

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

ANTONIO PASINI

**Some fixed point theorems of the mappings  
of partially ordered sets**

*Rendiconti del Seminario Matematico della Università di Padova,*  
tome 51 (1974), p. 167-177

[http://www.numdam.org/item?id=RSMUP\\_1974\\_\\_51\\_\\_167\\_0](http://www.numdam.org/item?id=RSMUP_1974__51__167_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1974, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques*  
<http://www.numdam.org/>

## Some Fixed Point Theorems of the Mappings of Partially Ordered Sets.

ANTONIO PASINI (Firenze) (\*)

### 1. Introduction.

In this paper we give new simple proofs of some fixed point theorems, and strengthen others. The methods we shall use base themselves on two « strong » induction principles, we stated and utilized in [5]. We shall show, moreover, that one of them is equivalent to Axiom of Choice.

Let's now recall some results on the fixed points of a function defined from a partially ordered set  $\langle P; \leq \rangle$  into itself.

**PROPOSITION A.** *Let  $\langle P; \leq \rangle$  be a nonempty partially ordered set every well ordered subset of which has an upper bound. And let  $f$  be a function from  $P$  into  $P$  such that  $x \leq f(x)$  for every  $x$ ; then  $f$  has a fixed point.*

The preceding result is proved in [2] by using Axiom of Choice. As a corollary we get:

**PROPOSITION B.** *Let  $\langle P; \leq \rangle$  be a nonempty partially ordered set every well ordered subset of which has a least upper bound, and let  $f$  be a function just like proposition A's one; then  $f$  has a fixed point.*

The preceding proposition, however, may be proved independently, and without using Axiom of Choice. (see [1] and [2]).

---

(\*) Indirizzo dell'A.: Istituto Matematico « U. Dini » dell'Università di Firenze.

Lavoro eseguito nell'ambito dei gruppi di ricerca matematica del C.N.R.

**PROPOSITION C** (see [3]). *Let  $\langle P; \leq \rangle$  be a nonempty partially ordered set every well ordered subset  $B$  of which has an upper bound which is minimal in the set of upper bounds of  $B$ . Let  $f$  be a mapping from  $P$  into  $P$  such that  $a \leq f(a)$  for some  $a$  in  $P$ , and such that  $x \leq f(x) \leq y$  implies  $f(x) \leq f(y)$ , for every  $x$  and  $y$  in  $P$ ; moreover  $\{x, f(x)\}$  has a greatest lower bound in  $\langle P; \leq \rangle$ ; then  $f$  has a fixed point.*

**PROPOSITION D** (see [1]; see also [2]). *Let  $\langle P; \leq \rangle$  be a nonempty partially ordered set every well ordered subset of which has a least upper bound. And let  $f$  be a mapping from  $P$  into  $P$  such that  $a \leq f(a)$  for some  $a$  in  $P$ , and such that  $x \leq y$  implies  $f(x) \leq f(y)$ , for every  $x$  and  $y$  in  $P$ . Then  $f$  has a fixed point.*

Proposition D is proved in [1] and in [2] without using Axiom of Choice.

**PROPOSITION E** (see [4]). *Let  $\langle P; \leq \rangle$  be a nonempty well ordered set every subset of which has a least upper bound. Let  $f$  be just like in proposition D; then the set  $\{p | f(p) = p\}$  is nonempty and has a maximal element.*

Proposition E is proved in [4] without any use of Axiom of Choice. But this is possible because  $\langle P; \leq \rangle$  is well ordered.

We shall prove (but postpone the proofs):

**THEOREM A.** *Let  $\langle P; \leq \rangle$  be a nonempty partially ordered set every well ordered subset  $B$  of which has an upper bound which is minimal in the set of upper bounds of  $B$ . And let  $f$  be a function from  $P$  into  $P$  such that:*

- (1)  $x \leq f(x) \leq y$  implies  $f(x) \leq f(y)$  for every  $x$  and  $y$  in  $P$ .
- (2)  $\{x, f(x)\}$  has a greatest lower bound in  $P$ , for every  $x$  in  $P$ .
- (3) There is an element  $a$  of  $P$  such that  $a \leq f(a)$ .

*Then the set  $\{p | f(p) = p\}$  is nonempty and has a maximal element.*

Theorem A strengthens proposition C. Theorem A's proof makes use of Axiom of Choice.

**THEOREM B.** *Let  $\langle P; \leq \rangle$  be a nonempty partially ordered set every well ordered subset of which has a least upper bound. And let  $f$  be a mapping from  $P$  into  $P$  such that  $x \leq f(x) \leq y$  implies  $f(x) \leq f(y)$  for every  $x$  and  $y$  in  $P$ . And let  $a \leq f(a)$  for some  $a$  in  $P$ . Then the set  $\{p | f(p) = p\}$  (is nonempty and) has a maximal element.*

REMARK. Theorem B generalizes propositions E and D: in the fact (we require on  $P$  more general conditions than in  $E$  and) what we require on  $f$  is less strong than isotonicity: clearly if  $f$  is isotone  $x \leq f(x) \leq y$  implies  $f(x) \leq f(y)$ , but there are non isotone functions  $f$  such that  $x \leq f(x) \leq y$  implies  $f(x) \leq f(y)$ , and is  $a \leq f(a)$  for a convenient  $a$  in  $P$ . Let consider, for instance, the ordered set  $\langle \{1, 2, 3\}; \leq \rangle$ , where  $1 < 2 < 3$ , and let's pose  $f(1) = 3$ ,  $f(3) = 3$ ,  $f(2) = 1$ .

In theorem B we doesn't need Axiom of Choice in proving  $\{p | f(p) = p\} \neq \emptyset$ . So we get a proof of proposition D which doesn't use Axiom of Choice. Moreover, if we assume  $\langle P; \leq \rangle$  is a well ordered set (as in proposition E), our methods give a proof of the existence of a maximal element in  $\{p | f(p) = p\}$ , which doesn't use Axiom of Choice.

THEOREM C. *Let  $\langle P; \leq \rangle$  and  $f$  be just like in proposition A. Then  $\{p | f(p) = p\}$  (is nonempty and) has a maximal element.*

## 2. « Strong » transfinite induction principles.

Let be given a class  $A$  and a limit ordinal number  $\alpha_0$ . Let  $\alpha$  be a nonzero ordinal number less than  $\alpha_0$ , and  $\Phi_\alpha$  be a function from the set  $[0, \alpha)$  (i.e.: the set of all ordinal numbers less than  $\alpha$ ) into  $A$ . Let  $P(x, y)$  be a first order quantification scheme free on  $x, y$ , belonging to a language which formalizes set theory. And let  $P(\alpha, \Phi_\alpha)$  be the sentence we give from  $P(x, y)$  by substituting  $x$  with  $\alpha$  and  $y$  with  $\Phi_\alpha$ . Moreover let  $P(\alpha, \Phi_\alpha)$  be of the form:

$$(\forall \gamma_1) \dots (\forall \gamma_n) \left( ((\gamma_1 < \alpha) \& \dots \& (\gamma_n < \alpha)) \rightarrow Q(\Phi_\alpha, \gamma_1, \dots, \gamma_n) \right)$$

where  $\gamma_1, \dots, \gamma_n$  vary on  $[0, \alpha)$  and  $Q(\Phi_\alpha, \gamma_1, \dots, \gamma_n)$  satisfies the following condition:

CONDITION A. *For any choice of  $\gamma_1, \dots, \gamma_n$  less than  $\alpha$ ,  $Q(\Phi_\alpha, \gamma_1, \dots, \gamma_n)$  is true only if for every  $\beta$  greater than zero, less than  $\alpha$ , and such that is an upper bound for  $\{\gamma_1, \dots, \gamma_n\}$ ,  $Q(\Phi_{\alpha|[0, \beta)}, \gamma_1, \dots, \gamma_n)$  is true (where  $\Phi_{\alpha|[0, \beta)}$  is the restriction of  $\Phi_\alpha$  to  $[0, \beta)$ ).*

We proved in [5] (section 2, corollaries 2.1 and 2.2) the following results:

LEMMA A. *If for every  $\alpha$  less than  $\alpha_0$  and greater than 0, for every function  $\Phi_\alpha$  such that  $P(\alpha, \Phi_\alpha)$  is true, there is only one function  $\Phi_{\alpha+1}$  such that  $P(\alpha+1, \Phi_{\alpha+1})$  is true and  $\Phi_{\alpha+1|_{[0, \alpha]}} = \Phi_\alpha$ , and if there is a function  $\Phi_1$  such that  $P(1, \Phi_1)$  is true, then for every  $\beta$  (less than  $\alpha_0$  and greater than 0) there exists a function  $\Phi_\beta$  such that  $P(\beta, \Phi_\beta)$  is true.*

Lemma A's proof doesn't need Axiom of Choice. Let's now suppose  $A$  is a set. Then we have:

LEMMA B. *If for every  $\alpha$  less than  $\alpha_0$  and greater than 0, for every function  $\Phi_\alpha$  such that  $P(\alpha, \Phi_\alpha)$  is true, there is a function  $\Phi_{\alpha+1}$  such that  $P(\alpha+1, \Phi_{\alpha+1})$  is true and  $\Phi_{\alpha+1|_{[0, \alpha]}} = \Phi_\alpha$ , and if there is a function  $\Phi_1$  such that  $P(1, \Phi_1)$  is true, then for every  $\beta$  (less than  $\alpha_0$  and greater than 0) there exists a function  $\Phi_\beta$  such that  $P(\beta, \Phi_\beta)$  is true.*

In lemma B's proof we made use of Axiom of Choice (formulated as well ordering principle); conversely we shall see now that lemma B implies the well ordering principle, and then the Axiom of Choice. Let  $A$  be a nonempty set, and  $a$  an element of  $A$ ; let's consider the set  $A^* = A \cup \{A\}$ , and let  $\alpha_0$  be a limit ordinal of power greater than  $2^{|A^*|}$  (where  $|A^*|$  is the cardinality of  $A^*$ )<sup>(1)</sup>. Let  $P(\alpha, \Phi_\alpha)$  be the following statement:

For every  $\gamma < \alpha$ , if  $\{\Phi_\alpha(\delta) \mid \delta < \gamma\}$  is properly contained in  $A$ , then  $\Phi_\alpha(\gamma) \in A$ , and  $\Phi_\alpha(\delta) \neq \Phi_\alpha(\gamma)$  for every  $\delta < \gamma$ ;

It's easily seen that this statement satisfies condition A. Let now  $\Phi_\alpha$  be a function from  $A^*$  into  $A^*$  such that  $P(\alpha, \Phi_\alpha)$  is true. If  $\Phi_\alpha(\delta) = \{A\}$  for a  $\delta < \alpha$ , then let  $\delta_0$  be the least  $\delta$  such that  $\Phi_\alpha(\delta) = \{A\}$ .  $\{\Phi_\alpha(\gamma) \mid \gamma < \delta_0\} \subseteq A$ ; let's suppose  $\{\Phi_\alpha(\gamma) \mid \gamma < \delta_0\} \neq A$ . Then  $\Phi_\alpha(\delta_0) \in A$ , by proposition  $P(\alpha, \Phi_\alpha)$ ; so we get a contradiction, and have to admit  $\{\Phi_\alpha(\gamma) \mid \gamma < \delta_0\} = A$ . So we can define  $\Phi_{\alpha+1}$  by: ( $\Phi_{\alpha+1|_{[0, \alpha]}} = \Phi_\alpha$ , and)  $\Phi_{\alpha+1}(\alpha) = \{A\}$ . And it's easily seen that  $P(\alpha+1, \Phi_{\alpha+1})$  is true. Now we have to consider the second case, that's when  $\{A\} \notin \{\Phi_\alpha(\delta) \mid \delta < \alpha\}$ . We have two subcases:  $\{\Phi_\alpha(\delta) \mid \delta < \alpha\} = A$  and  $\{\Phi_\alpha(\delta) \mid \delta < \alpha\} \neq A$ . In the first subcase we pose  $\Phi_{\alpha+1}(\alpha) = \{A\}$ ; in the second subcase we equate  $\Phi_{\alpha+1}(\alpha)$  to an arbitrary element of  $A - \{\Phi_\alpha(\delta) \mid \delta < \alpha\}$ . And it's trivial to see that  $P(\alpha+1, \Phi_{\alpha+1})$  is true (in the first as well as in the second subcase). So, by lemma B, if  $\beta$

<sup>(1)</sup> It's known that this can be done without any use of Axiom of Choice.

is an ordinal of power greater than  $2^{|\mathcal{A}^*|}$  and less than  $\alpha_0$  (Obviously we can always choose  $\alpha_0$  such that such a  $\beta$  exists), there is a  $\Phi_\beta$  such that  $P(\beta, \Phi_\beta)$  is true. Let's suppose, by absurde,  $\{\Phi_\beta(\gamma) | \gamma < \beta\}$  properly contained in  $A$ . Then  $\Phi_\beta$  is injective, and so has as many values as the power of  $\beta$ ; this gets a contradiction. So we must have  $\{\Phi_\beta(\gamma) | \gamma < \beta\} \supseteq A$ . And it's trivial to check that  $\Phi_\beta$  defines a well ordering on  $A$ ; as we wished to prove. Then:

*Lemma B is equivalent to Axiom of Choice.*

### 3. Proofs.

PROOF OF THEOREM A. Let  $a$  be an element of  $P$  such that  $f(a) \geq a$ . Let  $\alpha_0$  be a limit ordinal whose power is greater than  $2^{|\mathcal{P}|}$  ( $|\mathcal{P}|$  is  $P$ 's cardinality). Let  $\Phi_\alpha$  indicate a function from  $[0, \alpha)$  ( $0 < \alpha < \alpha_0$ ) into  $P$ , and let  $P(\alpha, \Phi_\alpha)$  be the following statement:

$\Phi_\alpha(0) = a$ . And for every ordinal number  $\gamma$ , if  $\gamma < \alpha$ , then is  $\gamma + 1 < \alpha$ ,

$\Phi_\alpha(\gamma + 1) = f(\Phi_\alpha(\gamma))$ , and if  $\gamma$  is a limit ordinal, then the set  $B_\gamma$  of upper bounds of  $\{\Phi_\alpha(\delta) | \delta < \gamma\}$  is nonempty and has a minimal element, and  $\Phi_\alpha(\gamma)$  is such an element.

It's easily seen that  $P(\alpha, \Phi_\alpha)$  satisfies condition A. Let's now suppose  $\Phi_\alpha$  is a function verifying  $P(\alpha, \Phi_\alpha)$ . Now we prove that there is a function  $\Phi_{\alpha+1}$  such that  $P(\alpha + 1, \Phi_{\alpha+1})$  is true and  $\Phi_{\alpha+1}|_{[0, \alpha)} = \Phi_\alpha$ . We have to distinguish two cases:  $\alpha = \delta + 1$  for some ordinal number  $\delta$ , or  $\alpha$  is a limit ordinal. First we verify  $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$  is a well ordered subset of  $\langle P; \leq \rangle$ . We start by proving (by transfinite induction on  $\gamma < \alpha$ ) the following statement  $S(\gamma)$ :

$\Phi_\alpha(\delta) \leq \Phi_\alpha(\gamma)$  for every  $\delta$ , with  $\delta \leq \gamma$ .

Let's now suppose  $S(\gamma)$  true for any  $\gamma < \mu$  ( $\mu$  is a given ordinal number less than  $\alpha$ ). We shall prove  $S(\mu)$  is true. We must distinguish two cases: there is an ordinal number  $\nu$  such that  $\mu = \nu + 1$ , or  $\mu$  is a limit ordinal. If  $\mu = \nu + 1$ , then  $\Phi_\alpha(\mu) = f(\Phi_\alpha(\nu))$ . We have two subcases:  $\nu = \lambda + 1$  or  $\nu$  is a limit ordinal; if  $\nu = \lambda + 1$ ,  $\Phi_\alpha(\nu) = f(\Phi_\alpha(\lambda))$ , and then  $\Phi_\alpha(\mu) = f(f(\Phi_\alpha(\lambda)))$ . But  $f(\Phi_\alpha(\lambda)) = \Phi_\alpha(\nu) \geq \Phi_\alpha(\lambda)$  (by inductive hypothesis). So  $\Phi_\alpha(\lambda) \leq f(\Phi_\alpha(\lambda)) = \Phi_\alpha(\nu)$ ; then we get  $f(\Phi_\alpha(\lambda)) \leq f(\Phi_\alpha(\nu))$ , that's  $\Phi_\alpha(\mu) \geq \Phi_\alpha(\nu)$ . And this suffices to prove  $S(\mu)$ .

Let's now consider the second subcase: that's when  $\nu$  is a limit ordinal. We have  $\Phi_\alpha(\nu) \geq \Phi_\alpha(\lambda)$  for every  $\lambda < \nu$ , and  $f(\Phi_\alpha(\lambda)) = \Phi_\alpha(\lambda + 1) \leq \Phi_\alpha(\nu)$  (by inductive hypothesis). And  $\Phi_\alpha(\lambda) < f(\Phi_\alpha(\lambda))$  (by inductive hypothesis). So  $\Phi_\alpha(\lambda) < f(\Phi_\alpha(\lambda)) \leq \Phi_\alpha(\nu)$ : then we get  $f(\Phi_\alpha(\lambda)) < f(\Phi_\alpha(\nu))$ ; then  $\Phi_\alpha(\lambda) < \Phi_\alpha(\mu)$ . So:  $\Phi_\alpha(\mu) \geq \Phi_\alpha(\lambda)$  for every  $\lambda < \nu$ ; hence  $\Phi_\alpha(\mu)$  is an upper bound of  $\{\Phi_\alpha(\lambda) | \lambda < \nu\}$ . But  $\Phi_\alpha(\mu) = f(\Phi_\alpha(\nu))$ ; then does exists the greatest lower bound of  $\{\Phi_\alpha(\nu), \Phi_\alpha(\mu)\}$ , and it's clearly an upper bound of  $\{\Phi_\alpha(\lambda) | \lambda < \nu\}$ ; but  $\Phi_\alpha(\nu)$  is a minimal upper bound of  $\{\Phi_\alpha(\lambda) | \lambda < \nu\}$ . Then  $\Phi_\alpha(\nu) = \text{g.l.b. } \{\Phi_\alpha(\nu), \Phi_\alpha(\mu)\}$ , that's  $\Phi_\alpha(\nu) \leq \Phi_\alpha(\mu)$ , as we wished to prove; finally we have  $S(\mu)$ .

Let's now consider the second case, that's when  $\mu$  is a limit ordinal; then, by  $P(\alpha, \Phi_\alpha)$ ,  $\Phi_\alpha(\mu)$  is an upper bound for  $\{\Phi_\alpha(\lambda) | \lambda < \mu\}$ , and  $S(\mu)$  is trivially verified; then  $S(\gamma)$  is true for every  $\gamma < \alpha$ . This implies  $\Phi_\alpha$  is a monotone function from  $[0, \alpha)$  into  $P$ ; and therefore  $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$  is a well ordered set; then, by hypothesis, does exists a minimal upper bound  $b$  of  $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$ , and we can pose  $\Phi_{\alpha+1}(\alpha) = b$ .

And so, by lemma  $B$ , for every  $\beta < \alpha_0$  is definable a  $\Phi_\beta$  such that  $P(\beta, \Phi_\beta)$  is true. We have already seen that  $\Phi_\beta$  is a monotone function from  $[0, \beta)$  into  $\langle P; \leq \rangle$ . Let's now suppose  $x \neq f(x)$  for every  $x$ ; in  $P$ ; then  $\Phi_\beta$  has as many values as the power of  $\beta$ . And we get an absurde because we can choose  $\beta$  with power greater than  $|P|$ . So we must admit  $\bar{p} = f(\bar{p})$  for a convenient  $\bar{p} \in P$ .

We remark that we can choose  $\bar{p}$  greater than (or equal to)  $\alpha$ .

We shall now prove that the set of fixed points of  $f$  has a maximal element. Let's consider another proposition  $P'(\alpha, \Phi'_\alpha)$  as follows:

for every  $\gamma_1$  and  $\gamma_2$  less than  $\alpha$ , if  $\gamma_1 < \gamma_2$ , then, if  $\Phi'_\alpha(\gamma_1)$  is maximal in  $\{p | f(p) = p\}$ , is  $\Phi'_\alpha(\gamma_1) = \Phi'_\alpha(\gamma_2)$ . And, if it isn't maximal in  $\{p | f(p) = p\}$ , is  $\Phi'_\alpha(\gamma_1) < \Phi'_\alpha(\gamma_2)$ . Moreover, for every  $\gamma$  less than  $\alpha$ ,  $\Phi'_\alpha(\gamma)$  is a fixed point for  $f$ . And is  $\Phi'_\alpha(0) = \bar{p}$  (where  $\bar{p}$  is the before found fixed point of  $f$ ).

It's a trivial question to check that condition  $A$  is satisfied by  $P'(\alpha, \Phi'_\alpha)$ . Now we must see that, if there's a function  $\Phi'_\alpha$  such that  $P'(\alpha, \Phi'_\alpha)$  is true, then we can construct a function  $\Phi'_{\alpha+1}$  such that  $P'(\alpha + 1, \Phi'_{\alpha+1})$  is true and  $\Phi'_{\alpha+1}|_{[0, \alpha]} = \Phi'_\alpha$ : We distinguish two cases: when is  $\alpha = \delta + 1$  for a convenient  $\delta$ , and when  $\alpha$  is a limit ordinal.

Let's suppose is  $\alpha = \delta + 1$ ; we have two subcases:  $\Phi'_\alpha(\delta)$  is maximal in  $\{p | f(p) = p\}$ , or nct. If  $\Phi'_\alpha(\delta)$  is such a maximal element, then we set  $\Phi'_{\alpha+1}(\alpha) = \Phi'_\alpha(\delta)$  (and  $\Phi'_{\alpha+1}|_{[0, \alpha]} = \Phi'_\alpha$ ). It's easily seen that  $P'(\alpha + 1, \Phi'_{\alpha+1})$  is true. Let's now consider the subcase when  $\Phi'_\alpha(\delta)$

isn't maximal in  $\{p|f(p) = p\}$ : then there is a fixed point  $q$  of  $f$  such that  $q > \Phi'_\alpha(\delta)$ . If we set  $\Phi'_{\alpha+1}(\alpha) = q$  (and  $\Phi'_{\alpha+1|_{[0,\alpha)}} = \Phi'_\alpha$ ) we get a function satisfying  $P'(\alpha + 1, \Phi'_{\alpha+1})$ . Let's now consider the case when  $\alpha$  is a limit ordinal. We have two subcases: there is an ordinal  $\gamma$  less than  $\alpha$  such that  $\Phi'_\alpha(\gamma)$  is maximal in  $\{p|f(p) = p\}$ , or there's no such ordinal. In the first subcase is  $\Phi'_\alpha(\lambda) = \Phi'_\alpha(\gamma)$  for every  $\lambda$  (less than  $\alpha$  and) greater than  $k$ . So we can set  $(\Phi'_{\alpha+1|_{[0,\alpha)}} = \Phi'_\alpha$  and  $\Phi'_{\alpha+1}(\alpha) = \Phi'_\alpha(\gamma)$ .  $\Phi'_{\alpha+1}$ , so defined, verifies  $P'(\alpha + 1, \Phi'_{\alpha+1})$ . In the second subcase  $\{\Phi'_\alpha(\gamma)|\gamma < \alpha\}$  is a well ordered set. Then by hypothesis, it has a minimal upper bound  $b$ . We shall see now that there is a fixed point  $q$  greater than (or equal to)  $b$ . We have  $\Phi'_\alpha(\gamma) = f(\Phi'_\alpha(\gamma))$ , and  $b > \Phi'_\alpha(\gamma)$ , for every  $\gamma$  less than  $\alpha$ . That's:  $\Phi'_\alpha(\gamma) \leq f(\Phi'_\alpha(\gamma)) \leq b$ . We get:  $f(\Phi'_\alpha(\gamma)) \leq f(b)$ ; that's  $\Phi'_\alpha(\gamma) \leq f(b)$ , for every  $\gamma$  less than  $\alpha$ . So  $f(b)$  is an upper bound for the set  $\{\Phi'_\alpha(\gamma)|\gamma < \alpha\}$ ; but the set  $\{b, f(b)\}$  has a greatest lower bound. This, and the minimality of  $b$  in the set of upper bounds of  $\{\Phi'_\alpha(\gamma)|\gamma < \alpha\}$ , give  $b \leq f(b)$ . By the first part of the proof of this theorem, we can find a fixed point (of  $f$ )  $q$ , such that  $q \geq b$ . If we set  $(\Phi'_{\alpha+1|_{[0,\alpha)}} = \Phi'_\alpha$  and  $\Phi'_{\alpha+1}(\alpha) = q$ , we have a function  $\Phi'_{\alpha+1}$  satisfying  $P(\alpha + 1, \Phi'_{\alpha+1})$ . Then, by lemma B, for every nonzero ordinal  $\beta$  less than  $\alpha_0$ , we have a function  $\Phi'_\beta$  such that  $P'(\beta, \Phi'_\beta)$  is true. Let's choose  $\beta$  of power greater than  $|P|$ . If no  $\Phi'_\beta(\gamma)$  is maximal in the set of fixed points of  $f$ ,  $\Phi'_\beta$  is an injection, and therefore has as many values as the power of  $\beta$ ; therefore we get an absurde. And we have to admit that there is an ordinal  $\gamma$  such that  $\Phi'_\beta(\gamma)$  is maximal in  $\{p|f(p) = p\}$ . As we wished to prove.

PROOF OF THEOREM B. The proof is quite similar to theorem A's one; then various details will be omitted.  $\alpha_0$  is an ordinal number of power greater than  $2^{|P|}$ ,  $\Phi_\alpha^a$  a function from  $[0, \alpha)$  into  $P$ ,  $a$  an element of  $P$  such that  $a \leq f(a)$ , and  $P(\alpha, \Phi_\alpha^a)$  is the following statement:

$\Phi_\alpha^a(0) = a$ . And for every ordinal number  $\gamma$ , if  $\gamma < \alpha$ , then if  $\gamma + 1 < \alpha$ ,  $\Phi_\alpha^a(\gamma + 1) = f(\Phi_\alpha^a(\gamma))$ , and if  $\gamma$  is a limit ordinal, then does exists in  $P$  the least upper bound of  $\{\Phi_\alpha^a(\lambda)|\lambda < \gamma\}$ , say it l.u.b.  $(\Phi_\alpha^a(\lambda)|\lambda < \gamma)$ , and is l.u.b.  $(\Phi_\alpha^a(\lambda)|\lambda < \gamma) = \Phi_\alpha^a(\gamma)$ .

It's easily seen that  $P(\alpha, \Phi_\alpha^a)$  satisfies condition A. Let now  $\Phi_\alpha^a$  be a function verifying  $P(\alpha, \Phi_\alpha^a)$ . We shall show that there is (only) one function  $\Phi_{\alpha+1}^a$  such that  $P(\alpha + 1, \Phi_{\alpha+1}^a)$  is true and is  $\Phi_{\alpha+1|_{[0,\alpha)}}^a = \Phi_\alpha^a$ ; We have to distinguish two cases:  $\alpha = \delta + 1$  for some ordinal number  $\delta$ , or  $\alpha$  is a limit ordinal. If  $\alpha = \delta + 1$  we define  $\Phi_{\alpha+1}^a$  by pos-



ing  $(\Phi_{\alpha+1}^a|_{[0,\alpha]} = \Phi_\alpha^a$  and  $\Phi_{\alpha+1}^a(\alpha) = f(\Phi_\alpha^a(\delta))$ . And it's easily seen that  $P(\alpha+1, \Phi_{\alpha+1}^a)$  is true. Let's now suppose  $\alpha$  is a limit ordinal. We have to see  $\{\Phi_\alpha^a(\lambda) | \lambda < \alpha\}$  is a well ordered subset of  $\{P; \leq\}$ . As we did do in theorem A, let's consider the following statement  $S(\gamma)$ :

$$\Phi_\alpha^a(\delta) \leq \Phi_\alpha^a(\gamma) \text{ for every } \delta(\delta < \gamma < \alpha).$$

Now, given  $\mu < \alpha$ , let's suppose that  $S(\gamma)$  is true for every  $\gamma < \mu$ . We shall prove that  $S(\mu)$  is true. We must distinguish two cases:  $\mu = \nu + 1$  or  $\mu$  is a limit ordinal. Let's now suppose  $\mu = \nu + 1$ . If  $\nu = \lambda + 1$ ,  $S(\mu)$  is proved in quite a similar manner as in theorem A's proof. Let's now suppose  $\nu$  is a limit ordinal. As in Theorem A's proof, we get  $\Phi_\alpha^a(\mu) \geq \Phi_\alpha^a(\lambda)$  for every  $\lambda < \nu$ ; then  $\Phi_\alpha^a(\mu) \geq \text{l.u.b.} \cdot (\Phi_\alpha^a(\lambda) | \lambda < \nu)$ , that's  $\Phi_\alpha^a(\mu) \geq \Phi_\alpha^a(\nu)$ . Hence  $S(\mu)$  is true. Let's now consider the second case, that's when  $\mu$  is a limit ordinal: now  $S(\mu)$  is trivial, by  $P(\alpha, \Phi_\alpha^a)$ . So  $S(\mu)$  is true in any case. Hence  $S(\gamma)$  is true for every  $\gamma < \alpha$ ; this implies that  $\Phi_\alpha^a$  is a monotone function from the well ordered set  $[0, \alpha)$  into  $\langle P; \leq \rangle$ ; then  $\{\Phi_\alpha^a(\gamma) | \gamma < \alpha\}$  is a well ordered subset of  $\langle P; \leq \rangle$ , and  $\text{l.u.b.} (\Phi_\alpha^a(\gamma) | \gamma < \alpha)$  does exists. So we can define  $\Phi_{\alpha+1}^a(\alpha) = \text{l.u.b.} (\Phi_\alpha^a(\gamma) | \gamma < \alpha)$  (and  $\Phi_{\alpha+1}^a|_{[0,\alpha]} = \Phi_\alpha^a$ ).  $\Phi_{\alpha+1}^a$  verifies  $P(\alpha+1, \Phi_{\alpha+1}^a)$ . Obviously there is only one such  $\Phi_{\alpha+1}^a$ . By lemma A it follows that for every  $\beta < \alpha_0$  is definable a function  $\Phi_\beta^a$  such that  $P(\beta, \Phi_\beta^a)$  is true. Now the fact that  $f$  has a fixed point  $p$  follows as in theorem A's proof. (condition  $a \leq p$  may be required on  $p$ ). In this first part of the proof no use of Axiom of Choice is done.

The existence of a maximal element in  $\{p | f(p) = p\}$  is proved similarly as in theorem A; therefore we omit this proof; it needs Axiom of Choice.

REMARK. If we suppose  $\langle P; \leq \rangle$  is well ordered, we don't need Axiom of Choice. First we note that in the preceding proof we get a unique  $\Phi_\beta^a$ , for every  $\beta$  less than  $\alpha_0$ . Let  $\beta_0$  be the first ordinal number  $\beta$  such that  $f(\Phi_\beta^a(\gamma)) = \Phi_\beta^a(\gamma)$  for a convenient  $k$ . And let  $\gamma_0$  be the first ordinal  $\gamma$  such that  $f(\Phi_{\beta_0}^a(\gamma)) = \Phi_{\beta_0}^a(\gamma)$ ; let's pose  $p(a) = \Phi_{\beta_0}^a(\gamma_0)$ . So we define a mapping  $p$  from the set  $\{x | x \leq f(x)\}$  into the set  $\{x | x = f(x)\}$ . Clearly is  $x \leq p(x)$  for every  $x$ . Let's now consider a proposition  $P(\alpha, \psi_\alpha)(\psi_\alpha: [0, \alpha) \rightarrow P)$  as follows:

For every  $\gamma_1$  and  $\gamma_2$  less than  $\alpha$ , if  $\gamma_1 < \gamma_2$ , then, if  $\psi_\alpha(\gamma_1)$  is maximal in  $\{x | f(x) = x\}$ , is  $\psi_\alpha(\gamma_1) = \psi_\alpha(\gamma_2)$ , and, if it isn't such a maximal element, is  $\psi_\alpha(\gamma_1) < \psi_\alpha(\gamma_2)$ . Moreover, for every  $\gamma$  less than  $\alpha$ ,  $\psi_\alpha(\gamma)$  is a fixed point of  $f$ . And is  $\psi_\alpha(0) = p(a)$ . And if  $\gamma + 1 < \alpha$ , if  $\psi_\alpha(\gamma)$

is not maximal in  $\{x|f(x) = x\}$ , is  $\psi_\alpha(\gamma + 1) = \text{g.l.b.}(x|f(x) = x \text{ and } x > \psi_\alpha(\gamma))$ . And if  $\gamma$  is a limit ordinal, then  $\psi_\alpha(\gamma) = p(\text{l.u.b.} \cdot (\psi_\alpha(\delta)|\delta < \gamma))$ .

It may be seen that  $P(\alpha, \psi_\alpha)$  so defined verifies condition A, and every  $\psi_\alpha$  such that  $P(\alpha, \psi_\alpha)$  is true may be extended in a unique manner to a  $\psi_{\alpha+1}$  such that  $P(\alpha + 1, \psi_{\alpha+1})$  is true. So we can find, by lemma A (and, therefore, without using Axiom of Choice) a  $\psi_\beta$  for every  $\beta < \alpha_0$ . If  $|P| < |\beta|$ , is  $\psi_\beta(\gamma)$  maximal in  $\{x|f(x) = x\}$  for an ordinal  $\gamma$  ( $|\beta|$  is the power of  $\beta$ ).

PROOF OF THEOREM C. Let  $\alpha_0$  be a limit ordinal whose power is greater than  $2^{|P|}$ . Let  $\Phi_\alpha$  indicate a function from  $[0, \alpha)$  ( $0 < \alpha < \alpha_0$ ) into  $P$ . Let  $P(\alpha, \Phi)_\alpha$  be the following statement:

for every ordinal number  $\gamma$  in  $[0, \alpha)$ , if  $\gamma + 1 < \alpha$ ,  $\Phi_\alpha(\gamma + 1) = f(\Phi_\alpha(\gamma))$ , and, if  $\gamma$  is a limit ordinal, then does exist an upper bound of  $\{\Phi_\alpha(\lambda)|\lambda < \gamma\}$  in  $\langle P; \leq \rangle$ , and  $\Phi_\alpha(\gamma)$  is such an upper bound.

It's easily seen that  $P(\alpha, \Phi_\alpha)$  verifies condition A. Moreover, given  $\Phi_\alpha$  verifying  $P(\alpha, \Phi_\alpha)$ , there is a  $\Phi_{\alpha+1}$  verifying  $P(\alpha + 1, \Phi_{\alpha+1})$  and such that  $\Phi_{\alpha+1}|_{[0, \alpha)} = \Phi_\alpha$ . We must distinguish two cases:  $\alpha = \delta + 1$  for a convenient ordinal  $\delta$ , or  $\alpha$  is a limit ordinal. If  $\alpha = \delta + 1$ , we pose  $\Phi_{\alpha+1}(\alpha) = f(\Phi_\alpha(\delta))$ ; if  $\alpha$  is a limit ordinal, then we set  $\Phi_{\alpha+1}(\alpha)$  equal to an upper bound of  $\{\Phi_\alpha(\gamma)|\gamma < \alpha\}$ ; and such an upper bound does exist because is  $\Phi_\alpha(\gamma) \leq \Phi_\alpha(\lambda)$  whenever is  $\gamma \leq \lambda$ , and so  $\Phi_\alpha$  realizes an order-homomorphism from the well-ordered set  $[0, \alpha)$  into  $\langle P; \leq \rangle$ , and then  $\{\Phi_\alpha(\gamma)|\gamma < \alpha\}$  is a well-ordered subset of  $\langle P; \leq \rangle$ . Then it follows from lemma B that for every  $\beta < \alpha_0$  there is a function  $\Phi_\beta$  such that  $P(\beta, \Phi_\beta)$  is true. As  $\alpha_0$  has power greater than  $2^{|P|}$ , we can choose  $\beta$  in  $[0, \alpha_0)$  of power greater than to  $2^{|P|}$ . Let's now suppose (by absurde)  $x < f(x)$  for every  $x$  in  $P$ ; then we get  $\Phi_\beta(\gamma) < \Phi_\beta(\mu)$  for every choose of  $\gamma, \mu$  such that  $\gamma < \mu$ . So  $\Phi_\beta$  is injective, and therefore has more than  $2^{|P|}$  values. But, as  $\Phi_\beta$ 's values are in  $P$ ,  $\Phi_\beta$  has at most  $|P|$  values; we get an absurde. This constrains us to admit there is a fixed point of  $f$ , say it  $p$ .

Now we shall prove that the nonempty set of fixed points of  $f$  has a maximal element. Let's consider the statement  $P'(\alpha, \Phi'_\alpha)$  as follows:

For every  $\gamma$  less than  $\alpha$ ,  $\Phi'_\alpha(\gamma)$  is a fixed point for  $f$ . And is  $\Phi'_\alpha(0) = p$ . And, for every  $\gamma_1, \gamma_2$  less than  $\alpha$ , if  $\gamma_1 < \gamma_2$ . Then, if

$\Phi'_\alpha(\gamma_1)$  is maximal in  $\{x|f(x) = x\}$ , is  $\Phi'_\alpha(\gamma_1) = \Phi'_\alpha(\gamma_2)$ ; and, if it isn't such a maximal element, is  $\Phi'_\alpha(\gamma_1) < \Phi'_\alpha(\gamma_2)$ .

$P'(\alpha, \Phi'_\alpha)$  verifies condition A. Now we shall see that, given a function  $\Phi'_\alpha$  such that  $P'(\alpha, \Phi'_\alpha)$  is true, we can construct a function  $\Phi'_{\alpha+1}$  which extends  $\Phi'_\alpha$  and such that  $P'(\alpha + 1, \Phi'_{\alpha+1})$  is also true. If  $\alpha$  isn't a limit ordinal, proof is just like in theorem A. If  $\alpha$  is a limit ordinal, but there's a  $\gamma$  less than  $\alpha$  and such that  $\Phi'_\alpha(\gamma)$  is maximal in  $\{x|f(x) = x\}$ , proof is just like in theorem A. The only case we must check is when ( $\alpha$  is a limit ordinal and)  $\Phi'_\alpha(\gamma)$  isn't maximal in the set of fixed points of  $f$ , for every  $\gamma$  less than  $\alpha$ . In this case the set  $\{\Phi'_\alpha(\gamma)|\gamma < \alpha\}$  is a well ordered set. Then there exists an upper bound  $b$  of it. In the ordered set  $Q$  of all elements of  $P$  greater or equal to  $b$ ,  $f$  has a fixed point  $q$ . If we set  $\Phi'_{\alpha+1}(\alpha) = q$ , we get a function as we required. Now the existence of a maximal element in  $\{x|f(x) = x\}$  follows as in theorem A.

As we wished to prove.

REMARK. Let's assume on  $P$  and  $f$  the hypothesis of proposition B. Then, if we substitute in the preceding proof the statement  $P(\alpha, \Phi_\alpha)$  with the statement:

for every ordinal  $\gamma$  in  $[0, \alpha)$ , if  $\lambda + 1 < \alpha$ ,  $\Phi_\beta(\gamma + 1) = f(\Phi_\alpha(\gamma))$  and, if  $\gamma$  is a limit ordinal,  $\Phi_\alpha(\gamma) = \text{l.u.b.}(\Phi_\alpha(\lambda)|\lambda < \gamma)$  we have a proof of proposition A which utilizes only lemma A, and therefore doesn't need Axiom of Choice.

## BIBLIOGRAPHY

- [1] S. ABIAN - A. BROWN, *A theorem on partially ordered sets, with application to fixed point theorems*, Can. J. Math., **13**, no I (1961).
- [2] A. ABIAN (= S. ABIAN), *Fixed point theorems on the mappings of partially ordered sets*, Rendiconti del Circolo Matematico di Palermo, Serie II, Tomo XX (1971), pp. 139-142.
- [3] R. SALINAS BALTASAR, *Fixed points in ordered sets*, In Honor of Prof. Dr. Iniguez y Almech on the Occasion of his Seventieth Birthday and

Academic Retirement, pp. 163-172, Consejo Sup. Investigacion Ci. Fac. C. Zaragoza, 1969 (see Mathematical Reviews, 1970, rep. n. 4171).

- [4] A. PELCZAR, *On the invariant points of a transformation*, Ann. Pol. Math., **II** (1961), pp. 199-202.
- [5] A. PASINI, *Sulla sottoalgebra di Frattini* (it has to appear on «Le Matematiche »).

Manoscritto pervenuto in redazione il 24 luglio 1973.