}v_2\|\leqslant\frac{1}{\lambda}\|v_1-v_2\|$. Hence $\bar{B}$ is m-accretive.

Proof of Theorem 2. The proof is exactly similar to that of Theorem 2 of [12] so we omit it.

Proof of Theorem 3. It is enough to prove that, under such hypotheses, the operator B is closed. This follows from

Lemma 3. *Let the hypotheses of Lemma 2 be satisfied and let $U(T,0)X\subset D$. If the equation (6) has a strict solution for each $v(t)\in Y$ wheu $\lambda=0$, then the equation*

$$\text{(10)}\qquad\begin{cases}\dfrac{du(t)}{dt}+A(t)u(t)=v(t)\,,\\ \varphi(u(0))=u(T)\,,\end{cases}$$

has a unique strict solution.

Proof. From conditions 4) and 7) it follows that

$$(\varphi-U(T,0))^{-1}\int_0^T U(t,s)v(s)\,ds$$

belongs to D. Thus the problem

$$\begin{cases} \dfrac{du(t)}{dt} + A(t)u(t) = v(t) , \\ u(0) = (\varphi - U(T, 0))^{-1}\displaystyle\int_0^T U(T, s)v(s)\, ds , \end{cases}$$

has a strict solution given by (9). Therefore (9) is a strict solution of equation (10).

REMARK 1. We can relax the condition that the domain $D_{A(t)}$ of $A(t)$ is independent of t and suppose that there exists a set D dense in X such that $D \subset D_{A(t)}$ for each $t \in [0, T]$.

REMARK 2. If in condition 1) we replace the hypothesis that $-A(t)$ is a generator of a contraction semigroup with the condition that $-A(t)$ belong to $K^M(X)$ [5], i.e.

$$|(\lambda + A(t))^{-1}|_{B(X,X)} \leqslant \frac{M}{\lambda} , \ \forall \lambda > 0 ,$$

with M independent of t and it is regular, and we suppose that the evolution operator satisfies $|U(t, s)|_{B(X,X)} \leqslant M$ instead of $|U(t, s)|_{B(X,X)} \leqslant 1$, then we can prove that the closure, $-\bar{B}$, of the graph of $-B$ belongs to $K^M_L(Y)$, i.e. for each $\lambda > 0$, $(\lambda + \bar{B})^{-1}$ is an operator from Y to Y such that $\|(\lambda + \bar{B})^{-1}\|_L \leqslant M/\lambda$ for each $\lambda > 0$.

4. Applications.

Let Ω be a bounded open set of $\mathbb{R}^n$ with a « sufficient smooth » boundary, $\partial\Omega$ (cfr. [6]). Let Δ be the operator $\sum_1^n {}_i \partial^2/\partial x_i^2$ and let ψ be a twice continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that $\psi(0) = 0$ and $|\psi(t_1) - \psi(t_2)| \geqslant |t_1 - t_2|$, $t_1, t_2 \in \mathbb{R}$. For each $p > 1$ we define the operators on $L^p(\Omega)$

$$D_{A_p} = \{u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\} , \qquad A_p u = -\Delta u ,$$

$$D_{\psi_p} = \{u \in L^p(\Omega), \ \psi_p u \in L^p(\Omega)\} , \qquad (\psi_p u)(x) = \psi(u(x)) ,$$

and the operators on $L^p((0, T)\times\Omega)$

$$\begin{cases} D_{B_p} = \{u \in L^p(0, T; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)) \cap W^{1,p}(0, T; L^p(\Omega)), \\ \qquad\qquad\qquad\qquad \psi_p^{\sim}(u(0)) = u(T)\}, \\ B_p u = \dfrac{du}{dt} - \Delta u . \end{cases}$$

In [12] we proved that the operators A_p and ψ_p satisfy the hypotheses of Theorem 2 if $p > n/2$. Thus, from Lemma 3, it follows that

THEOREM 4. *For each $v \in L^p((0,T)\times\Omega)$, $p > n/2$, there exists a unique function u belonging to $L^p(0, T; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)) \cap W^{1,p}(0, T; L^p(\Omega))$ such that*

$$\begin{cases} \dfrac{\partial u(t, s)}{\partial t} - \Delta u(t, s) = v(t, s) , \\ \psi(u(0, x)) = u(T, x) . \end{cases}$$

We know that the closure of the graph of B_p is m-accretive [12]. From Theorem 4 it follows that

LEMMA 4. *If $p > n/2$, B_p is m-accretive.*

Let $g(t)$ be a continuous accretive function from $\mathbb{R}$ to $\mathbb{R}$ such that $g(0) = 0$. If

$$D_{g_p} = \{u \in L^p((0, T)\times\Omega);\ g_p u \in L^p((0, T)\times\Omega)\}, \qquad (g_p u)(t, x) = g(u(t, x)),$$

then g_p is a m-accretive function on $L^p((0, T)\times\Omega)$.

LEMMA 5. *The operator $B_p + g_p$ is m-accretive if $p > n/2$ and $p \geqslant 2$.*

PROOF. It is known that the closure of the graph of $B_p + g_p$ is m-accretive [12]. Thus we have to prove that $B_p + g_p$ is closed.

Let $\{u_n\} \in D_{B_p} \cap D_{g_p}$ such that $u_n \to u$ and $B_p u_n + g_p u_n \to v$. From a Da Prato's result [2], if $p \geqslant 2$ we have $\|g_p u_n\| \leqslant \|B_p u_n + g_p u_n\|$. Therefore $\|g_p u_n\| \leqslant \text{cost}$, and hence [7] $g_p u_n \rightharpoonup g_p u$ and $B_p u_n \rightharpoonup B_p u$ (we denote by $\rightharpoonup$ weak convergence). Thus, as g_p and B_p are closed in $L^p((0T)\times\Omega)\times L^p_{\text{weak}}((0T)\times\Omega)$, we have $B_p u + g_p u = v$.

THEOREM 5. *Let g and ψ be the functions from $\mathbb{R}$ to $\mathbb{R}$ defined above. Let Ω be a convex bounded open set of $\mathbb{R}^n$. Then, for each $v \in L^p((0T) \times \Omega)$ with $p > n/2$ and $p \geqslant 2$, there exists a unique function u belonging to $L^p(0, T; W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)) \cap W^{1,p}((0, T; L^p(\Omega))$ such that*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \Delta u(t,x) + g(u(t,x)) = v(t,x) \,, \\ \psi(u(0,x)) = u(T,x) \,. \end{cases}$$

PROOF. We know that, as Ω is convex, there exists $\lambda_p > 0$ such that $-\lambda_p + B_p + g_p$ is m-accretive ([3], [12). The result thus follows from Lemma 5.

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