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Statistical Study of Navier-Stokes Equations, II.

C. FOIAS (*)

6. Stationary statistical solutions, invariant measures and time averages.

1. *a*) Let us recall that beginning with § 5 throughout the sequel of the paper the right term $f(\cdot)$ of (the abstract form of) the Navier-Stokes equations is supposed to be time-independent, that is $f(t) = f(0)$ for all t, and belonging to N, i.e. $f \in N$. In this case for any (Borel) probability μ on N satisfying (3.5) there exists a measure-valued function $\mu_\cdot$ defined on the whole $(0, \infty)$ enjoying the following property: For any $T \in (0, \infty)$, the family $\{\mu_t\}_{0<t<T}$ is a statistical solution of the Navier-Stokes equations with initial data μ, satisfying a.e. the strengthened energy inequality; the family $\{\mu_t\}_{0<t<\infty}$ will be called a statistical solution *on* $(0, \infty)$ with initial data μ. In order to avoid some tedious inessential difficulties let us sketch the construction of a statistical solution on $(0, \infty)$ in case the initial data μ is with bounded support in N. To this aim take inductively a statistical solution $\{\mu_t^{(n)}\}_{0<t<2}$ constructed as in Sec. 3.2, with initial data μ for $n = 1$ and $\mu_1^{(n-1)}$ for $n > 1$, and define $\mu_t = \mu_{t-n-1}^{(n)}$ for $t \in [n-1, n)$, $n = 1, 2, \ldots$. Theorem 2 and Lemma 6 in Sec. 3.3 imply that (for any $T \in (0, \infty)$) the family $\{\mu_t\}_{0<t<T}$ satisfies (3.13_{IV}) and (4.2) on $[0, T]$ (not only a.e.); moreover it enjoys Property (j) in Theorem 2, Sec. 3.3.c), and $\operatorname{supp} \mu_t$ is bounded in N, uniformly in $t \in [0, T]$. In particular, the

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above properties show that $\{\mu_t\}_{0<t<\infty}$ is a statistical solution on $(0, \infty)$ with initial data μ.

Let us conclude this discussion with two remarks. Firstly, that a statistical solution on $(0, \infty)$ is obliged, by definition, to satisfy the strengthened energy inequality (a.e. on $(0, \infty)$) and, secondly, that the above construction of a statistical solution on $(0, \infty)$ can be adapted in such a way that the solution should satisfy (5.32′) for all $t_2 \geqslant t_1 \geqslant 0$; plainly, this necessitates the use of Theorem 5 in Sec. 5.4.*a*).

DEFINITION. A *stationary statistical solution* of the Navier-Stokes is, by definition, a time-independent statistical solution on $(0, \infty)$, that is a (Borel) probability μ on $\boldsymbol{N}$ such that $\{\mu_t\colon \mu = \mu_t \text{ for a } t \in (0, \infty)\}$ is a statistical solution on $(0, \infty)$ with some initial data μ'.

Note that (3.16) implies the relations

$$\int_{\boldsymbol{N}} |u|^2 d\mu(u) < \infty\,, \qquad \int_{\boldsymbol{N}} \|u\|^2 d\mu(u) < \infty \tag{6.1}$$

where, as we already agreed, $\|u\| = \infty$ on $\boldsymbol{N} \setminus \boldsymbol{N}^1$. (Obviously the first relation in (6.1) is implied by the second one.) Moreover $\mu = \mu'$. Indeed from (3.13_{IV}) (satisfied a.e. on $(0, \infty)$) it results readily that

$$\int_{\boldsymbol{N}} \Phi(u)\, d\mu(u) = \int_{\boldsymbol{N}} \Phi(u)\, d\mu'(u) \tag{6.1'}$$

for all $\Phi \in \mathcal{C}_0^{\mathrm{ind}}$. Using the first relation (6.1) for μ and the same relation for μ' (since it is an initial data), we can deduce $\mu = \mu'$ reproducing almost the proof starting from formula (5.19) and finishing before Theorem 3′ in Sec. 5.2. With this remark, the relation (3.13_{IV}) obviously reduces to the equation

$$\int_{\boldsymbol{N}} \Big[\nu\big((u, \Phi'(u))\big) + b\big(u, u, \Phi'(u)\big) - (f, u)\Big] d\mu(u) = 0 \tag{6.2}$$

for any $\Phi \in \mathcal{C}_0^{\mathrm{ind}}$. However our solution satisfy also (4.2) a.e. on $(0, \infty)$. It is plain that this last relation for $\mu_t \equiv \mu$ becomes

$$\nu \int_{\boldsymbol{N}} \psi'(|u|^2)\, \|u\|^2 d\mu(u) \leqslant \int_{\boldsymbol{N}} \psi'(|u|^2)(f, u)\, d\mu(u) \tag{6.3}$$

for any $\psi \in C^1([0, \infty))$ such that ψ' is bounded and $\geqslant 0$.

Actually in (6.3), ψ' can be any bounded nonnegative function $\in C([0, \infty))$. Therefore it is easy to infer that

$$\nu \int_{\{u:\, u\in N,\ a\leq |u|\leq b\}} \|u\|^2 d\mu(u) \leq \int_{\{u:\, u\in N,\ a\leq |u|\leq b\}} (f, u)\, d\mu(u) \tag{6.3$'$}$$

for all reals $b \geq a \geq 0$. On the other hand it is obvious that if a (Borel) probability μ on N satisfies the second relation (6.1) as well as (6.2) and (6.3) then μ is a stationary statistical solution of the Navier-Stokes equations.

Thus a more direct definition of these solutions is the following: *A stationary statistical solution of the Navier-Stokes equations is a* (*Borel*) *probability on* N *satisfying* (6.1), (6.2) *for all* $\Phi \in \mathcal{T}^{\mathrm{ind}}$ (*or only* $\in \mathcal{T}_0^{\mathrm{ind}}$) *and* (6.3) *for all* ψ *as specified above.*

REMARKS. 1°. It is plain also that a Dirac measure δ_{u_0} is a stationary statistical solution if and only if u_0 is an (individual) stationary solution (see Sec. 2.2).

2°. By (2.6), the set S of all individual stationary solutions of the Navier-Stokes equations is a bounded set in N^1. Moreover it is easy to verify that it is also closed hence compact in N. Obviously any probability μ whose support is $\subset S$ is a stationary statistical solution of the Navier-Stokes equation. In a certain sense these solutions are to be considered as *trivial.*

We shall give now some (easy) properties of the stationary statistical solutions in the following

PROPOSITION 1. *There exist some constants* c_{85} *and* c'_{85} (*depending only on* f, Ω *and* ν) *such that for any stationary statistical solution* μ *the following relations hold:*

$$\int_N \|u\|^2 d\mu(u) \leq c_{85}\,, \tag{6.4}$$

$$\operatorname{supp} \mu \subset \{u:\ u \in N,\ |u| \leq c'_{85}\}\,. \tag{6.4$'$}$$

PROOF. From (6.3) with $\psi(\xi) \equiv \xi$, $\xi \geq 0$, we deduce

$$\nu \int_N \|u\|^2 d\mu(u) \leq |f| \cdot \int_N |u|\, d\mu(u) \leq |f| \Big(\int_N |u|^2 d\mu(u)\Big)^{\frac12} \leq |f|\, \lambda_1^{-\frac12} \Big(\int_N \|u\|^2 d\mu(u)\Big)^{\frac12},$$

thus (6.4) holds with

$$c_{85} = \frac{|f|^2}{\nu^2 \lambda_1}. \tag{6.4''}$$

Let now $r_0 > 0$ and let ψ in (6.3) be any function satisfying moreover the condition

$$\psi(\xi) = 0 \quad \text{for} \quad 0 \leqslant \xi \leqslant r_0^2 \quad \text{and} \quad \psi'(\xi) > 0 \quad \text{for} \quad \xi > r_0^2. \tag{6.5}$$

Then (6.3) and (6.5) yield

$$\nu \int\limits_{\{u:\, u \in N,\ |u| > r_0\}} \psi'(|u|^2) \|u\|^2 d\mu(u) = \nu \int\limits_N \psi'(|u|^2) \|u\|^2 d\mu(u) \leqslant$$

$$\leqslant |f| \cdot \int\limits_N \psi'(|u|^2) |u|\, d\mu(u) = |f| \cdot \int\limits_{\{u:\, u \in N,\ |u| > r_0\}} \psi'(|u|^2) |u|\, d\mu(u) \leqslant$$

$$\leqslant \frac{|f|}{r_0} \int\limits_{\{u:\, u \in N,\ |u| > r_0\}} \psi'(|u|^2) |u|^2 d\mu(u) \leqslant \frac{|f|}{r_0 \lambda_1} \int\limits_{\{u:\, u \in N,\ |u| > r_0\}} \psi'(|u|^2) \|u\|^2 d\mu(u),$$

so that if we put

$$c'_{85} = \frac{|f|}{\nu \lambda_1} \tag{6.4'''}$$

we obtain

$$\int\limits_{\{u:\, u \in N,\ |u| > c'_{85}\}} \psi'(|u|^2) \|u\|^2 d\mu(u) = 0,$$

whence (6.4′).

PROPOSITION 2. *(a) Let the space dimension $n = 2$. Then for any stationary statistical solution μ we have*

$$\begin{cases} \mu(\omega) = \mu(S(t)^{-1}\omega), \\ \textit{for all } t \geqslant 0 \textit{ and Borel subsets } \omega \textit{ of } N. \end{cases} \tag{6.6$_2$}$$

(b) Let $n = 3$. Then for any stationary statistical solution μ carried by a bounded set in N^1 we have

$$\begin{cases} \mu(\omega) = \mu(T(t)^{-1}\omega), \\ \textit{for all } t \geqslant 0 \textit{ and Borel subsets } \omega \textit{ of } N. \end{cases} \tag{6.6$_3$}$$

The point (a) follows readily from the initial definition of a stationary statistical solution and from Proposition 1 above together with Remark 1° in Sec. 5.1.b); while (6) is obtained, using, instead these two (Prop. 1 above and Remark 1°, Sec. 5.1.b)), Theorem 2 in Sec. 5.1.c).

REMARKS. 3°. There is an essential difference between the points (a) and (b) in Proposition 2, namely: (6.6_2) concerns *all* stationary solutions in dimension $n = 2$, while (6.6_3) concerns *some* stationary statistical solutions in dimension $n = 3$. Obviously, the additional condition in (b) is satisfied by the trivial solutions (i.e. those whose support lies in $\boldsymbol{S}$; see Remark 2° above) but for these solutions, (6.6_3) is trivially verified.

4°. If $f = 0$, then the only stationary statistical solution is δ_0. Indeed, (6.3), with $\psi(\xi) \equiv \xi$, gives

$$\nu \int_{\boldsymbol{N}} \|u\|^2 d\mu(u) = 0 ,$$

thus $\operatorname{supp} \mu \subset \{0\}$, hence $\mu = \delta_0$. However, in this case $u_0 = 0$ is the only stationary (individual) solution, so that in this case all (in fact the only one) stationary statistical solution μ are (in fact) trivial, i.e. $\operatorname{supp} \mu \subset \boldsymbol{S}$ (see Remark 2° above).

b) The existence of stationary statistical solutions which are trivial (assured by $\boldsymbol{S} \neq \emptyset$; see Remark 2° in the preceding Section and Sec. 2.2), naturally changes the nature of the existence problem for stationary statistical solutions. Moreover the Remark 4° above shows that non trivial stationary statistical solution do not always exist. Therefore more precise statements than a simple existence assertion have to be made in case of stationary statistical solutions.

To this purpose let μ be *any* (Borel) probability on $\boldsymbol{N}$ satisfying the condition (3.5), i.e.

$$\int_{\boldsymbol{N}} |u|^2 d\mu(u) < \infty .$$

Let $\{\mu_t\}_{0<t<\infty}$ be *any* statistical solution of the Navier-Stokes equations, constructed as in Sec. 3.2 (see also the beginning of Sec. 6.1.a)), with initial data μ; in particular it satisfies the strengthened energy inequality (4.2), hence also (4.5) for all $r \geqslant c_{52} = 2\nu^{-1}\lambda_1^{-\frac{1}{2}}\|f\|_{\boldsymbol{N}^{-1}}$ (see the proof of Corollary 2 in Sec. 4.1). It results

$$\text{(6.7)}\qquad \int\limits_{N}(1+|u|^2)\,d\mu_t(u) \leqslant 1+(c_{52})^2+\int\limits_{\{u:\,u\in N,\ |u|>1+c_{52}\}}(1+|u|^2)\,d\mu_t(u) \leqslant$$

$$\leqslant 1+(c_{52})^2+2\int\limits_{\{u:\,u\in N,\ |u|>c_{52}\}}|u|^2\,d\mu_t(u) \leqslant 1+(c_{52})^2+2\cdot\int\limits_{\{u:\,u\in N,\ |u|>c_{52}\}}|u|^2\,d\mu(u) \leqslant$$

$$\leqslant 1+(c_{52})^2+2\cdot\int\limits_{N}|u|^2\,d\mu(u)=c_{86}\,,$$

where c_{86} is independent of t; moreover for $t>0$

$$\text{(6.7}'\text{)}\qquad \int\limits_0^t\left[\int\limits_N \|u\|^2\,d\mu_\tau(u)\right]d\tau \leqslant \frac{2}{\nu}\int\limits_N |u|^2\,d\mu(u)+\frac{|f|^2}{\nu\lambda_1}t$$

so that for $\Phi\in\mathfrak{C}_{1,1}$ we have for all $t>0$

$$\int\limits_0^t\left[\int\limits_N \Phi(u)\,d\mu_\tau(u)\right]d\tau \leqslant \|\Phi\|_{\mathfrak{C}_{1,1}}\cdot\int\limits_0^t\left[\int\limits_N (1+|u|\cdot\|u\|)\,d\mu_\tau(u)\right]d\tau \leqslant$$

$$\leqslant \|\Phi\|_{\mathfrak{C}_{1,1}}\cdot\int\limits_0^t\left[\int\limits_N (1+\lambda_1^{-\frac{1}{2}}\|u\|^2)\,d\mu_\tau(u)\right]d\tau \leqslant \|\Phi\|_{\mathfrak{C}_{1,1}}(c'_{86}+tc''_{86})$$

where $c'_{86}-c''_{86}$ are constants (independent on t). Therefore if we define on $\mathfrak{C}_{1,1}$ the functionals $\mathfrak{M}_t$ $(0<t<\infty)$ by

$$\text{(6.8)}\qquad \mathfrak{M}_t(\Phi)=\frac{1}{t}\int\limits_0^t\left[\int\limits_N \Phi(u)\,d\mu_\tau(u)\right]d\tau$$

we shall have

$$\text{(6.8}'\text{)}\qquad \mathfrak{M}_t\in\mathfrak{C}_{1,1}^*\quad\text{and}\quad \|\mathfrak{M}_t\|_{\mathfrak{C}_{1,1}^*}\leqslant\tilde{c}_{86}\quad\text{for all } t\geqslant 1\,,$$

where $\tilde{c}_{86}=(c'_{86}+c''_{86})$ is independent of t.

Thus the directed set $\{\mathfrak{M}_t\}_{1\leqslant t<\infty}$ has at least a w^*-cluster point in $\mathfrak{C}_{1,1}^*$, i.e. belonging to the intersection $\mathcal{M}(\{\mu_t\}_{0<t<\infty})$ of the w^*-closure of $\{\mathfrak{M}_t\}_{t_0 \leqslant t < \infty}$ for all $t_0 \geqslant 1$. Note that if $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ then $\mathfrak{M}^*|\mathcal{C}_2$ is also a w^*-cluster point in $\mathcal{C}_2^*$ of the directed set $\{\mathfrak{M}_t|\mathcal{C}_2\}_{1\leqslant t<\infty}$, bounded in $\mathcal{C}_2^*$ too.

In virtue of these remarks the following theorem is not with an empty content.

THEOREM 1. *For any* $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ *there exists a stationary statistical solution* μ^*, *such that*

$$\mathfrak{M}^*(\Phi) = \int_N \Phi(u)\, d\mu^*(u) \qquad \textit{for all } \Phi \in \mathcal{C}_2\,. \tag{6.9}$$

PROOF. Let us first prove the representation (6.9) for a certain (Borel) probably μ^* on N satisfying (6.3). For this purpose let us note that (4.2) readily implies

$$\frac{1}{t}\int_0^t \left[\int_N \|u\|^2 d\mu_\tau(u)\right] d\tau \leqslant \frac{2}{t\nu}\int_N |u|^2 d\mu(u) + \frac{|f|^2}{\nu\lambda_1} \tag{6.10}$$

a.e. on $(0, \infty)$, thus, *on* $(0, \infty)$, whence

$$\mathfrak{M}^*(\|P_m\cdot\|^2) \leqslant c_{87} = \frac{|f|^2}{\lambda_1 \nu} \qquad \text{for all } m = 1, 2, \dots . \tag{6.10'}$$

Moreover from (6.10) we obtain also the relations

$$\mathfrak{M}_t(\Phi) = \frac{1}{t}\int_0^t \left[\int_{b_p} \Phi(u)\, d\mu_\tau(u)\right] d\tau + \frac{1}{t}\int_0^t \left[\int_{N\setminus b_p} \Phi(u)\, d\mu_\tau(u)\right] d\tau \leqslant \tag{6.10''}$$

$$\leqslant \max_{b_p} |\Phi| + \|\Phi\|_{\mathcal{C}_0}\cdot \frac{1}{t}\int_0^t \left[\int_{N\setminus b_p} \frac{\|u\|^2}{p^2}\, d\mu_\tau(u)\right] d\tau \leqslant \max_{b_p} |\Phi| + \|\Phi\|_{\mathcal{C}_0}\cdot \frac{c'_{87}}{p^2}$$

where $t \geqslant 1$, $\Phi \in \mathcal{C}_0$, $p = 1, 2, \dots$ are arbitrary and $c'_{87} = (2/\nu)\int_N |u|^2 d\mu(u) + c_{87}$. (Let us recall that $b_p = \{u\colon u \in N^1,\ \|u\| \leqslant p\}$; see (3.31).) The relation (6.10″) implies

$$\text{(6.10‴)} \qquad \mathfrak{M}^*(\Phi) \leqslant \max_{b_p} |\Phi| + \|\Phi\|_{\mathfrak{C}_0} \frac{c'_{87}}{p^2}\,.$$

Now if $\{\Phi_k\}_{k=1}^{\infty}$ is any nonincreasing sequence in $\mathfrak{C}_0$ such that $\Phi_k(u) \to 0$ for all $u \in \boldsymbol{N}$, then, $\boldsymbol{b}_p$ being compact in $\boldsymbol{N}$, $\max\limits_{b_p} |\Phi_k| \to 0$ for $k \to \infty$. On the other hand it is obvious that

$$\text{(6.11)} \qquad \mathfrak{M}^* \geqslant 0\,, \quad \text{i.e.} \quad \mathfrak{M}^*(\Phi) \geqslant 0 \quad \text{if} \quad \Phi \in \mathfrak{C}_{1,1}\,, \ \Phi \geqslant 0\,.$$

Therefore

$$\mathfrak{M}^*(\Phi_1) \geqslant \mathfrak{M}^*(\Phi_2) \geqslant \ldots \geqslant 0$$

and by (6.10‴)

$$0 \leqslant \lim_{k\to\infty} \mathfrak{M}^*(\Phi_k) \leqslant \|\Phi_1\|_{\mathfrak{C}_0} \frac{c'_{87}}{p^2}$$

for all $p = 1, 2, \ldots$; thus the limit is 0. The functional $\mathfrak{M}^*$ satisfies Daniell's condition so that there exists a Borel measure μ^* on $\boldsymbol{N}$ such that

$$\text{(6.9′)} \qquad \mathfrak{M}^*(\Phi) = \int_{\boldsymbol{N}} \Phi(u)\, d\mu^*(u)\,, \qquad \text{for all } \Phi \in \mathfrak{C}_0\,.$$

Plainly, μ^* is a probability on $\boldsymbol{N}$.

Put, for $\varrho > 0$,

$$\psi_\varrho(\xi) = \begin{cases} \xi & \text{for } 0 \leqslant \xi \leqslant \varrho\,, \\ \varrho & \text{for } \varrho \leqslant \xi\,. \end{cases}$$

Then $\psi_\varrho(\|P_m \cdot\|^2) \in \mathfrak{C}_0$ for all $m = 1, 2, \ldots,\ \varrho > 0$; by (6.9′), (6.10′) and (6.11) we obtain

$$\int_{\boldsymbol{N}} \psi_\varrho(\|P_m u\|^2)\, d\mu^*(u) = \mathfrak{M}^*\big(\psi_\varrho(\|P_m \cdot\|^2)\big) \leqslant \mathfrak{M}^*(\|P_m \cdot\|^2) \leqslant c_{87}$$

whence (letting first $\varrho \to \infty$ and afterwards $m \to \infty$),

$$\text{(6.12)} \qquad \int_{\boldsymbol{N}} \|u\|^2\, d\mu^*(u) \leqslant c_{87}\,.$$

Moreover, again working on (4.2),

$$\nu\mathfrak{M}_t\big(\psi'(|\cdot|^2)\|P_m\cdot\|^2\big)=\frac{\nu}{t}\int_0^t\Big[\int_N\psi'(|u|^2)\|P_m u\|^2 d\mu_\tau(u)\Big]d\tau\leqslant$$

$$\leqslant\frac{\nu}{t}\int_0^t\Big[\int_N\psi'(|u|^2)\|u\|^2 d\mu_\tau(u)\Big]d\tau\leqslant$$

$$\leqslant\frac{1}{t}\int_N\psi(|u|^2)\,d\mu(u)+\frac{1}{t}\int_0^t\Big[\int_N\psi'(|u|^2)(f,u)\,d\mu_\tau(u)\Big]d\tau=$$

$$=\frac{1}{t}\int_N\psi(|u|^2)\,d\mu(u)+\mathfrak{M}_t\big(\psi'(|\cdot|^2)(f,\cdot)\big)$$

from where we can infer

$$\nu\mathfrak{M}^*\big(\psi'(|\cdot|^2)\|P_m\cdot\|^2\big)\leqslant\mathfrak{M}^*\big(\psi'(|\cdot|^2)(f,\cdot)\big)\,.$$

Take a nondecreasing sequence $\{\psi_j\}_{j=1}^{\infty}\subset C^1([0,\infty))$ satisfying the following conditions: $\psi_j(\xi)=\psi(\xi)$ for $0\leqslant\xi\leqslant j$, $0\leqslant\psi_j(\xi)\leqslant\psi(\xi)$ for $0\leqslant\xi<\infty$ and $\psi_j'(\xi)=0$ for $j+1\leqslant\xi<\infty$, $j=1,2,\ldots$. Then $\psi_j'(|u|^2)\|P_m u\|^2$ and $\psi_j'(|u|^2)(f,u)$ as functions of u, belong to $\mathfrak{C}_0$; therefore (6.12′) can be written for $\psi=\psi_j$ under the form

$$\nu\int_N\psi_j'(|u|^2)\|P_m\|^2 d\mu^*(u)\leqslant\int_N\psi_j'(|u|^2)(f,u)\,d\mu^*(u)\,.$$

Letting firstly $j\to\infty$ and afterwards $m\to\infty$, we obtain

$$\nu\int_N\psi'(|u|^2)\|u\|^2 d\mu^*(u)\leqslant\int_N\psi'(|u|^2)(f,u)\,d\mu^*(u)\tag{6.12′}$$

for any $\psi\in C^1([0,\infty))$, such that ψ' is bounded and $\geqslant 0$. Consequently as in the proof of Proposition 1 in Sec. 6.1.a), it results

$$\operatorname{supp}\mu^*\subset\{u\colon u\in \boldsymbol{N},\ |u|\leqslant c_{85}'\}\,.\tag{6.13}$$

On the other hand if $\Phi\in\mathfrak{C}_2$, $\Phi\geqslant 0$, and Φ is 0 for $|u|\leqslant r$ where r is

$\geqslant c_{52}$ then, in virtue of (4.8),

$$\mathfrak{M}_t(\Phi) = \frac{1}{t}\int_0^t \left[\int_N \Phi(u)\, d\mu_\tau(u)\right] d\tau = \frac{1}{t}\int_0^t \left[\int_{\{u:\, u\in N,\ |u|>r\}} \Phi(u)\, d\mu_\tau(u)\right] d\tau \leqslant$$

$$\leqslant \|\Phi\|_{\mathfrak{C}_2}\cdot\frac{1}{t}\int_0^t \left[\int_{\{u:\, u\in N,\ |u|>r\}} (1+|u|^2)\, d\mu_\tau(u)\right] d\tau \leqslant$$

$$\leqslant \|\Phi\|_{\mathfrak{C}_2}\cdot\frac{1}{t}\int_0^t \left[\int_{\{u:\, u\in N,\ |u|>r\}} (1+|u|^2)\, d\mu(u)\right] d\tau =$$

$$= \|\Phi\|_{\mathfrak{C}_2}\cdot \int_{\{u:\, u\in N,\ |u|>r\}} (1+|u|^2)\, d\mu(u)$$

from where we can easily deduce

$$\mathfrak{M}^*(\Phi) \leqslant \|\Phi\|_{\mathfrak{C}_2}\cdot \int_{\{u:\, u\in N,\ |u|>r\}} (1+|u|^2)\, d\mu(u) \tag{6.14}$$

for any

$$\Phi\in\mathfrak{C}_2\,,\quad \Phi\geqslant 0\,,\quad \text{such that } \Phi(u)=0 \text{ for } |u|\leqslant r\,, \tag{6.14$'$}$$

where $r\geqslant c_{52}$. Let now $\Phi\in\mathfrak{C}_2$, $\Phi\geqslant 0$, and for $r\geqslant 2$, let $\varphi_r(\cdot)$ be defined as in (3.41). Applying (6.14) to $\varphi_r(\cdot)\Phi(\cdot)-\Phi(\cdot)$ for $r\geqslant c_{52}$ it results (since $\|\varphi_r\Phi-\Phi\|_{\mathfrak{C}_2}\leqslant\|\Phi\|_{\mathfrak{C}_2}$) that

$$\lim_{r\to\infty}\mathfrak{M}^*\big(\varphi_r(\cdot)\Phi(\cdot)\big) = \mathfrak{M}^*\big(\Phi(\cdot)\big)\,.$$

But for $r > c_{52}$ we have $\varphi_r(u)\Phi(u) = \Phi(u)$ μ^*—a.e.; therefore, since $\varphi_r\Phi\in\mathfrak{C}_0$, we finally can conclude that (6.9$'$) holds for all $\Phi\in\mathfrak{C}_2$, $\Phi\geqslant 0$, hence also for all $\Phi\in\mathfrak{C}_2$.

This finishes the first part of the proof. It must be still proved that actually μ^* is a stationary statistical solution. In virtue of (6.12)-(6.12$'$), it remains to prove only that (6.2) is verified by $\mu=\mu^*$ for all $\Phi\in\mathfrak{T}_0^{\text{ind}}$. To this aim put

$$\psi_\Phi(u) = \nu\big(\!\big(u, \Phi'(u)\big)\!\big) + b\big(u, u, \Phi'(u)\big) - \big(f, \Phi'(u)\big) \tag{6.15}$$

and note that for $\Phi \in \mathcal{C}_0^{\text{ind}}$, the functional ψ_Φ belongs to $\mathfrak{C}_{1,1}$. Therefore from the relations (consequences of (3.13_{IV}))

$$|\mathfrak{M}_t(\psi_\Phi)| = \left|\frac{1}{t}\int_0^t \left[\int_N \psi_\Phi(u)\, d\mu_\tau(u)\right] d\tau\right| =$$
$$= \frac{1}{t}\left|\int_N \Phi(u)\, d\mu(u) - \int_N \Phi(u)\, d\mu_t(u)\right| \leqslant \frac{2\|\Phi\|_{\mathfrak{C}_0}}{t},$$

it results

$$\mathfrak{M}^*(\psi_\Phi) = 0\,. \tag{6.16}$$

In virtue of (6.15), (6.16) will coincide with (6.2) once we proved that

$$\mathfrak{M}^*(\psi_\Phi) = \int_N \psi_\Phi(u)\, d\mu^*(u)\,.$$

Since $\nu((\cdot, \Phi'_\square(\boxdot)))$ and $(f, \Phi'_\square(\boxdot)) \in \mathfrak{C}_2$, it remains to prove that

$$\mathfrak{M}^*\big(b(\cdot, \cdot, \Phi'_\square(\boxdot))\big) = \int_N b\big(u, u, \Phi'(u)\big)\, d\mu^*(u)\,. \tag{6.17}$$

For $m = 1, 2, \ldots$ define

$$\psi_m(u) = b\big(P_m u, u, \Phi'(u)\big)\,, \qquad u \in N^1\,.$$

Clearly ψ_m is defined μ^*—a.e. and coincides on N^1 with the functional $-\,b\big(P_m\cdot, \Phi'_\square(\boxdot), \cdot\big)$ belonging to $\mathfrak{C}_2$. Thus

$$\mathfrak{M}^*(\psi_m) = \int_N \psi_m(u)\, d\mu^*(u) = \int_N b\big(P_m u, u, \Phi'(u)\big)\, d\mu^*(u)\,, \tag{6.17$_m$}$$

for all $m = 1, 2, \ldots$. But

$$\left|\int_N b\big(u, u, \Phi'(u)\big)\, d\mu^*(u) - \int_N \psi_m(u)\, d\mu^*(u)\right| \leqslant$$
$$\leqslant \int_N \left|b\big(u - P_m u, \Phi'(u), u\big)\right| d\mu^*(u) \leqslant$$
$$\leqslant c_{88}\int_N |(I - P_m)u|\cdot\|u\|\, d\mu^*(u) \leqslant c'_{88}\left(\int_N |(I - P_m)u|^2\, d\mu^*(u)\right)^{\frac{1}{2}}$$

(with some convenient constants $c_{88}-c'_{88}$), and $|(I-P_m)u|\leqslant|u|$ for all $m=1,2,\ldots$, so that, since $|(I-P_m)u|\to 0$ together with m^{-1} for all $u\in N^1$, i.e. μ^*—a.e. we can apply Lebesgue's dominated convergence theorem, obtaining

$$\int_N \psi_m(u)\,d\mu^*(u)\to\int_N b\big(u,u,\Phi'(u)\big)\,d\mu^*(u)\,.$$

Taking into account (6.17$_m$), $m=1,2,\ldots$, to establish (6.17) it remains still to prove the convergence

$$\mathfrak{M}^*(\psi_m)\to\mathfrak{M}^*\big(b(\cdot,\cdot,\Phi'_\square(\boxdot))\big)\,,\qquad\text{for } m\to\infty\,.$$

However this is a direct consequence of the following computations (valid for all $t\geqslant 1$ and $m=1,2,\ldots$)

$$\Big|\mathfrak{M}_t(\psi_m)-\mathfrak{M}_t\big(b(\cdot,\cdot,\Phi'_\square(\boxdot))\big)\Big|\leqslant\frac{1}{t}\int_0^t\Big[\int_N|b(u-P_mu,\Phi'(u),u)|\,d\mu_\tau(u)\Big]d\tau\leqslant$$

$$\leqslant c_{88}\frac{1}{t}\int_0^t\Big[\int_N|(I-P_m)u|\cdot\|u\|\,d\mu_\tau(u)\Big]d\tau\leqslant c_{88}\cdot$$

$$\cdot\frac{1}{t}\Big\{\int_0^t\Big[\int_N|(I-P_m)u|^2d\mu_\tau(u)\Big]d\tau\Big\}^{\frac12}\cdot\Big\{\int_0^t\Big[\int_N\|u\|^2d\mu_\tau(u)\Big]d\tau\Big\}^{\frac12}=$$

$$=c_{88}\big[\mathfrak{M}_t\big(|(I-P_m)\cdot|^2\big)\big]^{\frac12}\cdot\Big\{\frac{1}{t}\int_0^t\Big[\int_N\|u\|^2d\mu_\tau(u)\Big]d\tau\Big\}^{\frac12}\leqslant$$

$$\leqslant c_{88}\big[\mathfrak{M}_t\big(|(I-P_m)\cdot|^2\big)\big]^{\frac12}c''_{88}\qquad\text{where } (c''_{88})^2=\frac{2}{\nu}\int_N|u|^2d\mu(u)+\frac{|f|^2}{\nu\lambda_1},$$

see (6.7′);

indeed from them we infer

$$(6.16_m)\qquad\Big|\mathfrak{M}^*(\psi_m)-\mathfrak{M}^*\big(b(\cdot,\cdot,\Phi'_\square(\boxdot))\big)\Big|\leqslant c_{88}\cdot c''_{88}\big[\mathfrak{M}^*\big(|(I-P_m)\cdot|^2\big)\big]^{\frac12}=$$

$$=c_{88}\cdot c''_{88}\Big[\int_N|(I-P_m)u|^2d\mu^*(u)\Big]^{\frac12},$$

where the last relation follows from (6.9), since $|(I-P_m)\cdot|^2 \in C_2$. But as we already have shown the last integral in (6.16_m) tends to 0 for $m \to \infty$. This finishes the proof of the theorem.

REMARKS. 1°. The first part of the preceding proof is similar with that of Lemma 3 in Sec. 3.2.a), while the second part with that of the Corollary to this Lemma 3.

2°. Since individual solutions « are » also statistical solutions (see Sec. 3.4), the preceding Theorem 1 also concerns the asymptotic behaviour of individual solutions.

3°. Let Φ_0 be any particular functional belonging to $\mathcal{C}_{1,1}$ and let

$$\lim_{k\to\infty} \mathfrak{M}_{t_k}(\Phi_0) = \lambda\,. \tag{6.17}$$

for a certain sequence $t_1 < t_2 < \ldots < t_k < \to \infty$; obviously there exists an $\mathfrak{M}_0^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ such that

$$\mathfrak{M}_0^*(\Phi_0) = \lambda\,. \tag{6.17$'$}$$

In particular if $\Phi_0 \in C_2$, the stationary statistical solution μ_0^* corresponding to $\mathfrak{M}_0^*$ satisfies

$$\int_N \Phi_0(u)\, d\mu_0^*(u) = \lambda\,. \tag{6.17$''$}$$

Theorem 1 together with this last Remark have the following

COROLLARY. *There exists a stationary statistical solution* μ^* *which is non trivial* (i.e. *such that* $\operatorname{supp}\mu^* \not\subset S$) *and corresponds* (*by* (6.9)) *to an* $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ *if and only if*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mu_\tau(N\setminus O)\, d\tau > 0 \tag{6.18}$$

for some open set O *in* N *including the set* S *of all stationary individual solutions.*

PROOF. Let $d_S(\cdot)$ be the distance in N from u to S. It is easy to verify that $d_S \in \mathcal{C}_2$.

We shall prove first the following fact (of a certain independent interest) that (6.18) *holds for some open* (*in* $\boldsymbol{N}$) *set* $\boldsymbol{O} \supset \boldsymbol{S}$, *if and only if*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \left[\int d_S(u)\, d\mu_\tau(u)\right] d\tau > 0 . \tag{6.18'}$$

Indeed if (6.18) is valid for some convenient $\boldsymbol{O}$, then (since $\boldsymbol{S}$ is compact in $\boldsymbol{N}$) $\inf\limits_{\boldsymbol{N}\setminus\boldsymbol{O}} d_S(u) = \varepsilon_{\boldsymbol{O}} > 0$ so that

$$\int_0^t \left[\int_{\boldsymbol{N}} d_S(u)\, d\mu_\tau(u)\right] d\tau \geqslant \varepsilon_{\boldsymbol{O}} \int_0^t \mu_\tau(\boldsymbol{N}\setminus\boldsymbol{O})\, d\tau$$

and consequently (6.18′) is valid.

Suppose that now (6.18′) is valid and that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mu_\tau(\boldsymbol{N}\setminus\boldsymbol{O})\, d\tau = 0 \tag{6.19}$$

for any open (in $\boldsymbol{N}$) set $\boldsymbol{O} \supset \boldsymbol{S}$; in particular (6.19) holds for

$$\boldsymbol{O} = \boldsymbol{O}_q = \left\{u\colon\ u \in \boldsymbol{N},\ d_S(u) < \frac{1}{q}\right\}$$

for any $q = 1, 2, \ldots$. But

$$\begin{aligned}
&\int_{\boldsymbol{N}} d_S(u)\, d\mu_\tau(u) \leqslant \int_{\boldsymbol{O}_q} d_S(u)\, d\mu_\tau(u) + \int_{\boldsymbol{N}\setminus\boldsymbol{O}_q} d_S(u)\, d\mu_\tau(u) \leqslant \\
&\leqslant \frac{1}{q} + \|d_S\|_{\mathfrak{C}_2}\cdot \int_{\boldsymbol{N}\setminus\boldsymbol{O}_q} (1+|u|^2)\, d\mu_\tau(u) \leqslant \frac{1}{q} + \|d_S\|_{\mathfrak{C}_2}\cdot \int_{(\boldsymbol{N}\setminus\boldsymbol{O}_q)\cap\{u:\,u\in\boldsymbol{N},\ |u|\leqslant c'_{85}+1\}} (1+|u|^2)\, d\mu_\tau(u)\ (^*) + \\
&+ \|d_S\|_{\mathfrak{C}_2}\cdot \int_{(\boldsymbol{N}\setminus\boldsymbol{O}_q)\cap\{u:\,u\in\boldsymbol{N},\ |u|> c'_{85}+1\}} (1+|u|^2)\, d\mu_\tau(u) \leqslant \frac{1}{q} + c_{89}\mu_\tau(\boldsymbol{N}\setminus\boldsymbol{O}_q) + \|d_S\|_{\mathfrak{C}_2} \int_{\{u:\,u\in\boldsymbol{N},\ |u|> c'_{85}+1\}} (1+|u|^2)\, d\mu_\tau(u)
\end{aligned} \tag{6.20}$$

(*) For the constant c'_{85} see (6.13).

where c_{89} is a suitably constant (i.e. independent of $q = 1, 2, \ldots$ and $\tau \in (0, \infty)$). Let ψ be a continuous nonnegative function on $[0, \infty)$ such that $\psi(\xi) = 0$ for $0 \leqslant \xi \leqslant 1 + (c'_{85})^2$ and $\psi(\xi) = \xi$ for $\xi \geqslant 1 + (1 + c'_{85})^2$. Then in (6.20) we can insert ψ obtaining

$$\int_N d_S(u)\, d\mu_\tau(u) \leqslant \frac{1}{q} + c_{89}\mu_\tau(N \setminus O_q) + \|d_S\|_{\mathcal{C}_2} \cdot \int_N \psi(1 + |u|^2)\, d\mu_\tau(u)\,,$$

whence, for all $t > 0$,

$$\mathfrak{M}_t(d_S) \leqslant \frac{1}{q} + c_{89}\frac{1}{t}\int_0^t \mu_\tau(N \setminus O_q)\, d\tau + \|d_S\|_{\mathcal{C}_2} \cdot \mathfrak{M}_t\big(\psi(1 + |\cdot|^2)\big)\,, \tag{6.21}$$

where $\psi(1 + |\cdot|^2) \in \mathcal{C}_2$. Let λ denote the left term in (6.18′). By the above Remarks 3° there exists an $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ such that $\mathfrak{M}^*(d_S) = \lambda$. Moreover (6.21) implies for this $\mathfrak{M}^*$ that

$$\lambda \leqslant \frac{1}{q} + \|d_S\|_{\mathcal{C}_2} \cdot \mathfrak{M}^*\big(\psi(1 + |\cdot|^2)\big) = \frac{1}{q} + \|d_S\|_{\mathcal{C}_2} \cdot \int_N \psi(1 + |u|^2)\, d\mu^*(u) = \frac{1}{q}$$

for all $q = 1, 2, \ldots$, where the last equality follows from (6.13). It results $\lambda = 0$: Contradiction! This finishes the proof of the underlined above equivalence.

We pass now to the proof of the Corollary. Suppose again that the first term λ of (6.18′) is > 0. As above we deduce the existence of an $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ such that $\mathfrak{M}^*(d_S) = \lambda$. Let μ^* be the stationary statistical solution satisfying (6.9). The functional d_S belonging to $\mathcal{C}_2$ we shall have

$$\int_N d_S(u)\, d\mu^*(u) = \lambda$$

so that $\mu^*(N \setminus S) > 0$, thus μ^* is nontrivial. Conversely suppose that $\lambda = 0$, that is

$$\lim_{t\to\infty} \mathfrak{M}_t(d_S) = 0\,.$$

This implies that $\mathfrak{M}^*(d_S) = 0$, so that

$$\int_N d_S(u)\, d\mu^*(u) = 0$$

for any $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ and corresponding μ^*. Since $d_S > 0$ on $N \setminus S$ it results $\mu^*(N \setminus S) = 0$ for all such μ^*.

This finishes the proof of the Corollary.

REMARKS. 4°. There exists stationary statistical solutions which are nontrivial if and only if there exists a statistical solution $\{\mu_t\}_{0<t<\infty}$ with some initial data μ such that (6.18′) (or the equivalent relation involving (6.18)) holds. Indeed for the « only if » part of the assertion take $\mu_t = \mu^*$ and $\mu = \mu^*$, where μ^* is a desired stationary statistical solution. For the « if » part apply the Corollary.

5°. For statistical solution corresponding to individual solutions (see Sec. 3.4), the Corollary has a more intuitive formulation. To see this let $u(\cdot)$ be an individual solution on $(0, \infty)$ with initial data u and observe that for $\{\mu_t\}_{0<t<\infty}$ where $\mu_t = \delta_{u(t)}$ for $t > 0$, the integral

$$\frac{1}{t}\int_0^t \mu_\tau(\boldsymbol{\omega})\, d\tau\,, \qquad \text{where } \boldsymbol{\omega} \text{ is a Borel set } \subset N,$$

represents the average of the time $\tau \in (0, t)$ in which $u(t)$ belongs to ω. The individual solution $u(\cdot)$ is said to *converge asymptotically* to S if the above average tends to 1 for $t \to \infty$ whenever ω is an open neighbourhood (in N) of S (*). Therefore we can reformulate the Corollary in the particular case of Dirac measure valued statistical solutions in the following way: *Let $u(\cdot)$ be an individual solution on $(0, \infty)$ of the Navier-Stokes equations* (with some initial, nonsignificant, value u_0). *Then there exists an $\mathfrak{M}^* \in \mathcal{M}(\{\delta_{u(t)}\}_{0<t<\infty})$ for which the corresponding stationary statistical solution μ^* is nontrivial, if and only if $u(\cdot)$ does not converge asymptotically to S.*

6°. As it will be shown in Sec. 2 of this paragraph, in case the space dimension n is $= 2$, the statement made in Remark 4° above

(*) The notion of asymptotic convergence for dynamical systems belongs to Krylov-Bogoliubov [1].

can be restricted only to Dirac measure—valued solutions, i.e. relating the existence of nontrivial stationary statistical solution to the existence of some individual solutions which are not asumptotically convergent to $\boldsymbol{S}$. Unhappily in the more interesting three-dimensional case an analogue result is not yet established.

2. ***a***) In the case of space dimension $n = 2$, a (Borel) probability μ on $\boldsymbol{N}$ will be called *invariant* if

$$(6.22_2) \qquad \mu(\boldsymbol{\omega}) = \mu\big(S(t)^{-1}\boldsymbol{\omega}\big)$$

for all $t \geqslant 0$ and all Borel subsets $\boldsymbol{\omega}$ of $\boldsymbol{N}$. By proposition 2(a) in Sec. 6.1.a), ***any stationary statistical solution*** (*in the case* $n = 2$) ***is invariant.***

Conversely we have

PROPOSITION 3. *Let* $n = 2$ *and let* μ *be an invariant* (*Borel*) *probability on* $\boldsymbol{N}$. *Then* μ *is a stationary statistical solution carried by a bounded set in* $\boldsymbol{N}^1$. (Thus in the case of space-dimension $n = 2$ the stationary statistical solutions coincide with the invariant Borel probabilities on $\boldsymbol{N}$.)

PROOF. The fact that $\operatorname{supp}\mu$ is bounded in $\boldsymbol{N}^1$ was proved in Foias-Prodi [1] § 5. Therefore $\int_{\boldsymbol{N}} \|u\|^2 d\mu(u) < \infty$. By Sec. 3.1, the family of measures $\{\mu_t\}_{0<t<\infty}$ defined by (3.2) is a statistical solution of the Navier-Stokes equations (also satisfying the strengthened energy inequality). Since μ is invariant $\mu_t = \mu$ for all t, so that μ is a stationary statistical solution.

Though the proof of Proposition 3 is achieved, we want to sketch the proof of the fact established in Foias-Prodi [1] § 5, quoted above. Thus, for $\varepsilon > 0$ let χ_ε be the characteristic function (defined on $\boldsymbol{N}$) of the set $\{u : u \in \boldsymbol{N}^1, \|u\| \leqslant c_7 + \varepsilon\}$ (see Sec. 2.4 and formula (2.16′)). In virtue of Birkhoff's ergodic theorem (see Dunford-Schwartz [1], Ch. VIII, 7.4 and 7.5), for $t \to \infty$

$$\frac{1}{t}\int_0^t \chi_\varepsilon\big(S(\tau)u\big)\,d\tau \to \chi_\varepsilon^*(u)\,, \qquad \mu\text{ - a.e.}$$

and

$$\int_N \chi_\varepsilon(u)\, d\mu(u) = \int_N \chi_\varepsilon^*(u)\, d\mu(u)\,.$$

But (2.16′) implies that $\chi_\varepsilon^*(u) = 1$ for all $u \in N$, hence

$$\mu(\{u\colon u \in N,\ \|u\| \leqslant c_7 + \varepsilon\}) = \int_N \chi_\varepsilon(u)\, d\mu(u) = \mu(N) = 1\,.$$

Letting $\varepsilon \to 0$ we finally obtain

$$\mu(\{u\colon u \in N^1,\ \|u\| \leqslant c_7\}) = 1\,,$$

i.e.

$$\operatorname{supp}\mu \subset \{u\colon u \in N^1,\ \|u\| \leqslant c_7\})\,, \tag{6.23}$$

where c_7 is a constant (independent of μ; actually depending on Ω, ν and $|f|$). Using Birkhoff's ergodic theorem in a similar way in Foias-Prodi [1], § 5, was also proved

$$\int_N |Du|^2 d\mu(u) \leqslant c_7' \tag{6.23′}$$

where again c_7', like c_7, is independent of μ, and depends on Ω, ν and $|f|$.

PROPOSITION 4. *Let the space dimension n be $= 2$. Then all stationary statistical solutions* (or equivalently invariant probabilities) *are trivial, iff all individual solutions are asymptotically convergent to* S.

PROOF. In virtue of Remark 5° in Sec. 6.1.*b*) we have only to show that the last condition is sufficient for the « triviality » of all stationary statistical solutions. To this purpose, let μ be an invariant measure and let O be an open neighbourhood (in N) of S. Then if χ_O denotes the characteristic function of O, we have

$$\frac{1}{t}\int_N \chi_O(S(\tau)u)\, d\tau \to 1\,, \qquad \text{for } t \to N$$

and this for all $u \in N$, since *all* individual solutions are supposed to converge asymptotically to $\boldsymbol{S}$. By Birkhoff's ergodic theorem

$$\mu(\boldsymbol{O}) = \int_{\boldsymbol{N}} \chi_{\boldsymbol{O}}(u)\, d\mu(u) = \mu(\boldsymbol{N}) = 1$$

whence taking $\boldsymbol{O} = \boldsymbol{O}_q$ ($q = 1, 2, \ldots$; see the proof of the Corollary in Sec. 6.1.b)) and letting $q \to \infty$ we deduce $\mu(\boldsymbol{S}) = 1$ so that $\operatorname{supp}\mu \subset \boldsymbol{S}$, i.e. μ is trivial.

This finishes the proof.

Remarks. 1°. The first study on invariant measures (without relating them to statistical solutions) was done in Prodi [3], where was proved (without the assumption that $\partial\Omega$ is of class C^2, assumption which was always made in the present paper) that nontrivial invariant measures (in the case $n = 2$) exist if and only if there exists at least an individual solution which does not converge asymptotically to $\boldsymbol{S}$; there was also proved that the support of any invariant measure is bounded in $\boldsymbol{N}$. In our case we can already assert that it is compact in $\boldsymbol{N}$ since it is bounded in $\boldsymbol{N}^1$.

2°. There is an amazing characterization of stationary individual solutions by means of stationary statistical solutions. To see this note first that if $u_0 \in \boldsymbol{S}$ then δ_{u_0} is invariant and $\delta_{u_0}(\{u_0\}) = 1$. Conversely *if μ is a stationary statistical solution* (i.e. an invariant probability in $\boldsymbol{N}$) *and $\mu(\{u_0\}) \neq 0$ for a certain $u_0 \in \boldsymbol{N}$, then $u_0 \in \boldsymbol{S}$* (i.e. u_0 is a stationary individual solution). Indeed, since $S(t)$ is a continuous map in $\boldsymbol{N}$, for given ball $\boldsymbol{B}_t(\varrho) = \{v \colon v \in \boldsymbol{N},\ |v - S(t)u_0| < \varrho\}$ there exists a ball $\boldsymbol{B}_0(r) = \{v \colon v \in \boldsymbol{N},\ |v - u_0| < r\}$ (where $\varrho > 0$ and $r = r_\varrho > 0$ too), such that $\boldsymbol{B}_0(r) \subset S(t)^{-1}\boldsymbol{B}_t(\varrho)$; therefore

$$\mu(B_t(\varrho)) = \mu(S(t)^{-1}\boldsymbol{B}_t(\varrho)) \geqslant \mu(\boldsymbol{B}_0(r)) \geqslant \mu(\{u_0\}) .$$

It results

$$\mu(\{S(t)u_0\}) = \mu\left(\bigcap_{n=1}^{\infty} \boldsymbol{B}_t\left(\frac{1}{n}\right)\right) \geqslant \mu(\{u_0\})$$

hence $S(t)u_0$ as function of t can take only a finite number of values; being continuous it must be constant, that is: $u_0 \in \boldsymbol{S}$.

3°. Let μ be a stationary statistical solution; then

$$S(t)(\operatorname{supp}\mu) = \operatorname{supp}\mu \qquad \text{for all } t \geqslant 0 . \tag{6.24}$$

First, if $u_0 \in \operatorname{supp} \mu$ and $S(t_0)u_0 \notin \operatorname{supp} \mu$ for a certain $t_0 > 0$, take in $\boldsymbol{N}$ an enough small open ball $\boldsymbol{b}_0$ centred in $S(t_0)u_0$ such that $\boldsymbol{b}_0 \cap \cap \operatorname{supp} \mu = \emptyset$; one has $0 = \mu(\boldsymbol{b}_0) = \mu(S(t)^{-1}\boldsymbol{b}_0) > 0$ since $S(t)^{-1}\boldsymbol{b}_0$ is an open neighbourhood in $\boldsymbol{N}$ of $u_0 \in \operatorname{supp} \mu$. The contradiction obtained shows that no u_0 as above exists. Therefore

$$(6.24') \qquad S(t)(\operatorname{supp} \mu) \subset \operatorname{supp} \mu \qquad \text{for all } t \geqslant 0 .$$

Since $\operatorname{supp} \mu$ is compact in $\boldsymbol{N}$, so is $S(t)(\operatorname{supp} \mu)$, and hence morover (using also (6.24′))

$$1 \geqslant \mu\big(S(t)(\operatorname{supp} \mu)\big) = \mu\big(S(t)^{-1}[S(t)(\operatorname{supp} \mu)]\big) \geqslant \mu(\operatorname{supp} \mu) = 1$$

from where we infer that $\operatorname{supp} \mu \subset S(t)(\operatorname{supp} \mu)$, finishing the proof of (6.24).

b) In the case of space-dimension $n = 3$, there is no (or not yet available) such natural definition of an invariant probability as in the case of space-dimension $n = 2$ considered in the preceding Sec. 6.2.*a*). The only one already proposed (in Prodi [4]) is the following: In case $n = 3$, a (Borel) probability χ in $\boldsymbol{N}$ is called *invariant* if

$$(6.22_3) \qquad \mu(\boldsymbol{\omega}) = \mu\big(T(t)^{-1}\boldsymbol{\omega}\big)$$

for all $t > 0$ and all Borel subset $\boldsymbol{\omega}$ of $\boldsymbol{N}$. This definition needs some comments. Indeed $T(t)$ is not defined on whole $\boldsymbol{N}$ (see Sec. 5.1.*c*)) so that we have, in any case, for $t > 0$

$$(6.22_3') \qquad T(t)^{-1}\boldsymbol{\omega} = \{u\colon T(t)u \text{ makes sense and } T(t)u \in \boldsymbol{\omega}\} \subset \boldsymbol{N}^1 .$$

Therefore in the preceding definition it is also implicitely assumed that

$$(6.22_3'') \qquad \mu(\boldsymbol{N} \setminus \boldsymbol{N}^1) = 0 .$$

Moreover since

$$T(t)^{-1}\boldsymbol{N} \subset T(s)^{-1}\boldsymbol{N} \qquad \text{for } t \geqslant s$$

it results that the set

$$(6.25) \qquad \boldsymbol{\Omega} = \bigcap_{t>0} \big(T(t)^{-1}\boldsymbol{N}\big) = \bigcap_{n=1,2\ldots,} T(n)^{-1}\boldsymbol{N}$$

is a Borel subset of $\boldsymbol{N}$,

$$\boldsymbol{\Omega} \subset \boldsymbol{N}^1\,, \qquad \mu(\boldsymbol{\Omega}) = 1 \tag{6.25'}$$

and

$$T(t)^{-1}\boldsymbol{\Omega} = \boldsymbol{\Omega} \qquad \text{for all } t \geqslant 0\,. \tag{6.25''}$$

It is easy to verify on account of (6.25″) that for any $u_0 \in \boldsymbol{\Omega}$ there exists a *unique* individual solution $u(\cdot)$ defined on whole $[0, \infty)$ with values in $\boldsymbol{\Omega}$, having the initial data u_0, namely that is the function $u(t) = T(t)\,u_0$, $t \geqslant 0$.

In this manner the existence of nontrivial (i.e. with the support not included in $\boldsymbol{S}$) invariant measures (if possible with « large » supports) would give us the certitude that $\boldsymbol{\Omega}$ is large. However as shown in Sec. 6.1, there exists « many » stationary statistical solutions, but these may, *à priori*, not be invariant if $n = 3$ (in spite of the fact that in case $n = 2$, they are invariant). Of course, Proposition 2(b) (in Sec. 6.1.a)) implies readily (compare (6.6$_3$) with (6.22$_3$)) that *any stationary statistical solution carried by a bounded set in* $\boldsymbol{N}^1$ *is invariant.* In Sec. 6.2.a) it was proved that in case $n = 2$, any stationary statistical solution (being invariant) is carried by a bounded set in $\boldsymbol{N}^1$ (see (6.23)), but there is not (yet?) a similar result in the case $n = 3$.

We finish this discussion with the converse of the preceding underlined fact, namely with the following

PROPOSITION 5. *Let $n = 3$ and let μ be an invariant (Borel) probability in $\boldsymbol{N}$, carried by a bounded set in $\boldsymbol{N}^1$. Then μ is a stationary statistical solution.*

PROOF. We have that for an enough large r_2^0,

$$\operatorname{supp}\mu \subset \{u\colon u \in \boldsymbol{N}^1,\ \|u\| \leqslant r_2^0\} = \boldsymbol{b}_0^1$$

and that for $t \in [0, t_2]$ with a certain $t_2 > 0$, the operator $T(t)$ is defined on $\boldsymbol{b}_0^1$ with values in $\boldsymbol{b}^1 = \{u\colon u \in \boldsymbol{N}^1,\ \|u\| \leqslant r_2\}$ where r_2 is suitably chosen; moreover $T(t)$ maps $\boldsymbol{N}^1$-continuously $\boldsymbol{b}_0^1$ into $\boldsymbol{b}^1$. Therefore for a given $\Phi \in \mathcal{C}_0^{\text{ind}}$ we can consider the function φ in $t \in [0, t_2]$ defined by

$$\varphi(t) = \int_{\boldsymbol{N}} \Phi\big(T(t)\,u\big)\,d\mu(u) = \int_{\boldsymbol{b}_0^1} \Phi\big(T(t)\,u\big)\,d\mu(u)\,. \tag{6.26}$$

Since $\Phi'(u) = P_m \Phi'(u)$ for some m we have $\Phi'(u) \in \boldsymbol{N}^1$ for all $u \in \boldsymbol{N}$ thus taking the derivatives in (6.26) and going on as in Sec. 3.1 (what is possible in virtue of $T(t)u \in \boldsymbol{b}^1$ for all $u \in \boldsymbol{b}_0^1$ and $t \in [0, t_2]$) we arrive to

$$(6.26') \qquad \frac{d\varphi(t)}{dt} = \int_{\boldsymbol{b}_0^1} \Big[\big(f, \Phi'_\square(\boxed{T(t)u})\big) - \nu\big(\big(T(t)u, \Phi'_\square(\boxed{T(t)u})\big)\big) - b\big(T(t)u, T(t)u, \Phi'_\square(\boxed{T(t)u})\big) \Big] d\mu(u) .$$

But since μ is invariant $\varphi(\cdot)$ is constant so that taking (6.26′) in $t = 0$ we receive (6.2) for all $\Phi \in \mathcal{C}_0^{\mathrm{ind}}$ thus (since μ obviously satisfies (6.1)) also for all $\Phi \in \mathcal{C}^{\mathrm{ind}}$. Concerning (6.3), let us replace Φ in (6.2) by $\psi(|P_m \cdot|^2)$ where ψ is C^1-function on $[0, \infty)$ constant for all enough large real numbers. We receive

$$\int_{\boldsymbol{N}} \psi'(|P_m u|^2)\big[\nu \|P_m u\|^2 + b(u, u, P_m u)\big]\, d\mu(u) = \int_{\boldsymbol{N}} \psi'(|P_m u|^2)(f, P_m u)\, d\mu(u) .$$

Here we are permitted to let $m \to \infty$ and apply Lebesgue's dominated convergence theorem, since for all $u \in \operatorname{supp} \mu$

$$|b(u, u, P_m u)| \leqslant c_1 \|u\|^2 \cdot \|P_m u\| \leqslant c_1 (r_2^0)^3 .$$

We obtain finally

$$(6.27) \qquad \int_{\boldsymbol{N}} \psi'(|u|^2)\, \nu \|u\|^2 d\mu(u) = \int_{\boldsymbol{N}} \psi'(|u|^2)(f, u)\, d\mu(u)$$

from which (6.3) can be easily deduced. This finishes the proof.

REMARKS. 1°. We have actually also proved that a stationary statistical solution μ carried by a bounded set in $\boldsymbol{N}^1$ satisfies the strengthened energy equation, i.e.

$$(6.28) \qquad \nu \int_{\boldsymbol{N}} \varphi(|u|^2)\, \|u\|^2 d\mu(u) = \int_{\boldsymbol{N}} \varphi(|u|^2)(f, u)\, d\mu(u)$$

for any bounded continuous real function φ on $[0, \infty)$. Moreover this result (as its proof) is valid for both cases $n = 2$ and $n = 3$; by Sec. 6.2.a) in the first case ($n = 2$), it concerns all stationary statistical solutions.

2°. Let us show that *in the case* $n = 3$, *the property* (6.23′) *also holds for an invariant measure provided we assume that its support is bounded in* $\boldsymbol{N}^1$. To prove this we first note that we can consider μ as a Borel measure in $\boldsymbol{N}^1$ (*). Let us denote by $\boldsymbol{\sigma}_1$ its support as Borel measure in $\boldsymbol{N}^1$. By our assumption $\boldsymbol{\sigma}_1$ is bounded in $\boldsymbol{N}^1$. We can now prove as in Remark 3°, Sec. 6.2.a), that $T(t)\boldsymbol{\sigma}_1 \subset \boldsymbol{\sigma}_1$ for $t \in [0, t_2]$, where $t_2 > 0$ is small enough. By a convenient choice of t_2 we can suppose that (4.26$^{\text{v}}$)-(4.26$^{\text{vi}}$), in Sec. 5.1.c), are valid for all $u \in \boldsymbol{B}_0^1$ where $\boldsymbol{B}_0^1$ is a closed ball (with radius $r_2^0 < \infty$) in $\boldsymbol{N}^1$ containing $\boldsymbol{\sigma}_1$; in particular

$$\int_0^{t_2} |DT(t)u|^2 dt \leqslant c_0'' , \qquad \text{for all } u \in \boldsymbol{B}_0^1 . \tag{4.26$^{\text{vi}}_{\text{bis}}$}$$

The last result implies easily that if $u(\cdot)$ is any regular (individual) solution on $[0, t_0]$ (for any finite $t_0 > 0$) then $Du(\cdot) \in L^2(0, t_0; \boldsymbol{N})$ (see also Sec. 5.3). Consequently

$$\frac{1}{2}\|P_m u(t_0)\|^2 - \frac{1}{2}\|P_m u(0)\|^2 = \frac{1}{2}\int_0^t \frac{d\|P_m u(\tau)\|^2}{d\tau} d\tau =$$

$$= \int_0^{t_0} \left(\frac{du(\tau)}{d\tau}, DP_m u(\tau)\right) d\tau =$$

$$= \int_0^{t_0} [(f, DP_m u(\tau)) - \nu |DP_m u(\tau)|^2 - b(u(\tau), u(\tau), DP_m u(\tau)] \, d\tau \leqslant$$

$$\leqslant \int_0^{t_0} [|f| \cdot |DP_m u(\tau)| - \nu |DP_m u(\tau)|^2 + c_{67} \cdot c_{68} \|u(\tau)\|^{\frac{3}{2}} |Du(\tau)|^{\frac{1}{2}} |DP_m u(\tau)|] d\tau \leqslant$$

$$\leqslant \int_0^{t_0} [|f| \cdot |Du(\tau)| - \nu |DP_m u(\tau)|^2 + c_{67} \cdot c_{68} \|u(\tau)\|^{\frac{3}{2}} |Du(\tau)|^{\frac{3}{2}}] \, d\tau ,$$

(*) Note that on one side any Borel subset of $\boldsymbol{N}^1$ is a Borel set in $\boldsymbol{N}$, and that on the other side the intersection of $\boldsymbol{N}^1$ with any Borel set in $\boldsymbol{N}$, is a Borel set in $\boldsymbol{N}^1$.

whence

$$\tfrac{1}{2}\|P_m u(t_0)\|^2 - \tfrac{1}{2}\|P_m u(0)\|^2 + \nu\int_0^{t_0} |DP_m u(\tau)|^2 d\tau \leqslant$$

$$\leqslant \int_0^{t_0} [|f|\cdot|Du(\tau)| + c_{67}\cdot c_{68}\|u(\tau)\|^{\frac{3}{2}}\cdot|Du(\tau)|^{\frac{3}{2}}]\, d\tau \leqslant$$

$$\leqslant c_{90} + c'_{90}\int_0^{t} \|u(\tau)\|^6 d\tau + \frac{\nu}{2}\int_0^{t_0} |Du(\tau)|^2 d\tau;$$

letting $m \to \infty$, we finally obtain

$$\int_0^{t_0} |Du(\tau)|^2 d\tau \leqslant c_{90} + c'_{90}\int_0^{t_0} \|u(\tau)\|^6 d\tau \tag{6.29}$$

where $c_{90} - c'_{90}$ are constants (independent of the regular solution $u(\cdot)$ and $t_0 > 0$). For $u \in \boldsymbol{\sigma}_1$, $T(t)\,u \in \boldsymbol{\sigma}_1$ for any $t > 0$, so that applying (6.29) to $u(\cdot) = T(\cdot)\,u$ we obtain, for all $t_0 > 0$

$$\int_0^{t_0} |DT(t)\,u|^2 dt \leqslant c_{90} + c'_{90} t_0 (r_2^0)^6,$$

where (remember that) r_2^0 denote the radius (in N^1) of the $\boldsymbol{B}_0^1$. Thus

$$\limsup_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} |DT(t)\,u|^2 dt \leqslant c'_{90}(r_2)^6 = c_{91}\,, \qquad \text{for all } u \in \boldsymbol{\sigma}_1\,. \tag{6.30}$$

We shall use now an argument of Foias-Prodi [1], § 5: Take $\Phi_m(u) = |DP_m u|^2$ for $u \in \boldsymbol{\sigma}_1$ and apply Birkhoff's ergodic theorem to the system $\{T(t)|\boldsymbol{\sigma}_1\}_{0\leqslant t<\infty}$ and the μ-integrable function Φ_m. We obtain

$$\int_{\sigma_1} \Phi_m(u)\, d\mu(u) = \int_{\sigma_1} \Phi_m^*(u)\, d\mu(u)\,, \tag{6.31}$$

where

$$\Phi_m^*(u) = \lim_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} \Phi_m(T(t)\,u)\, dt \leqslant \limsup_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} |DT(t)\,u|^2 dt \leqslant c_{91}\,; \tag{6.31'}$$

by (6.31)-(6.31′)

$$\int_N |DP_m u|^2 \, d\mu(u) = \int_N \Phi_m(u)\, d\mu(u) \leqslant c_{91}\,,$$

whence letting $m \to \infty$ and applying Beppo Levi's theorem to the nondecreasing sequence $\{|DP_m \cdot|^2 = |P_m D \cdot|^2\}$ we receive

$$\int_N |Du|^2 \, d\mu(u) \leqslant c_{91}\,. \tag{6.23″}$$

This finishes the proof.

Note that contrary to the case $n = 2$ (see (6.23′)) in (6.26″) the constant c_{91} depends also on μ, thus the whole information furnished by (6.26″) is

$$\int_N |Du|^2 \, d\mu(u) < \infty\,. \tag{6.23‴}$$

c) The discussion in the preceding Section has not taken into account the fact that Sec. 5.4 provides us with accretive statistical solutions. We intend now to exploit the existence of these « nice » solutions. To this purpose for a (Borel) probability μ on $\boldsymbol{N}$, satisfying (3.5), let us introduce the notation $\mathfrak{M}(\mu)$ for the union of all sets $\mathfrak{M}(\{\mu_t\}_{0<t<\infty})$ where $\{\mu_t\}_{0<t<\infty}$ runs over all statistical solutions satisfying the strengthened energy inequality, with initial data μ. A simple overlook on the proof of Theorem 1 in Sec. 6.1.*b*), shows us that the conclusions of this Theorem 1 hold also for all $\mathfrak{M}^* \in \mathfrak{M}(\mu)$. Using Theorem 5′ in Sec. 5.4 we shall prove the following complement to Theorem 1, Sec. 6.1.*b*):

THEOREM 2. *Let* $n = 3$ *and suppose that* $\operatorname{supp}\mu$ *is bounded in* $\boldsymbol{N}$; *then there exists an* $\mathfrak{M}^* \in \mathfrak{M}(\mu)$ *such that its corresponding stationary statistical solution* χ^* *is accretive, i.e. it fulfils*

$$\mu(\boldsymbol{\omega}) \geqslant \mu\big(T(t)^{-1}\boldsymbol{\omega}\big) \tag{6.32}$$

for all $t \geqslant 0$ *and all Borel sets* $\boldsymbol{\omega}$ *in* $\boldsymbol{N}$.

PROOF. Let $\{\mu_t\}_{0<t<\infty}$ be the statistical solution with initial data μ which is yielded by Theorem 5′, Sec. 5.4. Since it verifies the strengthened energy inequality there exists a constant r_0 (see (4.5)) such that

$$\operatorname{supp}\mu_t \subset \boldsymbol{B}_0 = \{u\colon u \in \boldsymbol{N},\ |u| \leqslant r_0\}\ , \qquad \text{a.e. on } [0, \infty)\ .$$

To this r_0 there corresponds an r_1 such that any individual solution of the Navier-Stokes equations starting from a point of $\boldsymbol{B}_0$ does not leave $\boldsymbol{B}_1 = \{u\colon u \in \boldsymbol{N},\ |u| \leqslant r_1\}$ and moreover that

$$S_t^{(m)}\boldsymbol{B}_0 \subset \boldsymbol{B}_1 \qquad \text{for all } m = 1, 2, \ldots \text{ and } t \geqslant 0\ .$$

Let $t_0 > 0$ and let again $\boldsymbol{\omega}_0$ denote the set of those $u \in \boldsymbol{N}^1$ on which $T(t_0)$ is defined. As already remarked, $\boldsymbol{\omega}_0$ is open in $\boldsymbol{N}^1$. The property (5.32′) implies that

$$\int_{\boldsymbol{N}} \Phi(u)\, d\mu_{t_0+\tau}(u) \geqslant \int_{\boldsymbol{\omega}_0} \Phi\big(T(t_0)\,u\big)\, d\mu_\tau(u) \tag{6.33}$$

for any $\Phi \in \mathfrak{C}_0$, $\Phi \geqslant 0$. Let $\varphi \in \mathfrak{C}_{1,1}$ satisfy $0 \leqslant \varphi \leqslant 1$, $\varphi = 0$ outside $\boldsymbol{\omega}_0$. From (6.33) we deduce for any $t \geqslant t_0$

$$\begin{aligned} \mathfrak{M}_t(\Phi) + \frac{1}{t}\int_t^{t+t_0} \Big[\int_{\boldsymbol{N}} \Phi(u)\, d\mu_\tau(u)\Big]\, d\tau\, - \\ - \frac{1}{t}\int_0^{t_0} \Big[\int_{\boldsymbol{N}} \Phi(u)\, d\mu_\tau(u)\Big]\, d\tau \geqslant \mathfrak{M}_t\Big(\Phi\big(T(t_0)\cdot\big)\, \varphi(\cdot)\Big)\ . \end{aligned} \tag{6.33′}$$

Here $\Psi(\cdot) = \Phi\big(T(t_0)\cdot\big)\varphi(\cdot) \in \mathfrak{C}_{1,1}$ and

$$\Big|\int_s^{s+t_0} \Big[\int_{\boldsymbol{N}} \Phi(u)\, d\mu_\tau(u)\Big]\, d\tau\Big| \leqslant \|\Phi\|_{\mathfrak{C}_0} \cdot t_0\ , \qquad \text{for any } s \geqslant 0$$

so that (6.33′) implies

$$\int_{\boldsymbol{N}} \Phi(u)\, d\mu^*(u) = \mathfrak{M}^*(\Phi) \geqslant \mathfrak{M}^*(\Psi)\ . \tag{6.33″}$$

We have now to show that for a convenient choice of Φ and φ the value of the last term in (6.33″) can also be expressed $\int \Psi \, d\mu^*$. First note that for $\varrho > 0$

$$\boldsymbol{\omega}_\varrho = \left\{u\colon u \in \boldsymbol{\omega}_0, \sup_{0 \leqslant \tau \leqslant t_0} \|T(\tau)u\| < \frac{1}{\varrho}\right\} \tag{6.34}$$

is open in $\boldsymbol{N}^1$ and that

$$\boldsymbol{\omega}_0 = \bigcup_{n=1}^{\infty} \boldsymbol{\omega}_{n^{-1}} . \tag{6.34′}$$

Secondly put

$$\delta_\varrho(n) = \text{distance in } \boldsymbol{N}^1 \text{ from } n \text{ to } \boldsymbol{N}^1 \setminus \boldsymbol{\omega}_\varrho \tag{6.35}$$

for $\varrho > 0$, and

$$\varphi_{\varepsilon,\varrho}(u) = \begin{cases} 1 & \text{if } \delta_\varrho(u) \geqslant 2\varepsilon , \\ \text{linear in } \delta_\varrho(u) & \text{if } \varepsilon \leqslant \delta_\varrho(u) \leqslant 2\varepsilon \\ 0 & \text{if } \delta_\varrho(u) \leqslant \varepsilon , \end{cases} \tag{6.35′}$$

for $\varepsilon > 0$. Then for all $m = 1, 2, \ldots$

$$|\varphi_{\varepsilon,\varrho}(u) - \varphi_{\varepsilon,\varrho}(P_m u)| \leqslant \frac{1}{\varepsilon} |\delta_\varrho(u) - \delta_\varrho(P_m u)| \leqslant \frac{1}{\varepsilon} \|u - P_m u\| . \tag{6.35″}$$

Finally let us put

$$\Psi_m(u) = \Phi(S_{t_0}^{(m)} P_m u) \, \varphi_{\varepsilon,\varrho}(P_m u) , \quad \text{for } u \in \boldsymbol{N} \tag{6.35‴}$$

and all $m = 1, 2, \ldots$. It is plain that $\Psi_m \in \mathfrak{C}_0$, so that

$$\mathfrak{M}^*(\Psi_m) = \int_{\boldsymbol{N}} \Psi_m(u) \, d\mu^*(u) , \quad m = 1, 2, \ldots . \tag{6.36}$$

Moreover we have

$$\begin{aligned} |\Phi(T(t_0)u) \, \varphi_{\varepsilon,\varrho}(u) - \Psi_m(u)| \leqslant \frac{1}{\varepsilon} \|u - P_m u\| \cdot \|\Phi\|_{\mathfrak{C}_0} + \\ + \varphi_{\varepsilon,\varrho}(u) |\Phi(T(t_0)u) - \Phi(S_{t_0}^{(m)} P_m u)| , \quad \text{for } u \in \boldsymbol{N}^1 \end{aligned} \tag{6.37}$$

and $m = 1, 2, \ldots$. Applying the same technique as in the proof of the convergence property given in Sec. 5.1.c) (before Theorem 2′, Sec. 5.1.c)), one can establish (without major difficulty) that for $u \in \boldsymbol{\omega}_\varrho$ we have

$$(6.38)\qquad |S_t^{(m)} P_m u - T(t)u|^2 \leqslant c_\varrho |f - P_m f|^2 + c'_\varrho \int_0^{t_0} \|(I - P_m) T(t) u\|^2 dt \leqslant$$

$$\leqslant c_\varrho |f - P_m f|^2 + c'_\varrho \lambda_{m+1}^{-\frac{1}{2}} \int_0^t |(I - P_m) D T(t) u|^2 dt \leqslant$$

$$\leqslant c_\varrho |f - P_m f|^2 + c''_\varrho \lambda_{m+1}^{-\frac{1}{2}} ,$$

where $c_\varrho - c''_\varrho$ are some constants depending on $\varrho > 0$, but independent of $m = 1, 2, \ldots$ and $u \in \boldsymbol{\omega}_\varrho$. Therefore, for any $\varrho' > 0$,

$$(6.38')\qquad |S_{t_0}^{(m)} P_m u - T(t_0) u| \to 0 , \quad \text{for } m \to \infty$$

uniformly on $\boldsymbol{\omega}_{\varrho'}$, in particular on $\boldsymbol{\omega}_{\varrho/2}$. Since $\{u : u \in \boldsymbol{N}^1, \|u\| \leqslant 1/\varrho\} = \boldsymbol{b}^1$ is compact in $\boldsymbol{N}$, for $\delta > 0$ there exists an $\eta > 0$ such that $|v - v'| < \eta$, $v \in \boldsymbol{b}^1$ implies $|\Phi(v) - \Phi(v')| < \delta$. By (6.38)-(6.38′) and the definition (6.35)-(6.35′) we deduce from (6.37) that

$$(6.37')\qquad \left|\Phi\big(T(t_0)u\big)\varphi_{\varepsilon,\varrho}(u) - \Psi_m(u)\right| \leqslant \delta + \frac{1}{\varepsilon}\|u - P_m u\| \cdot \|\Phi\|_{\mathfrak{C}_0}$$

for all $u \in \boldsymbol{N}^1$ and $m \geqslant m(\delta)$ where $m(\delta)$ is an enough large number. (6.37′) implies that for $m \geqslant m(\delta)$ we have

$$\left|\mathfrak{M}_t\Big(\Phi\big(T(t_0)\cdot\big)\varphi_{\varepsilon,\varrho}(\cdot)\Big) - \mathfrak{M}_t(\Psi_m)\right| \leqslant \delta + \frac{1}{\varepsilon}\|\Phi\|_{\mathfrak{C}_0} \cdot \mathfrak{M}_t(\|\cdot - P_m \cdot\|) .$$

whence

$$(6.39)\qquad \left|\mathfrak{M}^*\Big(\Phi\big(T(t_0)\cdot\big)\varphi_{\varepsilon,\varrho}(\cdot)\Big) - \mathfrak{M}^*(\Psi_m)\right| \leqslant \delta + \frac{1}{\varepsilon}\|\Phi\|_{\mathfrak{C}_0} \cdot \mathfrak{M}^*(\|\cdot - P_m \cdot\|)$$

Plainly (6.39) shows that $\mathfrak{M}^*(\Psi_m) \to \mathfrak{M}^*\Big(\Phi\big(T(t_0)\cdot\big)\varphi_{\varepsilon,\varrho}(\cdot)\Big)$. Taking into account (6.36) and (6.37′) once again, we can finally deduce that

$$\mathfrak{M}^*\Big(\Phi\big(T(t_0)\cdot\big)\varphi_{\varepsilon,\varrho}(\cdot)\Big) = \int_{\boldsymbol{N}} \Phi\big(T(t_0)u\big)\varphi_{\varepsilon,\varrho}(u)\, d\mu^*(u) =$$

$$= \int_{\boldsymbol{\omega}_0} \Phi\big(T(t_0)u\big)\varphi_{\varepsilon,\varrho}(u)\, d\mu^*(u) .$$

Thus taking $\varphi = \varphi_{\varepsilon,\varrho}$ in (6.33″), we obtain

$$\int_N \Phi(u)\, d\mu^*(u) \geqslant \int_{\omega_0} \Phi(T(t_0)\, u)\, \varphi_{\varepsilon,\varrho}(u)\, d\mu^*(u)\, .$$

Letting first $\varepsilon \to \infty$ it results

$$\int_N \Phi(u)\, d\mu^*(u) \geqslant \int_{\omega_\varrho} \Phi(T(t_0)\, u)\, d\mu^*(u)\, .$$

Putting $\varrho = n^{-1}$, letting $n \to \infty$ and taking into account (6.34′) we finally obtain

$$\int_N \Phi(u)\, d\mu^*(u) \geqslant \int_{\omega_0} \Phi(T(t_0)\, u)\, d\mu^*(u)\, , \qquad \text{for all } \Phi \in \mathcal{C}_0\, . \tag{6.40}$$

It is now easy to deduce (since we already did a similar deduction) that (6.40) implies (is even equivalent to) the validity of (6.32) for $t = t_0$ and all Borel sets $\boldsymbol{\omega}$ in $\boldsymbol{N}$. Since $t_0 > 0$ is arbitrary, the proof of Theorem 2 is finished.

Remarks. 1°. Borel probabilities on $\boldsymbol{N}$ (actually on $\boldsymbol{N}^1$) satisfying (6.32) for all $t \geqslant 0$ and Borel sets $\boldsymbol{\omega} \subset \boldsymbol{N}$, were firstly attached to the study of Navier-Stokes equations in Prodi [4] (see also Prodi [5]), where they were called *semi-invariant*. A simple analysis of the proof of Theorem 6 in Sec. 5.4.*b*) shows us that *a semi-invariant* (*Borel*) *probability* μ *on* $\boldsymbol{N}$ *satisfying*

$$\int_N \|u\|^4\, d\mu(u) < \infty \tag{6.41}$$

is invariant. This fact was established (with proof different of that resulting from Sec. 5.4.*b*)) already in Prodi [4], [5].

2°. From the proof of the preceding Theorem 2, it is clear that its conclusion concerns any $\mathfrak{M}^*$ belonging to $\mathcal{M}(\{\mu_t\}_{0<t<\infty})$ if $\{\mu_t\}_{0<t<\infty}$ is chosen accordingly to Theorem 5′ in Sec. 5.4.*a*); but by Remark 3° in the same Section, any individual solution is, as statistical solution, of this kind. We can therefore conclude that *if* $u(\cdot)$ *is any individual solution on* $(0, \infty)$ (*with some initial data* $u_0 \in \boldsymbol{N}$), *then for any* $\mathfrak{M}^* \in$

$\in \mathcal{M}(\{\delta_{u(t)}\}_{0<t<\infty})$ *the corresponding stationary statistical solution* μ^* *is accretive*. This result includes the existence theorem for semi-invariant probabilities in Prodi [4].

3°. Concerning the discussion in the preceding two remarks let us mention that the present study connecting semi-invariant probabilities with stationary statistical solutions, involves new features which were not considered in Prodi [4].

3. ***a***) We shall prove now two propositions which will be used later in § 8.

Let us begin by noting that if μ is a stationary statistical solution then from (6.3) (with $\psi(\xi)=\xi$ for $\xi\geqslant 0$) it results readily the inequality

$$\int_N (1+\|u\|^2)\,d\mu(u)\leqslant 1+\frac{|f|^2}{\lambda_1\nu}=c_{92} \tag{6.41}$$

whence for all $\Phi\in\mathcal{C}_{1,1}$

$$\begin{aligned}\left|\int_N \Phi(u)\,d\mu(u)\right|&\leqslant\|\Phi\|_{\mathcal{C}_{1,1}}\int_N(1+|u|\cdot\|u\|)\,d\mu(u)\leqslant\\&\leqslant\max\{1,\lambda_1^{-\frac{1}{2}}\}\cdot\|\Phi\|_{\mathcal{C}_{1,1}}\cdot\int_N(1+\|u\|^2)\,d\mu(u)\leqslant c_{93}\|\Phi\|_{\mathcal{C}_{1,1}},\end{aligned} \tag{6.41'}$$

with a constant c_{93} independent of μ. Therefore if $\mathcal{C}_{2;1,1}$ denote the closure in $\mathcal{C}_{1,1}$ of $\mathcal{C}_2$, the map

$$\mathfrak{M}_\mu\colon\ \Phi\mapsto\int_N\Phi(u)\,d\mu(u)$$

on $\mathcal{C}_{2;1,1}$ is continuous and uniquely determined by its restriction to $\mathcal{C}_2$.

PROPOSITION 6. *Let* $\mathcal{S}$ *denote the set of all functionals* $\mathfrak{M}_\mu\in\mathcal{C}_{2;1,1}$ *corresponding to the stationary statistical solutions of the Navier-Stokes equations and let in case* $n=3$, $\mathcal{S}^a$ *be its subset formed by those* $\mathfrak{M}_\mu$ *for which* μ *is accretive. Then both* $\mathcal{S}$ *and* $\mathcal{S}^a$ *are convex sets in* $\mathcal{C}^*_{2;1,1}$ *compact in the* w^**-topology of* $\mathcal{C}^*_{2;1,1}$.

PROOF. The convexity of both $\mathcal{S}$ and $\mathcal{S}^a$ is obvious. Since by (6.41′) both sets are bounded in $\mathcal{C}^*_{2;1,1}$ it remains only to verify that they are closed in the w^*-topology of $\mathfrak{C}^*_{2;1,1}$. To this purpose let first $\{\mathfrak{M}_{\mu_\alpha}\}_\alpha$ be a directed set $\subset \mathfrak{S}$, converging in the w^*-topology of $\mathfrak{C}^*_{2;1,1}$ to some $\mathfrak{M} \in \mathfrak{C}^*_{2;1,1}$. It is plain that in virtue of (6.41′) we can regard $\mathfrak{M}_\mu$ (for any $\mu \in \mathfrak{S}$) as belonging to $\mathfrak{C}^*_{1,1}$ and verifying $\|\mathfrak{M}_\mu\|_{\mathfrak{C}^*_{1,1}} \leqslant c_{93}$. Thus we are allowed to suppose that $\{\mathfrak{M}_{\mu_\alpha}\}$ has a w^*-cluster point $\mathfrak{M}^*$ in $\mathfrak{C}^*_{1,1}$; obviously $\mathfrak{M} = \mathfrak{M}^*|\mathfrak{C}_{2;1,1}$. Reproducing the proof of Theorem 1 in Sec. 6.1.*b*) in the same way in which the proof of Theorem 1 in Sec. 4.2.*a*) reproduces that of Theorem 1 in Sec. 3.2, we can deduce that $\mathfrak{M} = \mathfrak{M}_{\mu^*}$ for some stationary statistical solutions μ^*.

Suppose now that $n = 3$ and that all μ_α are accretive. Then using the same notations as in the proof of Theorem 2 in Sec. 6.2.*c*)

$$\int_N \Phi(u)\, d\mu_\alpha(u) \geqslant \int_{\omega_0} \Phi\big(T(t_0)u\big)\, d\mu_\alpha(u) \geqslant \int_{\omega_0} \Phi\big(T(t_0)u\big)\, \varphi_{\varepsilon,\varrho}(u)\, d\mu_\alpha(u)\,,$$

whence

$$\int_N \Phi(u)\, d\mu^*(u) = \mathfrak{M}^*(\Phi) \geqslant \mathfrak{M}^*\big(\Phi(T(t_0)\cdot)\,\varphi_{\varepsilon,\varrho}(\cdot)\big) \tag{6.42}$$

for all $\Phi \in \mathfrak{C}_0$, $t_0 \geqslant 0$, and $\varepsilon, \varrho > 0$. Now, exactly as in the proof of Theorem 2 in Sec. 6.2.*c*) we can show that (6.42) implies that μ^* is accretive. We consider that this is sufficient to conclude the proof of Proposition 6.

REMARKS. 1°. Let $n = 2$. Then all stationary statistical solutions are invariant (and vice-versa). On the other hand it is well known (and easy to verify) that extremal invariant probabilities (see for instance Choquet [1], § 31) are the ergodic ones. In our case ergodicity of an invariant probability in $\boldsymbol{N}$ means that if for a Borel set $\boldsymbol{\omega} \subset \boldsymbol{N}$ we have $S(t)^{-1}\boldsymbol{\omega} = \boldsymbol{\omega}$ (up to a set of μ-measure 0) for all $t \geqslant 0$, then either $\mu(\boldsymbol{\omega}) = 1$ or $\mu(\boldsymbol{\omega}) = 0$. It is obvious that in our case if $u_0 \in \boldsymbol{S}$ (i.e. u_0 is a stationary individual solution) then δ_{u_0} is ergodic, and that if μ is an ergodic invariant probability on $\boldsymbol{N}$, then either $\mu(\boldsymbol{S}) = 0$ or $\mu = \delta_{u_0}$ for some $u_0 \in \boldsymbol{S}$.

Now let us suppose that all ergodic invariant probabilities in $\boldsymbol{N}$ are of the form δ_{u_0}, $u_0 \in \boldsymbol{S}$. Let $\mathfrak{M}_0$ be an extremal point of $\mathfrak{S}$, and let μ be the corresponding stationary statistical solution. Plainly if, as invariant probability, μ is not ergodic (i.e. extremal), neither $\mathfrak{M}_0$ could be extremal. Therefore $\mathfrak{M}_0 = \mathfrak{M}_{\delta_{u_0}}$ for some $u_0 \in \boldsymbol{S}$. Thus by the Krein-Milman theorem (see Dunford-Schwartz [1], Ch. V, 8.4) any $\mathfrak{M}_\mu \in \mathcal{S}$ is in the closed (in the w^*-topology of $\mathcal{C}^*_{2,1;1}$) convex hull of $\{\mathfrak{M}_{\delta_{u_0}}: u_0 \in S\}$; therefore there exists a directed set $\{\mathfrak{M}_{\mu_\alpha}\}_\alpha$ such that $\operatorname{supp} \mu_\alpha \subset S$ for all α, converging in the w^*-topology of $\mathcal{C}^*_{2;1,1}$ to $\mathfrak{M}_\mu$. Since if $d(u) =$ distance in N of u to S (for $u \in N$), then $d(\cdot) \in \mathcal{C}_2 \subset \mathcal{C}_{2;1,1}$ and $\mathfrak{M}_{\mu_\alpha}(d) = \int_N d(u)\, d\mu_\alpha(u) = 0$ for all α, we infer

$$\int_N d(u)\, d\mu(u) = \mathfrak{M}_\mu(d) = 0 ,$$

whence $\operatorname{supp} \mu \subset S$, i.e. μ is trivial. In virtue of Proposition 4 in Sec. 6.2.*a*) the fact that all stationary statistical solutions are trivial implies that all individual solutions converge asymptotically to S. Concluding, *if* $n = 2$ *either all individual solutions converge asymptotically to* S *or there exists a nontrivial ergodic invariant probability on* N.

2°. In case $n = 3$, since neither the stationary statistical solutions are known to be invariant, nor the invariant probabilities are known to be stationary statistical solutions the above discussion has no analogue. Moreover in this case we don't know if $\mathcal{S}^a$ really differs from $\mathcal{S}$.

b) Let be a (Borel) probability on N satisfying (3.5). For any $v \in N$ consider the integral

$$I(v) = \int_N (u, v)\, d\mu(u) .$$

Since

$$|I(v)| \leqslant |v| \cdot \int_N |u|\, d\mu(u) \leqslant |v| \cdot \Big[\int_N |u|^2 d\mu(u)\Big]^{\frac{1}{2}}, \qquad v \in N, \tag{6.43}$$

the map $v \mapsto I(v)$ is a continuous linear functional on N, thus there exists an element $U^\mu \in N$ such that $I(v) = (U^\mu, v)$, i.e.

$$(U^\mu, v) = \int_N (u, v)\, d\mu(u) , \qquad \text{for all } v \in N . \tag{6.44}$$

This U^μ is, by definition, the *barycenter* of μ. Actually, if we use the theory of vector valued integrable functions, we can write (see Dinculeanu [1], § 15)

$$U^{\mu} = \int_{N} u \, d\mu(u) \, . \tag{6.44'}$$

Let now $\{\mu_t\}_{0<t<\infty}$ be a statistical solution of the Navier-Stokes equations (with some initial data which needs not be specified); if

$$U_t = U^{\mu_t} \qquad \text{for } t > 0 \tag{6.45}$$

then by (6.43) and (6.7) we have

$$|U_t| \leqslant \Big(\int_{N} |u|^2 d\mu_t(u)\Big)^{\frac{1}{2}} \leqslant c_{94} \, , \qquad \text{a.e. on } (0, \infty) \, , \tag{6.45'}$$

where $c_{94} < \infty$ is independent of t. Sine the functional $(\cdot, v)$ is (for any $v \in \boldsymbol{N}$) $\boldsymbol{N}$-weakly continuous on $\boldsymbol{N}$, the integral

$$\int_{N} (u, v) \, d\mu_{\bullet}(u)$$

is measurable on $(0, \infty)$ (see (3.16″)), so that $(U_{\bullet}, v)$ is measurable on $(0, \infty)$ for any $v \in \boldsymbol{N}$; this means that $U_{\bullet}$ is weakly measurable on $(0, \infty)$. But $\boldsymbol{N}$ is separable so that $U_{\bullet}$ is also strongly measurable (see Hille-Phillips [1], Ch. III, § 1). Taking into account also (6.45′) we deduce that

$$\overline{U}_t = \frac{1}{t} \int_0^t U_\tau d\tau \quad \text{exists and} \quad |\overline{U}_t| \leqslant c_{94} \qquad \text{for all } t > 0 \, . \tag{6.46}$$

By the last inequality it results that the directed set $\{\overline{U}_t\}_{1 \leqslant t < \infty}$ has cluster points in the $\boldsymbol{N}$-weak topology. Thus the content of the next proposition is not empty.

PROPOSITION 7. *Let U be a cluster point in $\boldsymbol{N}_{\text{weak}}$ of the directed set $\{\overline{U}_t\}_{1 \leqslant t < \infty}$, where $\overline{U}_t$ were defined in* (6.46). *Then there exists an $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ such that U is the barycenter of the stationary statistical solution μ^* which corresponds to $\mathfrak{M}^*$.*

PROOF. Since, on $\{u: u \in N, |u| \leqslant c_{94}\}$, the weak topology of N is metrizable, there exists a sequence $1 \leqslant t_1 < t_2 < \ldots \to \infty$ such that

$$\overline{U}_{t_j} \to U \qquad \text{for } j \to \infty \text{, in } N_{\text{weak}} \text{,}$$

i.e.

$$\mathfrak{M}_{t_j}((\cdot, v)) = \frac{1}{t_j}\int_0^{t_j} U_\tau \, d\tau \to (U, v) \text{,} \qquad \text{for } j \to \infty \text{,}$$

where $v \in N$ is fixed but arbitrary. In virtue of Remark 3° in Sec. 6.1.*b*), there exists $\mathfrak{M}^* \in \mathcal{M}(\{\mu_t\}_{0<t<\infty})$ such that for the corresponding stationary statistical solution μ^* one has

$$(U, v) = \int_N (u, v) \, d\mu^*(v)$$

and this for all $v \in N$, thus U is the barycenter of μ^*. This finishes the proof.

REMARKS. 1°. Applying the preceding Proposition to the case $\mu_t = \delta_{u(t)}$, $t > 0$, where $u(\cdot)$ is an individual solution of the Navier-Stokes equations and taking into account Proposition 3 in Sec. 2.*a*) and Remark 1°-2° in Sec. 2.*c*) we readily deduce that *if u^* is a cluster point in N_{weak} of the directed set $\{1/t \int_0^t u(\tau)\, d\tau\}_{0<t<\infty}$, where $u(\cdot)$ is an individual solution, then u^* is the barycenter of a stationary statistical solution μ^* which is invariant in case $n = 2$ and semi-invariant in case $n = 3$.*

2°. In the case, considered in the preceding Remark, if $n = 2$, the cluster points in N_{weak} of $\{\overline{U}_t\}_{0<t\to\infty}$ are actually cluster points in N. Indeed in this case in virtue of (2.16′) we have

$$\limsup_{t\to\infty} \left\| \frac{1}{t}\int_0^t u(\tau)\, d\tau \right\| \leqslant c_7$$

so that if (see the proof of Proposition 7 above)

$$(6.47) \qquad \frac{1}{t_j}\int_0^{t_j} u(\tau)\, d\tau \to u^* \ (\text{in } N_{\text{weak}}) \text{,} \quad \text{for } j \to \infty$$

the sequence

$$\left\{\frac{1}{t_j}\int\limits_0^{t_j} u(\tau)\,d\tau\right\}_{j=1}^{\infty}$$

is contained in a bounded ball in $\boldsymbol{N}^1$, thus in a compact set in $\boldsymbol{N}$; therefore, in (6.47) the convergence holds even in $\boldsymbol{N}$.

3°. *Let* n (*the space dimension* $= 2$ *and let* $\boldsymbol{\Lambda}_0$ *denote the set of those* $u_0 \in \boldsymbol{N}$ *for which*

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t S(\tau)\,u_0\,d\tau$$

exists in $\boldsymbol{N}$. *Then* $\mu(\boldsymbol{\Lambda}_0) = 1$ *for all stationary statistical solutions.* To prove this assertion we shall exploit the fact that any such solution μ is an invariant probability in $\boldsymbol{N}$. Let now, for a $v \in \boldsymbol{N}$, $\boldsymbol{E}_v$ denotes the set of those $u_0 \in \boldsymbol{N}$ for which

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t \big(S(\tau)\,u_0,\, v\big)\,d\tau$$

does not exist. By Birkhoff's theorem $\mu(\boldsymbol{E}_v) = 0$. Choose a dense countable set $\{v_j\}_{j=1}^{\infty}$ in $\boldsymbol{N}$ and put $\boldsymbol{E} = \bigcup\limits_{j=1}^{\infty} \boldsymbol{E}_{v_j}$. Then for $u_0 \notin \boldsymbol{E}$, denoting by $u(\cdot)$ the individual solution $S(\cdot)u_0$, and using the preceding notations (see (6.44)-(6.46)) we have that

$$(\overline{U}_t,\, v_j) = \frac{1}{t}\int\limits_0^t \big(S(\tau)\,u_0,\, v_j\big)\,d\tau$$

is convergent for $t \to \infty$ and this for any $j = 1, 2, \ldots$. Since $\{\overline{U}_t\}_{0<t}$ is bounded in $\boldsymbol{N}$, the preceding conclusion implies that for $t \to \infty$, $\overline{U}_t$ is convergent in $\boldsymbol{N}_{\text{weak}}$. But repeating the argument from Remark 2°, $\{\overline{U}_t\}_{t_0\leq t}$ (for a $t_0 > 0$) is bounded in $\boldsymbol{N}^1$, thus the convergence for $t \to \infty$ of $\{\overline{U}_t\}$ in $\boldsymbol{N}_{\text{weak}}$ implies that in $\boldsymbol{N}$. In this manner, $\boldsymbol{N}\setminus\boldsymbol{E} \subset \boldsymbol{\Lambda}_0$. But $\mu(\boldsymbol{E}) = 0$; hence $\mu(\boldsymbol{\Lambda}_0) = 1$.

4°. Of course it would be very interesting if a similar fact to that underlined in Remark 3°, holds also in the case $n = 3$ for all accretive stationary statistical solutions. This would perhaps necessitate a nontrivial generalization of the classical Birkhoff ergodic theorem in a new direction, not yet tried (as far as we know).

5°. The results given in this Sec. 6.3.*b*) will be useful in § 8 for the study of Reynolds equations.

7. Study of the invariant measures and the corresponding dynamical systems.

1. We have shown in the preceding paragraph that the asymptotic behaviour of statistical or individual solutions of the Navier-Stokes equations is intimately connected with the existence and nature of stationary statistical solutions. In case the space dimension n (of the fluid, i.e. the dimension of Ω) is $= 2$, these solutions are precisely the invariant probabilities in $\boldsymbol{N}$. These facts justify the interest of studying such invariant measures. *All this paragraph 7 will be devoted to the study of the invariant probabilities on* $\boldsymbol{N}$, *for two-dimensional fluids*, i.e. to the study of those (Borel) probabilities μ in $\boldsymbol{N}$ which satisfy

$$\mu(\boldsymbol{\omega}) = \mu\big(S(t)^{-1}\boldsymbol{\omega}\big) \qquad \text{for all Borel set } \boldsymbol{\omega} \subset \boldsymbol{N}\,.$$

As already observed in Sec. 6.2, the support of any such μ is bounded in $\boldsymbol{N}^1$ and thus compact in $\boldsymbol{N}$. We shall complete this fact with the following

THEOREM 1. *Let μ be an invariant* (*Borel*) *probability in* $\boldsymbol{N}$. *Then*

$$\eta(\delta) = \operatorname{supp}\big\{\|u - v\| : u, v \in \operatorname{supp}\mu,\ |u - v| \leqslant \delta\big\} \tag{7.1}$$

tends to 0 *for* $\delta \to 0$. *Consequently on supp* μ *the topologies of* $\boldsymbol{N}$ *and* $\boldsymbol{N}^1$ *coincide, hence supp* μ *is also compact in* $\boldsymbol{N}^1$.

PROOF. It is plain that we have only to prove the first statement that is: $\eta(\delta) \to 0$ for $\delta \to 0$. To this aim let $r_1 = \sup\big\{\|u\| : u \in \operatorname{supp}\mu\big\}$. Note that $r_1 < \infty$ and $\|S(t)u\| \leqslant r_1$ for all $t \geqslant 0$ and $u \in \operatorname{supp}\mu$. Let now $u, v \in \operatorname{supp}\mu$ and put $U(t) = S(t)u$, $V(t) = S(t)v$ for $t \geqslant 0$. Then

$w(\cdot) = U(\cdot) - V(\cdot)$ is an N-valued absolutely continuous function satisfying a.e. on $(0, \infty)$ the relations

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 = -b(w, U, w) \leqslant \sqrt{2}|w|^{\frac{1}{2}}\cdot\|w\|^{\frac{1}{2}}\cdot\|U\|\cdot|w|^{\frac{1}{2}}\cdot\|w\|^{\frac{1}{2}} \leqslant$$
$$\leqslant \sqrt{2}r_1|w|\cdot\|w\| \leqslant \frac{\nu}{2}\|w\|^2 + \frac{r_1^2}{\nu}|w|^2$$

that is, $|w(\cdot)|^2$ satisfies the differential inequality

$$\frac{d}{dt}|w|^2 + \nu\|w\|^2 \leqslant \frac{2r_1^2}{\nu}|w|^2, \qquad \text{a.e. on } (0, \infty)\,.$$

It results readily (with $c_{95} = \exp[2\nu^{-1}r_1^2]$ and $c'_{95} = (r_1^2/\nu^2)\exp[2\nu^{-1}r_1^2] + +1$)

$$|w(t+s)|^2 \leqslant c_{95}|w(t)|^2 \tag{7.2}$$

$$\int_t^{t+s}\|w(\tau)\|^2 d\tau \leqslant c'_{95}|w(t)|^2 \tag{7.2'}$$

for all $t \geqslant 0$, $0 \leqslant s \leqslant 1$. On the other hand we have also a.e. on $(0, \infty)$

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu|Dw|^2 \leqslant |b(w, U, Dw)| + |b(V, w, Dw)| \leqslant \tag{7.3}$$
$$\leqslant 2^{\frac{1}{4}}c_4|w|^{\frac{1}{2}}\cdot\|w\|^{\frac{1}{2}}\cdot\|U\|^{\frac{1}{2}}\cdot|DU|^{\frac{1}{2}}\cdot|Dw| + 2^{\frac{1}{4}}c_4|V|^{\frac{1}{2}}\cdot\|V\|^{\frac{1}{2}}\cdot\|w\|^{\frac{1}{2}}\cdot|Dw|^{\frac{3}{2}} \leqslant$$
$$\leqslant c_{96}|DU|^2\cdot|w|^2 + c'_{96}\|w\|^2 + \frac{\nu}{2}|Dw|^2,$$

where $c_{96} - c'_{96}$ are some convenient constant depending only on r_1, Ω, ν and $|f|$. Analogous calculus, made on U instead w, provide on $(0, \infty)$ the following differential inequality for U:

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \nu|DU|^2 \leqslant c_{97}|f|^2 + c'_{97}\|U\|^2 + \frac{\nu}{2}|DU|^2 \qquad \text{a.e.},$$

where again $c_{97} - c'_{97}$ are constants depending on r_1, Ω, ν and $|f|$.

It results

$$(7.3') \qquad \int_t^{t+s} |DU(\tau)|^2 d\tau \leqslant \frac{2c_{97}}{\nu}|f|^2 + \frac{2(c'_{97}+1)}{\nu} r_1^2 = c''_{97}$$

for all $t \geqslant 0$, $0 \leqslant s \leqslant 1$; here obviously c''_{97} is a constant depending only on r_1, Ω, ν, $|f|$. Integrating (7.3) and taking into account (7.2)-(7.2′) we deduce easily that

$$(7.4) \qquad \|w(t+s)\|^2 + \nu \int_t^{t+s} |Dw(\tau)|^2 d\tau \leqslant c_{98}|w(t)|^2 + \|w(t)\|^2$$

for all $t \geqslant 0$, $0 \leqslant s \leqslant 1$; here again c_{98} is a constant depending only on r_1, Ω, ν, $|f|$. In (7.4) take $t+s = t_0+1$ where $t_0 \geqslant 0$, neglect the integral and then integrate in $s \in (0,1)$. It results

$$\|w(t_0+1)\|^2 \leqslant \int_{t_0}^{t_0+1} (c_{98}|w(t)|^2 + \|w(t)\|^2)\, dt \leqslant$$

$$\leqslant \left(\frac{c_{98}}{\lambda_1} + 1\right) \int_{t_0}^{t_0+1} \|w(t)\|^2 dt \leqslant \left(\frac{c_{98}}{\lambda_1} + 1\right) c'_{95}|w(t_0)|^2$$

where (7.2′) was used. Thus we can conclude with:

$$(7.5) \qquad \|S(t+1)u - S(t+1)v\| \leqslant c_{99}|u-v|$$

for all $u, v \in \operatorname{supp}\mu$ and $t \geqslant 0$, c_{99} being a constant depending on μ, Ω, ν and $|f|$.

Suppose now that $\eta(\delta) \not\to 0$ for $\delta \to 0$. Then there exists $\delta > 0$ $u_j, v_j \in \operatorname{supp}\mu$, $j = 1, 2, \ldots$, such that

$$(7.6) \qquad |u_j - v_j| \to 0\,, \qquad \text{for } j \to \infty$$

and

$$(7.6') \qquad \|u_j - v_j\| \geqslant \delta\,, \qquad \text{for all } j = 1, 2, \ldots .$$

In virtue of (6.24) (see Sec. 6.2.a), Remark 3°) we can suppose that there exists $u_j^0, v_j^0 \in \operatorname{supp}\mu$ such that $S(1)u_j^0 = u_j$, $S(1)v_j^0 = u_j$, $j = 1, 2, \ldots$. Since $\operatorname{supp}\mu$ is compact in $\boldsymbol{N}$ we can also suppose (passing to subsequences if necessary) that $u_j^0 \to u^0$, $v_j^0 \to v^0$ in $\boldsymbol{N}$ for $j \to \infty$. Since $S(1)$ is a continuous map in $\boldsymbol{N}$ for any $t \geq 0$, we deduce $u_j = S(1)u_j \to S(1)u_j^0$, $v_j = S(1)v_j^0 \to S(1)v^0$ in $\boldsymbol{N}$ for $j \to \infty$; by (7.6), $S(1)u^0 = S(1)v^0$. But (7.5) implies

$$\|u_j - S(1)u^0\| = \|S(1)u_j^0 - S(1)u^0\| \leqslant c_{66}|S(1)u_j^0 - S(1)u^0| \to 0\,,$$

$$\|v_j - S(1)v^0\| = \|S(1)v_j^0 - S(1)v^0\| \leqslant c_{99}|S(1)v_j^0 - S(1)v^0| \to 0\,,$$

for $j \to \infty$. Since $S(1)u^0 = S(1)v^0$, the preceding convergences contradict (7.6′).

The proof of the theorem is now finished.

REMARK. A precise evaluation of the function $\eta(\cdot)$ defined by (7.1) would be extremely interesting. For instance if

$$\eta(\delta) = O(\delta) \qquad \text{for } \delta \to 0 \tag{7.1′}$$

(that is if $\eta(\delta) \leqslant c_{100}\delta$ for all $\delta > 0$ sufficiently small and a certain fixed constant $c_{100} < \infty$), then $\operatorname{supp}\mu$ would be of finite dimension.

To verify this assertion take $u, v \in \operatorname{supp}\mu$ and remark that for all $m = 1, 2, \ldots$

$$\lambda_{m+1}|(I - P_m)(u - v)|^2 \leqslant \|(I - P_m)(u - v)\|^2 \leqslant \|u - v\|^2 \leqslant \eta(|u - v|^2) \leqslant$$
$$\leqslant c_{100}^2\,|u - v|^2 \leqslant c_{100}^2|P_m(u - v)|^2 + c_{100}^2|(I - P_m)(u - v|^2)$$

implies

$$(\lambda_{m+1} - c_{100}^2)|(I - P_m)(u - v)|^2 \leqslant c_{100}^2|P_m(u - v)|^2$$

so that, for m sufficiently large (i.e. such that $\lambda_{m+1} > c_{100}^2$) it results that $P_m|\operatorname{supp}\mu$ is injective, hence a homeomorphic map of $\operatorname{supp}\mu$ onto a compact set of the m-dimensional real Euclidean space.

2. ***a***) We have mentioned in Sec. 2.2, that if $\boldsymbol{S}$ denotes the set of all stationary individual solutions of the Navier-Stokes equations and if m is large enough then P_m is a homeomorphic map from $\boldsymbol{S}$ into $P_m\boldsymbol{N}$, i.e. $\boldsymbol{S}$ is a compact in $\boldsymbol{N}$ of finite dimension. In case the invariant measure μ is trivial i.e. $\operatorname{supp}\mu \subset \boldsymbol{S}$, $\operatorname{supp}\mu$ will also be a compact set in $\boldsymbol{N}$ of finite dimension. One of the main mathematical problems in this framework is that of the nature of $\operatorname{supp}\mu$ for any invariant measure μ, for instance, if $\operatorname{supp}\mu$ is always of finite dimension. We shall comment later this question, after we have studied some connections between $\operatorname{supp}\mu$ and P_m for an invariant measure μ and m large enough.

We begin our discussion by observing that if $u_0 \in S$, the invariant probability Dirac measure δ_{u_0} has the property that $P_m(\operatorname{supp}\mu)$ is reduced to a point (namely $\{P_m u_0\}$) and that for m large enough this is the only Dirac measure valued invariant probability for which $P_m(\operatorname{supp}\mu) = \{P_m u_0\}$. This trivial remark can be completed by the following:

PROPOSITION 1. *Let μ be an invariant (Borel) probability in $\boldsymbol{N}$ such that $P_m(\operatorname{supp}\mu)$ is reduced to a point for m large enough. Then μ is a Dirac measure, that is $\mu = \delta_{u_0}$ with $u_0 \in \boldsymbol{S}$.*

PROOF. Let

$$u_0 = U^\mu\Big(= \int_{\boldsymbol{N}} u \, d\mu(u)\Big) .$$

Then $u_0 \in \mathfrak{D}_D$ (see (6.23′), Sec. 6.2.a)) and for any $u \in \operatorname{supp}\mu$ we have

$$\begin{aligned}(7.7) \qquad \frac{1}{2}\frac{d}{dt}|S(t)u - u_0|^2 &= \Big(S(t)u - u_0, \frac{d}{dt}S(t)u\Big) = \\ = -\nu((S(t)u - u_0, S(t)u)) - b(S(t)u, S(t)u, S(t)u - u_0) &+ \\ + (f, S(t)u - u_0) = -\nu\|S(t)u - u_0\|^2 - b(S(t)u - u_0, u_0, S(t)u - u_0) &+ \\ + (f - \nu D u_0 - B(u_0, u_0), S(t)u - u_0)\,, &\quad \text{a.e. on } (0, \infty) .\end{aligned}$$

Let now $r_1 = \sup\{\|u\| : u \in \operatorname{supp}\mu\}$; then $r_1 < \infty$ and $\|u_0\| \leqslant r_1$. Let m be subjected to the condition

$$(7.7') \qquad \lambda_{m+1} \geqslant 8 r_1^2 \nu^{-2} .$$

Fix now an m such that $P_m(\operatorname{supp}\mu)$ is reduced to a point and that (7.7′) be valid. Plainly

$$(7.8) \qquad P_m(\operatorname{supp}\mu) = \{P_m u_0\} .$$

Consequently we have $S(t)u - u_0 = (I - P_m)(S(t)u - u_0)$ for all $t \geqslant 0$ and $u \in \operatorname{supp} \mu$, so that (7.7)-(7.7′) yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|S(t)u - u_0|^2 + \nu\|S(t)u - u_0\|^2 &\leqslant 2^{\frac{1}{2}}|S(t)u - u_0|\cdot\|S(t)u - u_0\|\cdot \\ &\cdot\|u_0\| + (u_1, S(t)u - u_0) = 2^{\frac{1}{2}}|(I - P_m)(S(t)u - u_0)|\cdot\|S(t)u - u_0\|\cdot \\ &\cdot\|u_0\| + (u_1, S(t)u - u_0) = (2\lambda_{m+1}^{-1})^{\frac{1}{2}}\|S(t)u - u_0\|^2\cdot\|u_0\| + \\ &+ (u_1, S(t)u - u_0) = (2r_1^2\lambda_{m+1}^{-1})^{\frac{1}{2}}\|S(t)u - u_0\|^2 + \\ &+ (u_1, S(t)u - u_0) \leqslant \frac{\nu}{2}\|S(t)u - u_0\|^2 + \frac{1}{2}(u_1, S(t)u - u_0),\end{aligned}$$

a.e. on $(0, \infty)$. Here above $2^{-1}u_1 = f - \nu D u_0 - B(u_0, u_0)$. We can conclude these computations with the relation

$$(7.7'') \qquad |S(t)u - u_0|^2 + \nu\int_0^t \|S(t)u - u_0\|^2 d\tau \leqslant |u - u_0|^2 + \int_0^t (u_1, S(\tau)u - u_0)\, d\tau$$

valid for all $t \in (0, \infty)$, $u \in \operatorname{supp} \mu$. Denote by $f(u)$, resp. $g(u)$, the functions (in $u \in \operatorname{supp} \mu$) $\|u - u_0\|^2$, resp. $(u_1, u - u_0)$. Plainly $f, g \in L^\infty(\mu) \subset L^1(\mu)$ so that by Birkhoff's ergodic theorem (see Dunford-Schwartz [1], Ch. VIII, 7.4 and 7.5)

$$f^*(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t f(S(\tau)u)\, d\tau, \qquad g^*(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t g(S(\tau)u)\, d\tau$$

exist μ - a.e. and

$$\int_N f^*(u)\, d\mu(u) = \int_N f(u)\, d\mu(u), \qquad \int_N g^*(u)\, d\mu(u) = \int_N g(u)\, d\mu(u).$$

But in virtue of (7.7″) we will have $f^*(u) \leqslant \nu^{-1} g^*(u)$ μ - a.e. so that

$$\begin{aligned}0 \leqslant \int_N \|u - u_0\|^2 d\mu(u) &= \int_N f^*(u)\, d\mu(u) \leqslant \nu^{-1}\int_N g^*(u)\, d\mu(u) = \\ &= \nu^{-1}\int_N (u_1, u - u_0)\, d\mu(u) = \nu^{-1}\Big(u_1, \int_N u\, d\mu(u) - u_0\Big) = 0,\end{aligned}$$

whence $u = u_0$ μ - a.e. This readily implies that $\operatorname{supp}\mu = \{u_0\}$ so that $\mu = \delta_{u_0}$ and consequently (since μ is a stationary statistical solution) $u_0 \in \boldsymbol{S}$.

The proof is now complete.

REMARKS. 1°. Comparing the preceding Proposition 1 with Remark 2° in (Sec. 6.2.*a*), it looks very plausible that the condition that $P_m(\operatorname{supp}\mu)$ be a singleton might be replaced by the weaker one that $\mu(P_m^{-1}\{\xi_0\}) > 0$ for a certain $\xi_0 \in P_m\boldsymbol{N}$. We taught once that this fact is rather obvious, but finally we couldn't prove it.

2°. Another open related question is the following: If for an invariant (Borel) probability we have $P_m(\operatorname{supp}\mu) \subset P_m\boldsymbol{S}$ for large enough m, does it follow that μ is trivial, i.e. that $\operatorname{supp}\mu \subset \boldsymbol{S}$?

b) We shall now give some estimations on the support of an invariant (Borel) probability in $\boldsymbol{N}$, which gives us an insight on the position in $\boldsymbol{N}$ of the support. To this aim we shall introduce the following definitions. For a compact set $\boldsymbol{\Sigma} \subset \boldsymbol{N}$ and $m = 1, 2, \ldots$, we shall put

$$(7.9) \qquad \begin{cases} h_m(\boldsymbol{\Sigma}) = \sup\{|(I - P_m)u| : u \in \operatorname{supp}\mu\}\,, \\ t_m(\boldsymbol{\Sigma}) = \sup\{|(I - P_m)(u - v)| : u, v \in \operatorname{supp}\mu,\ P_m u = P_m v\} \end{cases}$$

and will call $h_m(\boldsymbol{\Sigma})$, resp. $t_m(\boldsymbol{\Sigma})$, the *highness*, resp. the *thickness*, of $\boldsymbol{\Sigma}$ in the orthogonal direction to $P_m\boldsymbol{N}$. It is clear that $h_m(\boldsymbol{\Sigma}) = 0$ means that $\boldsymbol{\Sigma} \subset P_m\boldsymbol{N}$ and that $t_m(\boldsymbol{\Sigma}) = 0$ means that P_m is injective from $\boldsymbol{\Sigma}$ in $P_m\boldsymbol{N}$ (and hence that $\boldsymbol{\Sigma}$ is of finite dimension being homeomorphic with $P_m(\boldsymbol{\Sigma})$.

For the highness of the support of an invariant probability μ in $\boldsymbol{N}$, we have the following estimation

$$(7.10) \qquad h_m(\operatorname{supp}\mu) \leqslant C\lambda_{m+1}^{-\frac{3}{4}}\,, \qquad m = 1, 2, \ldots\,,$$

where C is a constant depending only Ω, ν and $|f|$ while $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ are the eigenvalue of D (see Sec. 2.1). The estimation (7.10) was proved in Foias-Prodi [1], § 5.

We shall improve now (7.10) by the following

PROPOSITION 2. *Let μ be an invariant (Borel) probability in $\boldsymbol{N}$. Then*

$$(7.10') \qquad \limsup_{m\to\infty} \frac{\log[h_m(\operatorname{supp}\mu)]}{\log\lambda_{m+1}} = -1\,.$$

PROOF. Let $\Phi_m(\cdot)$ be defined for $u \in \operatorname{supp}\mu$ by

$$\Phi_m(u) = \nu\|(I-P_m)u\|^2 + b\big(u, u, (I-P_m)u\big) - \big(f, (I-P_m)u\big),$$

and let $\varphi \in C^1([0, \infty))$. Plainly $\Phi_m(\cdot)$ and $\varphi\big(|(I-P_m)\cdot|^2\big)$ belong to $L^1(\mu)$. Note also that for any $u \in \operatorname{supp}\mu$ we have

$$\frac{1}{2}\frac{d}{dt}\varphi\big(|(I-P_m)S(t)u|^2\big) + \varphi'\big(|(I-P_m)S(t)u|^2\big)\,\Phi_m\big(S(t)u\big) = 0$$

so that

$$\left|\frac{1}{t}\int_0^t \varphi'\big(|(I-P_m)S(\tau)u|^2\big)\,\Phi_m\big(S(\tau)u\big)\,d\tau\right| =$$

$$= \frac{\left|\varphi\big(|(I-P_m)S(t)u|^2\big) - \varphi\big(|(I-P_m)u|^2\big)\right|}{2t} \leqslant \frac{c_{101}}{2t}, \qquad \text{for } t > 0,$$

where $c_{101} = \sup\{|\varphi(\xi)| : 0 \leqslant \xi \leqslant r_0^2\}$, $r_0 = \sup\{|u| : u \in \operatorname{supp}\mu\}$.

It results that

$$\Psi_m(u) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi'\big(|(I-P_m)S(\tau)u|^2\big)\,\Phi_m\big(S(\tau)u\big)\,d\tau = 0$$

for all $u \in \operatorname{supp}\mu$, thus μ - a.e. By Birkhoff's theorem (see Dunford-Schwartz [1], Ch. VIII, 7.4 and 7.5) we have

$$\int_N \varphi'\big(|(I-P_m)u|^2\big)\,\Phi_m(u)\,d\mu(u) = \int_N \Psi_m(u)\,d\mu(u) = 0\,. \tag{7.11}$$

In (7.11), φ' is actually any continuous real-valued function on $[0, \infty)$, therefore (7.11) implies

$$\int_{\{u:\, u\in N,\ a\leqslant|(I-P_m)u|\leqslant b\}} \Phi_m(u)\,d\mu(u) = 0$$

that is

$$\int_{\{u:\, u\in N,\ |a|\leqslant|(I-P_m)u|\leqslant b\}} \big[\nu\|(I-P_m)u\|^2 + b\big(u, P_m u, (I-P_m)u\big)\big]\,d\mu(u) = \tag{7.11'}$$

$$= \int_{\{u:\, u\in N,\ a\leqslant|(I-P_m)u|\leqslant b\}} \big(f, (I-P_m)u\big)\,d\mu(u)$$

for all $b \geqslant a \geqslant 0$. From (7.11′), (2.2), (2.3′) and (2.15) we infer

$$\nu \int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} \|(I-P_m)u\|^2 d\mu(u) \leqslant |f|\cdot \int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} |(I-P_m)u|\, d\mu(u) +$$

$$+ \int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} |u|_p \cdot |\operatorname{grad} P_m u|_{2p/(p-2)} |(I-P_m)u|\, d\mu(u) \leqslant$$

$$\leqslant |f|\cdot \int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} |(I-P_m)u|\, d\mu(u) +$$

$$+ c_3 \cdot \int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} \|u\| \cdot |D^{\frac{1}{2}+1/p} P_m u| \cdot |(I-P_m)u|\, d\mu(u) \leqslant$$

$$\leqslant \left(|f| + r_1^2 c_3 \lambda_m^{1/p}\right) \int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} |(I-P_m u)|\, d\mu(u)$$

where $r_1 = \sup\{\|u\| : u \in \operatorname{supp} \mu\}$, $m = 1, 2, \ldots$ and $b \geqslant a \geqslant 0$; it results

$$\int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} |(I-P_m)u|^2 d\mu(u) \leqslant \tag{7.11''}$$

$$\leqslant \frac{c_{102}(1+\lambda_m^{1/p})}{\lambda_{m+1}} \cdot \int_{\{u:\, u\in N,\ a\leqslant |(I-P_m)u|\leqslant b\}} |(I-P_m)u|\, d\mu(u)$$

where $m = 1, 2, \ldots,\ b \geqslant a \geqslant 0,\ p \geqslant 2$ are arbitrary and c_{102} is a constant (i.e. independent of them). If $h_m = h_m(\operatorname{supp} \mu) > 0$, take $b = h_m$ and $a = h_m/2$. Then

$$A_m = \{u:\ u \in N, a \leqslant |(I-P_m)u| \leqslant b\}$$

must satisfy $\mu(A_m) > 0$ and from (7.11″) we can deduce

$$\frac{h_m^2}{4} \cdot \mu(A_m) \leqslant \frac{c_{102}(1+\lambda_m^{1/p})}{\lambda_{m+1}} h_m \cdot \mu(A_m)\ ,$$

whence for all $m = 1, 2, \ldots$

$$h_m \leqslant 4 c_{102}(1+\lambda_m^{1/p}) \lambda_{m+1}^{-1} \leqslant 4 c'_{102} \lambda_{m+1}^{-1+1/p}$$

which obviously implies

$$\limsup_{m\to\infty} \frac{\log h_m}{\log \lambda_{m+1}} \leqslant -1 + \frac{1}{p}. \tag{7.12}$$

Letting $p \to \infty$ in (7.12) we obtain (7.10′), concluding the proof.

We proceed now to the main result of this Section, namely to the following

THEOREM 2. *Let μ be an invariant (Borel) probability in $\boldsymbol{N}$ and let $h_m = h_m(\operatorname{supp}\mu)$, $m = 1, 2, \ldots$. Then for any $\varepsilon \in (0, \frac{1}{2})$ there exists a constant c_ε (depending also on Ω, ν and $|f|$) such that*

$$|(I - P_m)(u - v)| \leqslant h_m \cdot \exp[-c_\varepsilon \lambda_m^{\frac{1}{2}-\varepsilon}] + c_\varepsilon \lambda_m^{\frac{1}{2}-\varepsilon} |P_m(u - v)| \tag{7.13}$$

whenever $u, v \in \sup\mu$ and $m \geqslant c_\varepsilon$.

In particular the thickness $t_m = t_m(\operatorname{supp}\mu)$ and highness $h_m = h_m(\operatorname{supp}\mu)$ of supp μ *satisfy*

$$t_m \leqslant h_m \cdot \exp[-c_\varepsilon \cdot \lambda_m^{\frac{1}{2}-\varepsilon}] \qquad \text{for } m \geqslant c_\varepsilon, \tag{7.13′}$$

hence

$$\liminf_{m\to\infty} \frac{\log\log h_m t_m^{-1}}{\log \lambda_m} \geqslant \frac{1}{2}. \tag{7.13″}$$

PROOF. Let $\Phi(\cdot)$ be a real functional on $\boldsymbol{N}$ with continuous Frechet differential. (This means, as usual, that

$$\Phi(w + \delta w) - \Phi(w) = (\Phi'(w), \delta w) + O(|\delta w|) \qquad \text{for } |\delta w| \to 0$$

where $w \in \boldsymbol{N}$ is arbitrary and Φ' as $\boldsymbol{N}$-valued function defined on $\boldsymbol{N}$ is continuous). Put $w(\cdot) = S(\cdot)u - S(\cdot)v$ with $u, v \in \operatorname{supp}\mu$. Then $\Phi(w(\cdot))$ is absolutely continuous on $(0, \infty)$ and

$$\frac{d}{dt}\Phi(w) + \nu(Dw, \Phi'(w)) + b(w, u, \Phi'(w)) + b(v, w, \Phi'(w)) = 0 \tag{7.14}$$

a.e. on $(0, \infty)$. Putting

$$\Psi(u, v, w; \Phi) = \nu(Dw, \Phi'(w)) + b(w, u, \Phi'(w)) + b(v, w, \Phi'(w)) \tag{7.14′}$$

and integrating (6.14), we obtain for all $t > 0$

$$\Phi(w(t)) - \Phi(w(0)) = \int_0^t \Psi(u(\tau), v(\tau), w(\tau); \Phi)\, d\tau ,$$

whence

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Psi(S(\tau)u, S(\tau)v, S(\tau)u - S(\tau)v; \Phi)\, d\tau = 0 . \tag{7.14''}$$

Applying Birkhoff's ergodic theorem to the dynamical system $S(t)\otimes$ $\otimes S(t)$ defined in $\operatorname{supp}\mu \times \operatorname{supp}\mu$ by $S(t)\otimes S(t)\{u, v\} = \{S(t)u, S(t)v\}$ and its invariant measure $\mu\otimes\mu$, from (7.14″) we infer as in the proof of the preceding Proposition 2 that

$$\iint_{N\times N} \Psi(u, v, u - v; \Phi)\, d\mu(u)\, d\mu(v) = 0 . \tag{7.15}$$

Let $f\in C^1((-\infty, \infty))$. By the definition (7.14′) we have

$$\Psi(u, v, w, f\circ\Phi) = f'\circ\Phi\cdot\Psi(u, v, w, \Phi)$$

therefore (7.15), applied to $f\circ\Phi$ instead of Φ, yields

$$\iint_{N\times N} f'(\Phi(u - v))\cdot\Psi(u, v, u - v; \Phi)\, d\mu(u)\, d\mu(v) = 0 . \tag{7.15'}$$

Since, in (7.15′), f' can be chosen any function $\in C((-\infty, +\infty))$, one can easily deduce from (7.15)

$$\iint_{\{\{u, v\}: \{u, v\}\in N\times N,\ \Phi(u-v)>a\}} \Psi(u, v, u - v; \Phi)\, d\mu(u)\, d\mu(v) = 0 \tag{7.15''}$$

for all reals a. We shall take Φ equal to the functional

$$\Phi_{m,F}(w) = \tfrac{1}{2}|(I - P_m)w|^2 - \tfrac{1}{2}F(|P_m w|^2) , \tag{7.16}$$

where $F\in C^1([0, \infty))$; then

$$\begin{aligned} \Psi(u, v, w; \Phi_{m,F}) = {} & [\nu\|(I - P_m)u\|^2 + \\ & + b(w, u, (I - P_m)w) + b(v, w, (I - P_m)w)] - \\ & - F'(|P_m w|^2)[\nu\|P_m w\|^2 + b(w, u, P_m w) + b(v, w, P_m w)] \end{aligned} \tag{7.16'}$$

where

$$(7.16'') \quad \begin{aligned} &\nu\|P_m w\|^2 + b(w, u, P_m w) + b(v, w, P_m w) \leqslant \\ &\leqslant \nu\|P_m w\|^2 + c_{103}|P_m w|\cdot\|P_m w\| + |b((1-P_m)w, P_m w, u)| + \\ &+ |b(v, P_m w, (1-P_m)w)| \leqslant \nu\|P_m w\|^2 + c_{103}|P_m w|\cdot\|P_m w\| + \\ &+ c_{104}|(1-P_m)w|\cdot(\|P_m w\| + \|P_m w\|^{\frac{1}{2}}|DP_m w|^{\frac{1}{2}}) \leqslant \\ &\leqslant \|P_m w\|[(\nu\lambda_m^{\frac{1}{2}} + c_{105})|P_m w| + c_{106}(1+\lambda_m^{\frac{1}{4}})|(I-P_m)w|] \end{aligned}$$

where $c_{103} - c_{106}$ are constants (depending only on Ω, ν and $|f|$). Consequently denoting by Ψ_m the function in u, v, w occuring in the first bracket [...] in (7.16′) and by Θ_m that occuring in the second bracket [...] in (7.16′), we shall have

$$(7.16''') \quad \Psi(u, v, w; \Phi_{m,F}) = \Psi_m(u, v, w) + F'(|P_m w|^2)\,\Theta_m(u, v, w)$$

where (by (7.16″))

$$(7.16'''') \quad |\Theta_m(u, v, w)| \leqslant c_{107}(\lambda_m + 1)|w|$$

with a constant c_{107} (depending on Ω, ν and $|f|$) large enough, and $w = u - v$, $u, v \in \operatorname{supp}\mu$. Let now

$$(7.17) \quad F \in C([0, \infty)) \cap C^1((0, \infty))$$

satisfy also

$$(7.17') \quad \limsup_{\xi \to +0} |F'(\xi)|\sqrt{\xi} < \infty\,.$$

Let $r_0 = \sup\{|u|: u \in \operatorname{supp}\mu\}$ and choose a sequence $\{F_p\}_{p=1}^{\infty} \subset C^1([0, \infty))$ such that

$$(7.17'') \quad \begin{cases} F_p(\xi) \to F(\xi)\,, & (\text{for } p \to \infty) \text{ uniformly on } [0, (2r_0)^2]\,, \\ F_p'(\xi) \to F'(\xi)\,, & (\text{for } p \to \infty) \text{ on } (0, (2r_0)^2]\,, \end{cases}$$

and

$$(7.17''') \quad \sup\{|F_p'(\xi)|\sqrt{\xi}: p = 1, 2, \ldots,\ \xi \in (0, (2r_0)^2]\} < \infty\,.$$

For instance define $F_p(0)=F(0)$ and $F'_p(x)=F'(1/p)$ if $0\leqslant x\leqslant 1/p$ and $F'_p(x)=F'(x)$ if $x\geqslant 1/p$. Then, for $p\to\infty$,

$$(7.18)\qquad \varepsilon_p=\sup\{|\Phi_{m,F_p}(u-v)-\Phi_{m,F}(u-v)|:\ u,v\in\operatorname{supp}\mu\}\to 0$$

and

$$(7.18')\qquad \Psi(u,v,u-v;\Phi_{m,F})\to\Psi_m(u,v;F)\qquad \text{on } \operatorname{supp}\mu\times\operatorname{supp}\mu$$

where, by definition we set on $\operatorname{supp}\mu\times\operatorname{supp}\mu$

$$(7.18'')\qquad \Psi_m(u,v;F)=$$

$$=\begin{cases}\Psi_m(u,v,u-v)+\\ \qquad\qquad +F'(|P_m(u-v)|^2)\,\Theta_m(u,v,u-v)\,, & \text{if } |P_m(u-v)|>0\,,\\ \Psi_m(u,v,u-v)\,, & \text{if } |P_m(u-v)|=0\,.\end{cases}$$

Moreover

$$(7.18''')\qquad |\Psi(u,v,u-v;\Phi_{m,F_p})|\leqslant c(m,F)<\infty$$

for all $p=1,2,\ldots$ and $u,v\in\operatorname{supp}\mu$. Put

$$A=\{\{u,v\}:\ \{u,v\}\in\operatorname{supp}\mu\times\operatorname{supp}\mu,\ \Phi_{m,F}(u-v)>a\}$$

and

$$A_p=\{\{u,v\}:\ \{u,v\}\in\operatorname{supp}\mu\times\operatorname{supp}\mu,\ \Phi_{m,F_p}(u-v)>a+\varepsilon_p\}\,.$$

Plainly $A\subset A_p$ $(p=1,2,\ldots)$ and

$$\bigcap_{q=1}^{\infty}\bigcup_{p=q}^{\infty}(A\setminus A_p)=\emptyset\,,$$

hence

$$(7.19)\qquad \lim_{p\to\infty}\mu\otimes\mu(A\setminus A_p)=0\,.$$

Using (7.18′), (7.15″), (7.18‴) and (7.19) (in this order) we obtain

$$\Big|\iint_A \Psi_m(u, v; F)\, d\mu(u)\, d\mu(v)\Big| \leqslant$$

$$\leqslant \limsup_{p\to\infty}\Big|\iint_A \Psi(u, v, u-v; \Phi_{m,F_p})\, d\mu(u)\, d\mu(v)\Big| =$$

$$= \limsup_{p\to\infty}\Big|\iint_A \Psi(u, v, u-v; \Phi_{m,F_p})\, d\mu(u)\, d\mu(v) -$$

$$-\iint_{A_p} \Psi(u, v, u-v; \Phi_{m,F_p})\, d\mu(u)\, d\mu(v)\Big| \leqslant \mathrm{const}\cdot\limsup_{p\to\infty} \mu\otimes\mu(A\setminus A_p) = 0\,,$$

that is

$$\iint\limits_{\{\{u,v\}:\{u,v\}\in \operatorname{supp}\mu\times\operatorname{supp}\mu,\ \Phi_{m,F}(u-v)>0\}} \Psi_m(u, v; F)\, d\mu(u)\, d\mu(v) = 0 \tag{7.20}$$

for all reals a and all functions F satisfying (7.17)-(7.17′).

Let us suppose that for a convenient choice of a function F satisfying the conditions (7.17)-(7.17′), the functionals $\Phi_{m,F}$ and $\Psi_m(u, v; F)$ enjoy the following property

$$\Phi_{m,F}(u, v) > 0 \quad \text{and} \quad u, v \in \operatorname{supp}\mu \;\Rightarrow\; \Psi_m(u, v; F) > 0\,. \tag{7.21}$$

Then the relation (7.20) (with $a = 0$) becomes

$$\mu\otimes\mu(\{\{u, v\}: u, v\in\operatorname{supp}\mu,\ |(I-P_m)(u-v)|^2 > F(|P_m(u-v)|^2)\}) = 0\,. \tag{7.20′}$$

If for certain $u_0, v_0 \in \operatorname{supp}\mu$ we have

$$|(I-P_m)(u_0-v_0)|^2 > F(|P_m(u_0-v_0)|^2)$$

then the same relation will hold for all $\{u, v\}\in \boldsymbol{U}_0\times\boldsymbol{V}_0$, where $\boldsymbol{U}_0$ and $\boldsymbol{V}_0$ are some open balls (in $\boldsymbol{N}$) centred in u_0, resp. v_0. By (7.20′),

$$\mu(\boldsymbol{U}_0)\cdot\mu(\boldsymbol{V}_0) = \mu\otimes\mu(\boldsymbol{U}_0\times\boldsymbol{V}_0) = 0\,,$$

thus at least one of u_0, v_0 does not belong to $\operatorname{supp}\mu$: Contradiction! Consequently we can conclude that if F satisfies (7.17)-(7.17′) and if (7.21) is also valid, then

$$(7.22)\qquad |(I-P_m)(u-v)|^2\leqslant F\big(|P_m(u-v)|^2\big)\qquad \text{for all } u, v\in \operatorname{supp}\mu\,.$$

It remains to find a convenient choice for F. To this aim note first the relations

$$\begin{aligned}
(7.23)\qquad & \Psi_m(u, v, w) = \nu\|(I-P_m)w\|^2 + b\big(w, u, (I-P_m)w\big) + \\
& + b\big(v, w, (I-P_m)w\big) = \nu\|(I-P_m)w\|^2 + b\big((I-P_m)w, u, (I-P_m)w\big) + \\
& + b\big(P_m w, u, (I-P_m)w\big) + b\big(v, P_m w, (I-P_m)w\big)\geqslant \\
& \geqslant \nu\|(I-P_m)w\|^2 - \sqrt{2}\|u\|\cdot|(I-P_m)w|\cdot\|(I-P_m)w\| - \\
& - \sqrt{2}|P_m w|^{\frac12}\cdot\|P_m w\|^{\frac12}\|u\|\,|(I-P_m)w|^{\frac12}\cdot\|(I-P_m)w\|^{\frac12} - \\
& - |v|_p\cdot|\operatorname{grad}(P_m w)|_{2p/(p-2)}\cdot|(I-P_m)w|\geqslant \\
& \geqslant \nu\|(I-P_m)w\|^2 - c_{108}|(I-P_m)w|\cdot\|(I-P_m)w\| - \\
& - c_{109}|P_m w|\,\lambda_m^{\frac12}\lambda_{m+1}^{-\frac12}\|(I-P_m)w\| - c_{110}\|v\|\cdot|D^{\frac12+1/p}P_m w|\cdot \\
& \cdot|(I-P_m)w|\geqslant \nu(I-P_m)w\|^2 - c_{108}|(I-P_m)w|\cdot\|(I-P_m)w\| - \\
& - c_{109}|P_m w|\cdot\|(I-P_m)w\| - c_{110}\lambda_m^{\frac12+1/p}|P_m w|\cdot\lambda_{m+1}^{-\frac12}\|(I-P_m)w\|\geqslant \\
& \geqslant \nu\|(I-P_m)w\|\big[\|(I-P_m)w\| - c_{108}|(I-P_m)w| - c'_{110}(\lambda_{m+1}^{1/p}+1)|P_m w|\big]\geqslant \\
& \qquad\qquad \geqslant \nu\|(I-P_m)w\|\cdot\big[(\lambda_{m+1}^{\frac12} - c_{108})\cdot|(I-P_m)w| - c'_{110}\lambda_m^{1/p}|P_m w|\big]
\end{aligned}$$

and similarly

$$\begin{aligned}
(7.23')\qquad & \Theta_m(u, v, w) = \nu\|P_m w\|^2 + b(w, u, P_m w) + b(v, w, P_m w)\leqslant \\
& \leqslant \nu\|P_m w\|^2 + \sqrt{2}\|u\|\cdot|P_m w|\cdot\|P_m w\| + \big|b\big((I-P_m)w, P_m w, u\big)\big| + \\
& + \big|b\big(v, (I-P_m)w, P_m w\big)\big|\leqslant \nu\|P_m w\|^2 + c_{108}|P_m w|\cdot\|P_m w\| + \\
& + |(I-P_m)w|\cdot|\operatorname{grad}(P_m w)|_{2p/(p-2)}\big(|u|_p + |v|_p\big)\leqslant \\
& \leqslant \nu\|P_m w\|^2 + c_{108}|P_m w|\cdot\|P_m w\| + c_{110}|(I-P_m)w|\cdot|D^{\frac12+1/p}P_m w|\leqslant \\
& \leqslant \nu\|P_m w\|^2 + c_{108}|P_m w|\cdot\|P_m w\| + c_{110}|(I-P_m)w|\cdot\lambda_m^{1/p}\|P_m w\|\leqslant \\
& \qquad\qquad \leqslant \nu\|P_m w\|\big[(\lambda_m^{\frac12} + c_{108})|P_m w| + c'_{110}\lambda_m^{1/p}|(I-P_m)w|\big]\,,
\end{aligned}$$

where we used (2.4), (2.3′) and (2.15′). In (7.23)-(7.23′), $m = 1, 2, \ldots$ and $p\in(2, \infty)$ are arbitrary, $c_{108} - c_{109}$ are suitable constants depending on Ω, ν and $|f|$, while $c_{110} - c'_{110}$ depend also on p, the last one being chosen sufficiently large. Choosing a suitable large constant c_{111} we have (see (7.18″) and (7.23)-(7.23′))

$$\begin{aligned}(7.24)\qquad \Psi_m(u, v;\, F) \geqslant \nu\|(I-P_m)w\|\cdot\big[(\lambda_m^{\frac12}-c_{111})|(I-P_m)w| -\\ - c_{111}\lambda_m^{1/p}|P_m w|\big] - F'(|P_m w|^2)\,\nu\|P_m w\|\cdot\big[(\lambda_m^{\frac12} +\\ + c_{111})|P_m w| + c_{111}\lambda_m^{1/p}|(I-P_m)w|\big]\end{aligned}$$

whenever

$$(7.24')\qquad |P_m w| > 0\,, \qquad w = u - v \quad \text{and} \quad u, v \in \operatorname{supp}\mu\,.$$

Note that in (7.24)-(7.24′), $m = 1, 2, \ldots$ and $p \in (2, \infty)$ are still arbitrary (of course with c_{111} depending on p).

Consider now the solution $\tilde{\omega}(\sigma)$ of the differential equation

$$(7.25)\qquad \frac{d\omega}{d\sigma} = \frac{(\lambda_m^{\frac12} - c)\,\omega - c\lambda_m^{1/p}\sigma}{(\lambda_m^{\frac12} + c)\,\sigma + c\lambda_m^{1/p}\omega}\,, \qquad (\text{where } c = c_{111})$$

in the interval $(0, \sigma_m]$ where

$$(7.25')\qquad \sigma_m = 2h_m(\lambda^{\frac12} - c)\,c^{-1}\lambda_m^{-1/p}, \qquad h_m = h_m(\operatorname{supp}\mu)\,,$$

determined by the initial condition

$$(7.25'')\qquad \tilde{\omega}(\sigma_m) = 2h_m\,.$$

(We take already m sufficiently large in order that $\lambda_m > c^2$ and $\sigma_m > 0$.) It is easy to verify that actually $\tilde{\omega}$ extends by continuity on $[0, \sigma_m]$ and that this extension, still denoted by $\tilde{\omega}(\cdot)$, belongs to $C^1([0, \sigma_m])$. Finally we shall put

$$(7.26)\qquad F(\xi) = \begin{cases} \tilde{\omega}(\sqrt{\xi})^2\,, & \text{for } \xi \in [0, \sigma_m^2] \\ (2h_m)^2, & \text{for } \xi \in [\sigma_m^2, \infty)\,. \end{cases}$$

This function will satisfy the conditions (7.17)-(7.17′). We shall now prove that the relation (7.21) is also valid for this function F (i.e. defined by (7.26)). To this purpose, observe first that if for a point $\{\sigma, \omega\} \in R^2$ we have $0 \leqslant \sigma \leqslant \sigma_m$ and $\omega > \tilde{\omega}(\sigma)$, then

$$(\lambda_m^{\frac12} - c)\,\omega - c\lambda_m^{1/p}\sigma > (\lambda_m^{\frac12} - c)\,\tilde{\omega}(\sigma) - c\lambda_m^{1/p}\sigma \geqslant 0\;;$$

therefore if

$$\Phi_{m,F}(w) = |(I - P_m)w|^2 - F\big(|P_m w|^2\big) > 0 \quad \text{and} \quad |P_m w| \leqslant \sigma_m \tag{7.27}$$

where $w = u - v$ and $u, v \in \operatorname{supp} \mu$, we have

$$(\lambda_m^{\frac{1}{2}} - c)|(I - P_m)w| - c\lambda_m^{1/p}|Pw| > 0\,,$$

so that, if moreover

$$|P_m w| > 0\,, \tag{7.27$'$}$$

the relation (7.24) will yield

$$\begin{aligned}
\Psi_m(u, v; F) &\geqslant \nu\lambda_m^{\frac{1}{2}}|(I - P_m)w|\cdot \\
&\cdot[(\lambda_m^{\frac{1}{2}} - c)|(I - P_m)w| - c\lambda_m^{1/p}|P_m w|] - F'\big(|P_m w|^2\big)\nu\lambda_m^{\frac{1}{2}}|P_m w|\cdot \\
&\cdot[(\lambda_m^{\frac{1}{2}} + c)|P_m w| + c\lambda_m^{1/p}|(I - P_m)w|] = \nu\lambda_m^{\frac{1}{2}}[\,|(I - P_m)w| - \tilde{\omega}\big(|P_m w|\big)]\cdot \\
&\cdot[(\lambda_m^{\frac{1}{2}} - c)|(I - P_m)w| - c\lambda_m^{1/p}|P_m w|] > 0\,.
\end{aligned} \tag{7.27$''$}$$

If instead (7.27$'$) we have

$$|P_m w| = 0\,, \tag{7.27$'''$}$$

then, by (7.23),

$$\Psi_m(u, v; F) \geqslant \nu\lambda_m^{\frac{1}{2}}(\lambda_m^{\frac{1}{2}} - c)|(I - P_m)w|^2 > 0 \tag{7.27$''''$}$$

(since otherwise $w = 0$, so that $\Phi_{m,F}(w) = -F(0)^2 < 0$). Finally since $|P_m w| > \sigma_m$ implies

$$\begin{aligned}
\Phi_{m,F}(w) = |(I - P_m)w|^2 - (2h_m)^2 &\leqslant \big(|(I - P_m)u| + |(I - P_m)v|\big)^2 - \\
&- (2h_m)^2 \leqslant (2h_m)^2 - (2h_m)^2 = 0\,,
\end{aligned}$$

we can conclude that the last condition $|P_m w| \leqslant \sigma_m$ in (7.27) is redundant; therefore the verification of (7.21) is achieved.

By (7.22) and (7.26) it results

$$|(I - P_m)(u - v)| \leqslant \tilde{\omega}\big(|P_m(u - v)|\big) \tag{7.28}$$

for

$$\text{(7.28}'\text{)} \qquad \text{all } u, v \in \operatorname{supp} \mu \, .$$

An elementary integration of (7.25)-(7.25″) yields (with some sufficiently large constant c_{112} depending on p, Ω, ν and $|f|$ (but, of course independent of m))

$$\text{(7.28}''\text{)} \qquad \tilde{\omega}(0) \leqslant 2h_m \exp[-c_{112}\lambda_m^{\frac{1}{2}-1/p}] \quad \text{and} \quad \tilde{\omega}(\sigma) \leqslant \tilde{\omega}(0) + c_{112}\lambda_m^{\frac{1}{2}-1/p}\sigma$$

for all sufficiently large m.

Taking $\varepsilon = 1/p$ and choosing the constant c_ε large enough, from (7.28)-(7.28″) we deduce readily (7.13), finishing the proof of Theorem 2.

REMARKS. 1°. The relation (7.13′) was, in the particular case $\varepsilon = 1/4$, already given together with a sketch of its proof (actually almost the present one) in FOIAS [1], § 8; the only improvement in our present proof lies in the use of (2.3′) and (2.15) (instead of (2.1) and (2.3)) in the relations (7.23)-(7.23′). However, as it will be seen in the next Section (see Sec. 7.3) this will sensibly ameliorate our previous limitation of the ε-entropy of the support of an invariant probability in $\boldsymbol{N}$.

2°. Since $\lambda_m \to \infty$ for $m \to \infty$, the relation (7.13′) shows that the support of an invariant (Borel) probability in $\boldsymbol{N}$ has, for $m \to \infty$, the tendency to take the shape of a slice once it is regarded in $\boldsymbol{N}$ considered as the Cartesian product $P_m\boldsymbol{N} \times (I - P_m)\boldsymbol{N}$.

c) It is plain that any efficient estimation on the eigenvalues $\{\lambda_m\}_{m=1}^{\infty}$ of D will provide a more useful form for (7.13)-(7.13′). Such an estimation is the following

$$\text{(7.29)} \qquad c_{113}m \leqslant \lambda_m \leqslant c'_{113}m \qquad \text{for all } m = 1, 2, \dots ,$$

where $c_{113} - c'_{113}$ are some suitable constants (depending only on Ω). Let us recall that the relation (7.29), as well as the whole content of § 7, concerns the case when Ω is a bounded domain (with a C^2-boundary) in $\boldsymbol{R}^2$, i.e. the space dimension n is $= 2$.

Proof of (7.29) (*). We start by observing that, by a Sobolev's type inequality (see Agmon [1], § 13, Lemma 13.2), any $u \in \boldsymbol{H}^2$ can be considered as a continuous function from $\bar{\Omega}$ to R^2 satisfying

$$(7.30) \qquad |u(x)| \leqslant c_{114} \|u\|_2^{\frac{1}{2}} |u|^{\frac{1}{2}} \qquad \text{for all } x \in \bar{\Omega}$$

where c_{114} is a constant depending only on Ω Taking into account (2.14), (7.30) can be given the form

$$(7.30') \qquad |u(x)| \leqslant c_{114} c_2^{-1} |Du|^{\frac{1}{2}} |u|^{\frac{1}{2}} \qquad \text{for all } x \in \bar{\Omega},\ u \in \boldsymbol{D}_D .$$

In particular for $u = \sum_{j=1}^{m} a_j w_j$ where $\{w_j\}_{j=1}^{\infty}$ is the orthogonal basis, formed by eigenvectors of D, introduced in Sec. 2.1, and $\{a_1, a_2, \ldots, a_m\} \in R^m$ is arbitrary, we shall have

$$\left|\sum_{j=1}^{m} a_j w_j(x)\right|^2 \leqslant c_{115} \left|D \sum_{j=1}^{m} a_j w_j\right|^{\frac{1}{2}} \left|\sum_{j=1}^{m} a_j w_j\right|^{\frac{1}{2}} \leqslant c_{115}\, \lambda_m \left|\sum_{j=1}^{m} a_j w_j\right|^2 = c_{115}\, \lambda_m \sum_{j=1}^{m} a_m^2 .$$

Since this is valid for all $\{a_1, \ldots, a_m\} \in R^m$ we must have

$$\sum_{j=1}^{m} |w(x)|^2 \leqslant 2 c_{115} \lambda_m$$

and this holds for all $x \in \Omega$ and $m = 1, 2, \ldots$. Integrating over Ω, we finally obtain

$$m = \sum_{j=1}^{m} |w_j|^2 \leqslant 2 c_{115} \operatorname{meas}(\Omega) \cdot \lambda_m = c_{116} \cdot \lambda_m$$

which yields the first relation (7.29) with $c_{113} = c_{116}^{-1}$:

To prove the second relation (7.29), we consider an open (again with a C^2-boundary) domain $\tilde{\Omega} \subset \Omega$. All the entities $\boldsymbol{N}, \boldsymbol{N}^l, D, \lambda_m$, etc. will be denoted by $\tilde{\boldsymbol{N}}, \tilde{\boldsymbol{N}}^l, \tilde{D}, \tilde{\lambda}_m$, etc. if they refer to $\tilde{\Omega}$, instead Ω.

(*) Actually (7.29) results also from some deep and general asymptotic spectral formulae for elliptic systems; as sample see Grisvard [1].

With this convention we shall prove first

$$\lambda_m \leqslant \tilde{\lambda}_m \qquad \text{for all } m = 1, 2, \ldots . \tag{7.31}$$

To this aim observe that

$$\left|\tilde{D}^{\frac{1}{2}} \sum_{j=1}^{m} a_j \tilde{w}_j\right| = \left|D^{\frac{1}{2}} \sum_{j=1}^{m} a_j \tilde{w}_j\right|$$

(since $\tilde{N}^1$ can be considered a subspace of N^1), thus

$$\left|D^{\frac{1}{2}} \sum_{j=1}^{m} a_j \tilde{w}_j\right|^2 \leqslant \tilde{\lambda}_m \left|\sum_{j=1}^{m} a_j \tilde{w}_j\right|^2 \tag{7.32}$$

where all norms are taken in N, and $\{a_1, \ldots, a_m\} \in R^m$ is arbitrary. Let $\lambda_{m'} \leqslant \tilde{\lambda}_m < \lambda_{m'+1}$. Then from (7.32) we infer

$$(\lambda_{m'+1} - \tilde{\lambda}_m)\left|(I - P_{m'}) \sum_{j=1}^{m} a_j \tilde{w}_j\right|^2 \leqslant (\tilde{\lambda}_m - \lambda_1)\left|P_{m'} \sum_{j=1}^{m} a_j \tilde{w}_j\right|^2,$$

so that if $P_{m'} \sum_{j=1}^{m} a_j w_j = 0$ we must have also $(I - P_{m'}) \sum_{j=1}^{m} a_j w_j = 0$ i.e. $\sum_{j=1}^{m} a_j w_j = 0$, that is, $P_{m'}|\tilde{P}_m N$ is injective. Therefore the dimension of $P_{m'} N$ is $\geqslant$ than that of $\tilde{P}_m N$; this means that $m' \geqslant m$ hence $\lambda_m \leqslant \lambda_{m'} \leqslant \tilde{\lambda}_m$, which establishes (7.31). To complete the proof of (7.29), in virtue of (7.31) it is sufficient to show that $\tilde{\lambda}_m = 0(m)$ (for $m \to \infty$) for a suitable choice of $\tilde{\Omega}$. Since we can suppose that the origin $\{0, 0\}$ belongs to $\tilde{\Omega}$ we can henceforth take $\tilde{\Omega} = \{x : x = \{x_1, x_2\} \in R^2, \varrho_1 < |x| < \varrho_2\}$ with some $\varrho_2 > \varrho_1 > 0$. Using the invariance to rotations of $\tilde{\Omega}$ one can obtain, after some tedious computations, that $\{\tilde{\lambda}_m\}_{m=1}^{\infty}$ are precisely the eigenvalues λ in the following boundary problem for a differential equation

$$\begin{cases} \varrho^2 v''(\varrho) + \varrho v'(\varrho) - v(\varrho) + \lambda v(\varrho) = 0 , \qquad \text{on } [\varrho_1, \varrho_2] \\ \text{and} \\ v(\varrho_1) = v(\varrho_2) = 0; \end{cases} \tag{7.33}$$

therefore $\tilde{\lambda}_m = 0(m)$ for $m \to \infty$ (see for instance Sansone [1], Ch. IV. § 7). In virtue of (6.31), the last conclusion achieves the proof of (7.29), Introducing the estimation (7.29) in (7.13)-(7.13‴) we can supplement Theorem 2, in Sec. 7.2.*b*), with the following

THEOREM 2_{bis}. *Under the assumptions of Theorem 2, Sec. 7.2.b), we have, for any* $\varepsilon \in (0, \frac{1}{2})$,

$$(7.13_{\text{bis}}) \qquad |(I - P_m)(u - v)| \leqslant \exp[-c'_\varepsilon m^{\frac{1}{2}-\varepsilon}] + c'_\varepsilon m^{\frac{1}{2}-\varepsilon}|P_m(u - v)|$$

whenever $u, v \in \operatorname{supp}\mu$ *and* $m \geqslant c'_\varepsilon$, *where* c'_ε *is a convenient large constant (depending on* ε, Ω, ν *and* $|f|$); *also*

$$(7.13''_{\text{bis}}) \qquad \liminf_{m\to\infty} \frac{\log\log h_m t_m^{-1}}{\log m} \geqslant \frac{1}{2}\,.$$

Though the proof of Theorem 2_{bis} is straightforward (and therefore will be omitted), in the sequel we shall use Theorem 2_{bis} instead of Theorem 2.

2. One of the main questions concerning the invariant (Borel) probabilities in N is that of the dimension of their supports, that is if these are of finite dimension (i.e. homeomorphic to compact subsets of some R^m)? This question was raised in an explicit manner by Prodi [3] in the early sixthies. It seems to be one of the most deep open problems in our approach which though of a purely mathematical character has also an important interpretation in the theory of turbulence (see the next § 8). In this Section we present our contribution in the study of this question. It seems convenient to use for this purpose the notion ε-entropy of Kolmogorov (Kolmogorov-Tihomirov [1], § 1).

Let us recall this notion. Let A be a totally bounded metric space. For any $\varepsilon \in (0, \frac{1}{2})$, let $\mathfrak{N}(\varepsilon; A)$ denote the smallest number of subsets $\subset A$ of diameter $\leqslant 2\varepsilon$ covering A. Then the function

$$(7.34) \qquad \mathfrak{H}(\varepsilon; A) = \log \mathfrak{N}(\varepsilon; A)\,,$$

in ε, is the ε-entropy of A (*). If A is a bounded set in R^m, $m = 1, 2, \dots$, (endowed with the usual metric of R^m), then a rough

(*) Actually the logarithm is taken usually in the basis 2. For our purposes it is more convenient to work in the natural basis *e*.

elementary estimate is

$$\mathfrak{H}(\varepsilon; A) \leqslant m\left(c_A + \frac{1}{2}\cdot\log m + \log\frac{1}{\varepsilon}\right), \qquad 0 < \varepsilon \leqslant \frac{1}{2}; \tag{7.34'}$$

conversely it is a well known fact that if for an arbitrary A we have

$$\mathfrak{H}(\varepsilon; A) \leqslant c'_A \log\frac{1}{\varepsilon}, \qquad \text{for } \varepsilon \to 0 \tag{7.34''}$$

then A is homeomorphic with a bounded set in some R^m.

It is plain that (7.34″) implies

$$\limsup_{\varepsilon \to +0} \frac{\log \mathfrak{H}(\varepsilon; A)}{\log\log 1/\varepsilon} \leqslant 1. \tag{7.34'''}$$

Therefore an estimation of

$$d(A) = \limsup_{\varepsilon \to +0} \frac{\log \mathfrak{H}(\varepsilon; A)}{\log\log 1/\varepsilon} \tag{7.35}$$

will provide some knowledge on how for or near is A to be of finite dimension (*).

After these short preliminaries we can state the following

THEOREM 3. *Let μ be an invariant (Borel) probability in $\boldsymbol{N}$. Then*

$$d(\operatorname{supp}\mu) \leqslant 3 \tag{7.36}$$

where supp μ *is endowed with the metric of* $\boldsymbol{N}$.

(*) For instance if A is the subset of $C([0, 1])$ endowed with the usual supremum metric, formed by:

Case (a): All functions φ which can be extended to analytic functions on $\{z: z \in C, |z - \frac{1}{2}| < 1\}$ and which satisfy $\max\limits_{0 \leqslant x \leqslant 1} |\varphi(x)| \leqslant 1$.

Case (b): All functions $\varphi \in C^p([0, 1])$ satisfying $\max\limits_{0 \leqslant x \leqslant 1} |\varphi^{(p)}(x)| \leqslant 1$,

then $d(A) = 2$ in Case (a), and $d(A) = \infty$ in Case (b) (see Kolmogorov-Tihomirov [1], § 3).

PROOF. Let $\delta, \varepsilon \in (0, \frac{1}{2})$ and let m be the first integer satisfying

$$m \geqslant \max\{c'_\delta, (4c'_\delta)^{-2/(1-2\delta)}\} \quad \text{and} \quad 4\exp[-c'_\delta m^{\frac{1}{2}-\delta}] \leqslant \varepsilon\,. \tag{7.37}$$

Set

$$\eta = \frac{1}{c'_\delta m^{\frac{1}{2}-\delta}} \exp[-c'_\delta m^{\frac{1}{2}-\delta}] \tag{7.37'}$$

and let d and N be the first integers satisfying

$$\sqrt{m}\, d^{-1} \leqslant \eta \quad \text{and} \quad N \geqslant \sup\{|u| : u \in \operatorname{supp} \mu\}\,. \tag{7.37''}$$

For $i_1, i_2, \ldots, i_m \in \{0, \pm 1/d, \pm 2/d, \ldots, \pm Nd/d\}$, let $\boldsymbol{q}(i_1, \ldots, i_m)$ denote the hypercube

$$\left\{u : u = \sum_{j=1}^{m} \xi_j w_j \in P_m \boldsymbol{N},\ i_j \leqslant \xi_j < i_j + 1,\ j = 1, 2, \ldots, m\right\}. \tag{7.38}$$

Set $\boldsymbol{p}(i_1, \ldots, i_m) = \boldsymbol{q}(i_1, \ldots, i_m) \cap P_m(\operatorname{supp} \mu)$, and if $\boldsymbol{p}(i_1 \ldots i_m) \neq \emptyset$ choose $u(i_1, i_2, \ldots, i_m) \in \operatorname{supp} \mu$ such that $P_m u(i_1, \ldots, i_m) \in \boldsymbol{p}(i_1, i_2, \ldots, i_m)$. Let $\boldsymbol{P}(i_1, \ldots, i_m)$ denote

$$\left\{u : u \in \operatorname{supp} \mu,\ P_m u \in \boldsymbol{p}(i_1, \ldots, i_m), |(I - P_m)(u - u(i_1, \ldots, i_m))| \leqslant \frac{\varepsilon}{2}\right\}. \tag{7.38'}$$

We shall show now that

$$\text{diameter of } \boldsymbol{P}\ (i_1, i_2, \ldots, i_m) \leqslant 2\varepsilon\,, \tag{7.39}$$

$$\operatorname{supp} \mu = \bigcup_{i_1, i_2, \ldots, i_m} \boldsymbol{P}(i_1, \ldots, i_m)\,. \tag{7.39'}$$

To this aim observe that if $u, v \in \boldsymbol{P}(i_1, \ldots, i_m)$ then

$$|P_m(u - v)| \leqslant \sqrt{m}\, d^{-1} \leqslant \eta$$

since the diameter of $\boldsymbol{q}(i_1, \ldots, i_m)$ is $= \sqrt{m}\, d^{-1}$. Consequently

$$\begin{aligned}|u - v| &\leqslant |P_m(u - v)| + |(I - P_m)(u - v)| \leqslant \eta + \\ &+ |(I - P_m)(u - u(i_1, \ldots, i_m))| + |(I - P_m)(u(i_1, \ldots, i_m) - v)| \leqslant \eta + \varepsilon \leqslant 2\varepsilon\,,\end{aligned}$$

since $\eta \leqslant \varepsilon$, by (7.37). This establishes (7.39). Let now $u \in \operatorname{supp} \mu$. Then $P_m u$ belongs to some $\boldsymbol{p}(i_1, i_2, \ldots, i_m)$ so that, on account of (7.13_{bis}) and (7.37)-(7.37′) we shall have

$$\begin{aligned}|(I - P_m)(u - u(i_1, \ldots, i_m))| &\leqslant \exp[-c'_\delta m^{\frac12 - \delta}] + \\ &+ c'_\delta m^{\frac12-\delta} \cdot |P_m(u - u(i_1, \ldots, i_m))| \leqslant \exp[-c'_\delta m^{\frac12-\delta}] + \\ &+ c'_\delta m^{\frac12-\delta} \cdot \eta = 2\exp[-c'_\delta m^{\frac12-\delta}] \leqslant \frac{\varepsilon}{2}\end{aligned}$$

thus $u \in \boldsymbol{P}(i_1, \ldots, i_m)$. This concludes the proof of (7.39′).

In virtue of (7.39)-(7.39′), $\mathfrak{N}(\varepsilon, \operatorname{supp}\mu)$ is $\leqslant$ than the number of the sets $\boldsymbol{P}(i_1, \ldots, i_m)$, that is $\leqslant (2Nd + 1)^m$, thus

$$\begin{aligned}\mathfrak{N}(\varepsilon, \operatorname{supp}\mu) &\leqslant [2N(\eta^{-1}\sqrt{m} + 1) + 1]^m \leqslant \\ &\leqslant (3N\sqrt{m})^m \eta^{-m} \leqslant (3N\sqrt{m})^m (c'_\delta m^{\frac12-\delta} \cdot \exp[c'_\delta m^{\frac12-\delta}])^m\end{aligned}$$

whence

$$\mathfrak{N}(\varepsilon, \operatorname{supp}\mu) \leqslant (3Nc'_\delta)^m m^m \exp[c'_\delta m^{\frac32-\delta}] \tag{7.40}$$

for all $\varepsilon, \delta \in (0, \frac12)$ and all m satisfying (7.37). If

$$\left(\frac{1}{c'_\delta} \cdot \log\frac{4}{\varepsilon}\right)^{2/(1-2\delta)} \geqslant \max\{c'_\delta, (4c'_\delta)^{-2/(1-2\delta)}\} + 2 \tag{7.40′}$$

(i.e. if ε is sufficient small) then necessarily we have

$$4\exp[-c'_\delta(m-1)^{\frac12-\delta}] > \varepsilon\,,$$

whence

$$m < \left(\frac{1}{c'_\delta}\log\frac{4}{\varepsilon}\right)^{2/(1-2\delta)} + 1 \leqslant 2\left(\frac{1}{c'_\delta}\log\frac{4}{\varepsilon}\right)^{2/(1-2\delta)}.$$

Introducing this estimation in (7.40), we obtain

$$\begin{aligned}\mathfrak{H}(\varepsilon, \operatorname{supp}\mu) &= \log \mathfrak{N}(\varepsilon; \operatorname{supp}\mu) \leqslant m(c_{117} + \log m) + c'_\delta m^{\frac32-\delta} \leqslant \\ &\leqslant \left(c'_{117} + c_{118}\log\log\frac{1}{\varepsilon}\right)\left(\log\frac{4}{\varepsilon}\right)^{2/(1-2\delta)} + c'_{118}\left(\log\frac{4}{\varepsilon}\right)^{(3-2\delta)/(1-2\delta)} \leqslant \\ &\leqslant c_{119} + c'_{119}\left(\log\frac{1}{\varepsilon}\right)^{(3-2\delta)/(1-2\delta)} \leqslant c''_{119}\left(\log\frac{1}{\varepsilon}\right)^{3/(1-2\delta)}\end{aligned}$$

for all sufficiently small $\varepsilon > 0$ satisfying (7.40′). Here $c_{117} - c''_{119}$ are sufficiently large constants depending on Ω, ν, $|f|$ and δ. Therefore

$$\limsup_{\varepsilon \to +0} \frac{\log \mathfrak{H}(\varepsilon, \operatorname{supp} \mu)}{\log \log 1/\varepsilon} \leqslant \frac{3}{1-2\delta}.$$

Finally letting $\delta \to +0$ in this last relation we arrive to (7.36), concluding the proof of Theorem 3.

REMARKS. 1°. Our previous result on $\mathfrak{H}(\varepsilon; \operatorname{supp} \mu)$ announced in Foias [1], § 8, was

$$\mathfrak{H}(\varepsilon, \operatorname{supp} \mu) \leqslant c_{120} \left(\log \frac{1}{\varepsilon}\right)^5 \quad \text{for } \varepsilon \to +0. \tag{7.41}$$

It is clear that our result (7.36) implies

$$\mathfrak{H}(\varepsilon; \operatorname{supp} \mu) \leqslant c'_{120}(\alpha) \cdot \left(\log \frac{1}{\varepsilon}\right)^\alpha \quad \text{for } \varepsilon \to +0, \tag{7.41′}$$

where α can be any number > 3; obviously (7.41′) is much stronger than (7.41) (which corresponds to $\alpha = 5$). Thus our estimation (7.36) improves our former result (7.41).

2°. One should observe that (7.36) is independent (at least explicitly) of the size of Ω, ν and $|f|$, thus independent of the so called Reynolds number. For physical reasons (see for instance Landau-Lifshitz [1], § 19) any definitive result concerning Navier-Stokes equations must depend in a certain sense of this Reynolds number; more explicitely the dependence on Ω, ν and $|f|$ must be visible (or at least possible to evaluate) in the definitive result. From this point of view, for instance the results (6.4)-(6.4′) have more definitive feature than (7.13$''_{\text{bis}}$) or (7.36), since it is easy to verify that the constants $c_{85} - c'_{85}$ are increasing in $1/\nu$, thus have a behaviour of the same kind as the Reynolds number. The above discussion justifies the hope that (7.36) can be substantially improved.

3°. It is worth to remark that if

$$\boldsymbol{A}_{p,q} = \{u : u \in \boldsymbol{D}_{D^p}, |D^p u| \leqslant q\} \tag{7.42}$$

(where $p, q = 1, 2, \ldots$), then

$$d(\boldsymbol{A}_{pq}) = \infty\ . \tag{7.42'}$$

Indeed, in virtue of (7.29) we have

$$\boldsymbol{A}_{p,q} \supset \boldsymbol{B} = \left\{u\colon u \in \boldsymbol{N}, \sum_{m=1}^{\infty} m^{2p} |(u, w_m)|^2 \leqslant c_{121}^2\right\}$$

where c_{121} is a convenient small constant (namely $c_{121} = q\cdot(c_{113})^{-1}$). So it suffices to show that $d(\boldsymbol{B}) = \infty$: For a fixed m let $k(m)$ denote the greatest integer $\leqslant c_{121}(3\varepsilon m^{p+\frac{1}{2}})^{-1}$, where $\varepsilon \in (0, \frac{1}{2})$. Then the set

$$\begin{aligned} \boldsymbol{B}(m) &= \\ &= \left\{u\colon u = \sum_{j=1}^{m} k_j w_j \text{ where } k_j = 0, \pm 3\varepsilon, \pm 2\cdot 3\varepsilon, \ldots, \pm k(m)\cdot 3\varepsilon\right\} \end{aligned} \tag{7.42''}$$

is contained in $\boldsymbol{B}$ and the distance (in $\boldsymbol{N}$) between any two distinct points of $\boldsymbol{B}(m)$ is $\geqslant 3\varepsilon > 2\varepsilon$. Thus $\mathfrak{N}(\varepsilon, \boldsymbol{B})$ must be $\geqslant$ than the cardinal of $\boldsymbol{B}(m)$ which is $(2k(m) + 1)^m$. Suppose now that

$$m^{p+\frac{1}{2}} \geqslant \varepsilon^{-\frac{1}{2}} > (m-1)^{p+\frac{1}{2}}\,.$$

Then $k(m) \geqslant c_{121}((m-1)/m)^{p+\frac{1}{2}}(3\varepsilon^{\frac{1}{2}})^{-1} - 1$, whence

$$\mathfrak{N}(\varepsilon, \boldsymbol{B}) \geqslant (2c'_{121}(3\varepsilon^{\frac{1}{2}})^{-1} - 1)^m \geqslant c''_{121}\left(\frac{1}{\varepsilon}\right)^{m/2} \geqslant c'''_{121}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}(1/\varepsilon)^{1/(p+\frac{1}{2})}},$$

so that $\mathfrak{H}(\varepsilon, \boldsymbol{B}) \geqslant \frac{1}{2}(1/\varepsilon)^{1/(2p+1)}\cdot\log 1/\varepsilon + c''''_{121}$. It results

$$\log \mathfrak{H}(\varepsilon, \boldsymbol{B}) \geqslant \frac{1}{2p+1}\log\frac{1}{\varepsilon}\,.$$

Therefore $d(\boldsymbol{B}) = \infty$. This finishes the proof of (7.42').

In virtue of (7.42') we can say that $\boldsymbol{A}_{pq} \setminus \operatorname{supp}\mu$ is « much more massive » than $\operatorname{supp}\mu$. Indeed it is plain that (7.42') and (7.36) imply

$$d(\boldsymbol{A}_{pq} \setminus \operatorname{supp}\mu) = \infty \tag{7.43}$$

for any $p, q = 1, 2, \ldots$ and any invariant probability μ on $\boldsymbol{N}$.

4°. Perhaps it is also useful to show that there is no hope to exploit in a better way the relation (7.13_{bis}) aiming at the improving of (7.36). This results readily from the following fact: *If* $1 > \alpha > 0$, $\beta, \gamma > 0$ *and*

$$(7.44) \qquad \boldsymbol{T}_{\alpha,\beta,\gamma} = \left\{u : u \in \boldsymbol{N},\ \sum_{m=1}^{\infty} \exp\,[2\beta m^{\alpha}]\,|(u, w_m)|^2 \leqslant \gamma^2\right\}$$

then

$$(7.44') \qquad |(I - P_m)(u - v)| \leqslant c(\alpha, \beta, \gamma) \exp\left[-\frac{\beta}{2} m^{\alpha}\right]$$

for all $u, v \in \boldsymbol{T}_{\alpha,\beta,\gamma}$, $m = 1, 2, \ldots$, *and*

$$(7.44'') \qquad \mathrm{d}(\boldsymbol{T}_{\alpha,\beta,\gamma}) \geqslant \frac{1+\alpha}{\alpha}\,.$$

PROOF. For $u, v \in \boldsymbol{T}_{\alpha,\beta,\gamma}$ we have

$$|(I - P_m)(u - v)| \leqslant \sum_{j=m+1}^{\infty} |(u - v, w_j)| \leqslant \left(\sum_{j=m+1}^{\infty} \exp\,[-2\beta j^{\alpha}]\right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\sum_{j=m+1}^{\infty} \exp\,[2\beta j^{\alpha}]\,|(u - v, w_j)|^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{j=m+1}^{\infty} \exp\,[-2\beta j^{\alpha}]\right)^{\frac{1}{2}} \cdot 2\gamma \leqslant$$

$$\leqslant \left(\int_m^{\infty} \exp\,[-2\beta x^{\alpha}]\,dx\right)^{\frac{1}{2}} \cdot 2\gamma \leqslant \exp\left[-\frac{\beta}{2} m^{\alpha}\right] \cdot \left(\int_0^{\infty} \exp\,[-\beta x^{\alpha}]\,dx\right)^{\frac{1}{2}} \cdot 2\gamma\,,$$

whence (7.44′). To prove (7.44″) we shall proceed as in Remark 3° above. Let thus $\varepsilon \in (0, \frac{1}{2})$ and $k(m)$ be the greatest integer smaller than

$$\gamma\left(3\varepsilon m^{\frac{1}{2}} \exp\,[\beta m^{\alpha}]\right)^{-1}$$

and define $\boldsymbol{B}(m)$ as in (7.42″). Then $\mathfrak{N}(\varepsilon, \boldsymbol{T}_{\alpha,\beta,\gamma}) \geqslant$ the cardinal of $\boldsymbol{B}(m)$, i.e. $\left(2k(m) + 1\right)^m$.

Suppose now that

$$\exp\,[-\beta(m+1)^{\alpha}] \leqslant \varepsilon^{\frac{1}{2}} < \exp\,[-\beta m^{\alpha}]\,.$$

Then

$$k(m) \geqslant \gamma(3\varepsilon^{\frac{1}{2}} m^{\frac{1}{2}})^{-1} - 1 \geqslant \gamma\left\{3\varepsilon^{\frac{1}{2}}\left[\left(\frac{1}{2\beta} \log \frac{1}{\varepsilon}\right)^{1/\alpha} - 1\right]\right\}^{-1} - 1 \geqslant c_{122}\,\varepsilon^{-\frac{1}{2}}$$

with a convenient constant c_{122}. Hence

$$\mathfrak{H}(\varepsilon, \boldsymbol{T}_{\alpha,\beta,\gamma}) \geqslant m \cdot \log (2c_{122}\varepsilon^{-\frac{1}{2}} + 1) \geqslant \frac{m}{2} \log \frac{1}{\varepsilon} + c'_{122} \geqslant$$

$$\geqslant \frac{1}{2}\left[\left(\frac{1}{2\beta}\log\frac{1}{\varepsilon}\right)^{1/\alpha} - 1\right]\log\frac{1}{\varepsilon} + c'_{122} \geqslant c''_{122}\left(\log\frac{1}{\varepsilon}\right)^{(1+\alpha)/\alpha} - c'''_{122}\,, \qquad c''_{122} > 0\,,$$

from where we can readily infer

$$\log \mathfrak{H}(\varepsilon, T_{\alpha,\beta,\gamma}) \geqslant \frac{1+\alpha}{\alpha} \log\log\frac{1}{\varepsilon} - c''''_{122}\,.$$

Therefore letting $\varepsilon \to \infty$ we will obtain (7.44″), finishing the proof.

Note that if $\alpha \leqslant \frac{1}{2}$, then (7.44″) yields $d(\boldsymbol{T}_{\alpha,\beta,\gamma}) \geqslant 3$. Consequently the conclusion of Theorem 3 is the best possible once only (7.13_{bis}) is used.

4. ***a***) The study of the stationary statistical behaviour of the Navier-Stokes equations in dimension $n = 2$, cannot be reduced only to that of the stationary statistical solutions of the Navier-Stokes equations, i.e. to that of the invariant probabilities in $\boldsymbol{N}$, but it must involve also the dynamical system $\{S(t)\}$. Therefore a more comprehensive study of the stationary statistical behaviour of the Navier-Stokes for plane-fluids (i.e. $n = 2$) must concern the measure-preserving dynamical system $\{S(t)\}_{0 \leqslant t < \infty}$ on supp μ whose invariant measure is μ, receiving thus a strong ergodic character. We were not yet able to exploit in this special topic the powerful tools developed in Ergodic Theory (except the now a days classical Birkhoff's ergodic theorem). However using the notion of isomorphism in Ergodic Theory we shall give a fact, which in spite of its simplicity, yields some mathematical consistency to the fact that a plane fluid behaves statistically as a random point in an Euclidean space R^m, where the number behaves similarly to the Reynolds number.

Before passing to our specific case let us introduce some definitions and conventions from Ergodic Theory. In Ergodic Theory, two dynamical systems $\{S'(t)\}_{0 \leqslant t < \infty}$ and $\{S''(t)\}_{0 \leqslant t < \infty}$ with invariant probabilities μ', resp. μ'', are called *isomorphic* if there exists two maps φ and ψ such that $\psi \circ \varphi =$ identical map μ' - a.e. and $\varphi \circ \psi =$ identical map μ'' - a.e., transforming μ' in μ'' and $S'(t)$ in $S''(t)$ for all $t \geqslant 0$, i.e.

$$\mu''(A'') = \mu'(\varphi^{-1}(A'')) , \qquad \mu'(A') = \mu''(\psi^{-1}(A'))$$

and

$$S''(t) \circ \varphi = \varphi \circ S'(t) \ (\mu' \text{ - a.e.}) , \qquad S'(t) \circ \psi = \psi \circ S''(t) \ (\mu'' \text{ - a.e.})$$

for all $t \geqslant 0$ and measurable sets A', A'' (with respect to μ', resp. μ'') (see Jacobs [1], Ch. X). Let $\{\xi(t)\}_{0 \leqslant t < \infty}$ be a stationary stochastic process with values in some R^m; let us recall that this means that there exists a probability P on a σ-algebra $\mathfrak{E}$ of subsets of a certain set E such that for any $t \geqslant 0$, $\xi(t)$ is a function on E, measurable with respect to $\mathfrak{E}$, and that

$$P(\{\varepsilon\colon \varepsilon \in E, \ \{\xi(t_1), \xi(t_2), \dots, \xi(t_k)\} \in A\}) =$$
$$= P(\{\varepsilon\colon \varepsilon \in E, \ \{\xi(t_1 + t), \dots, \xi(t_k + t)\} \in A\})$$

for all $t, t_1, t_2, \dots, t_k \geqslant 0$, $k = 1, 2, \dots$, and Borel subset A of R^{mk} (see Doob [1], Ch. 2, § 8). To such a process there is a standard representation in a product space (see Doob [1], App., § 2, Ex. 2.3 and Nelson [1]), namely in $X = \prod_{0 \leqslant t < \infty} X_t$, where X_t is for any t the Alexandroff one-point compactification of R^m and X is endowed with the Tihonoff product topology. Indeed for any real function Φ on X of the form

$$\Phi(\{x_t\}_{0 \leqslant t < \infty}) = \varphi(x_{t_1}, x_{t_2}, \dots, x_{t_k})$$

where φ is continuous on $X_{t_1} \times .. \times X_{t_k}$, set

$$J(\Phi) = \int_E \varphi(\xi(t_1), \dots, \xi(t_k)) \, dP(\varepsilon) .$$

It is easy to see that J extends uniquely to a Radon integral on X, thus

$$J(\Phi) = \int_X \Phi(x) \, d\pi(x)$$

with a certain regular probability π on X (see Dunford-Schwartz [1], Ch. IV, 6.3). Defining on X the map

$$\tau(t_0)\big(\{x_t\}_{0\leqslant t<\infty}\big)=\{x_{t+t_0}\}_{0\leqslant t<\infty}$$

we obtain a dynamical system $\{\tau(t_0)\}_{0\leqslant t<\infty}$ preserving the probability π on X. A dynamical system $\{S'(t)\}_{0\leqslant t<\infty}$ preserving the probability μ' is said to correspond to the stationary R^m-valued stochastic process $\{\xi(t)\}_{0\leqslant t<\infty}$ if it is isomorphic to $\{S''(t)\}_{0\leqslant t<\infty}$ with $S''(t)=\tau(t)$, $t\geqslant 0$ and $\mu''=\pi$.

b) We are now in state to return to the study of the Navier-Stokes equations.

Let μ be an invariant (Borel) probability in $\boldsymbol{N}$ and let $m=1,2,\ldots$. Set

$$E=\operatorname{supp}\mu\,,\qquad P=\mu\quad\text{ and }\quad \xi(t)=pr_m(t)$$

where

$$pr_m(t)(u)=\big\{\big(S(t)u,w_1\big),\ldots,\big(S(t)u,w_m\big)\big\}\in R^m\,. \tag{7.45}$$

PROPOSITION 3. *The dynamical system* $\{S(t)\}_{0\leqslant t<\infty}$ *with invariant probability* μ, *corresponds to the stationary* R^m-*valued stochastic process* $\{pr_m(t)\}_{0\leqslant t<\infty}$ (defined above) *whenever*

$$m>c_{123}=\frac{2c_7^2}{c_{113}\nu^2}\,(^*)\,. \tag{7.46}$$

PROOF. Define the map φ: $\operatorname{supp}\mu\mapsto X$ (where X is the product introduced in the preceding Sec. 7.4.a)) by

$$\varphi(u)=\{pr_m(t)(u)\}_{0\leqslant t<\infty}\,.$$

It is plain that φ is continuous from $\operatorname{supp}\mu$ (endowed with the topology of $\boldsymbol{N}$) to X. Also

$$\varphi\big(S(t_0)u\big)=\big\{pr_m(t)\big(S(t_0)u\big)\big\}_{0\leqslant t<\infty}=\{pr_m(t+t_0)(u)\}_{0\leqslant t<\infty}=\tau(t_0)\big(\varphi(u)\big)\,,$$

(*) See formulae (2.16′) and (7.29).

i.e.

$$\varphi \circ S(t) = \tau(t) \circ \varphi \qquad \text{for all } t \geqslant 0 . \tag{7.47}$$

Moreover for any real function Φ on X defined by $\Phi(\{x_t\}_{0\leqslant t<\infty}) = \varphi(x_{t_1}, \dots, x_{t_k})$ where φ is continuous on $X_{t_1}\times\dots\times X_{t_k}$, we have

$$\int_X \Phi(x)\, d\pi(x) = J(\Phi) = \int_{\operatorname{supp}\mu} \varphi(pr_m(t_1)u, pr_m(t_2)u, \dots, pr_m(t_k)u)\, d\mu(u) = \int_{\operatorname{supp}\mu} \Phi \circ \varphi(u)\, d\mu(u) .$$

Since by the Weierstrass-Stone theorem the functions involved are dense in the Banach algebra $C(X)$ of all real continuous functions on X, we deduce that

$$\int_X \Phi(x)\, d\pi(x) = \int_{\operatorname{supp}\mu} \Phi \circ \varphi(u)\, d\mu(u)$$

for all $\Phi \in C(X)$. Therefore

$$\pi(A) = \mu(\varphi^{-1}(A)) \tag{7.47$'$}$$

for all Borel subsets A of X. Suppose now that for $u_0, v_0 \in \operatorname{supp}\mu$ we have $\varphi(u_0) = \varphi(v_0)$. This means that

$$P_m S(t) u_0 = P_m S(t) v_0 \qquad \text{for all } t \geqslant 0 . \tag{7.48}$$

In virtue of (6.24) there exists $u_{-1}, v_{-1} \in \operatorname{supp}\mu$ such that $S(1)u_{-1} = u_0$, $S(1)v_{-1} = v_0$. Repeating indefinitely this argument we obtain two individual solutions $u(\cdot)$ and $v(\cdot)$ of the Navier-Stokes equations on the whole $(-\infty, \infty) = R$ such that

$$u(t), v(t) \in \operatorname{supp}\mu \qquad \text{for all } t \in R \text{ and } u(0) = u_0,\ v(0) = u_0] . \tag{7.48$'$}$$

But $u(\cdot)$ and $v(\cdot)$ are N-valued analytic functions in t (see Masuda [1]) thus so are $P_m u(\cdot)$ and $P_m v(\cdot)$. By (7.48) we conclude

$$P_m u(t) = P_m v(t) \qquad \text{for all } t \in R . \tag{7.48$''$}$$

For $w(\cdot) = u(\cdot) - v(\cdot)$, we will have, by (6.23), and (7.48″)

$$\frac{1}{2}\frac{d|w|^2}{dt} + \nu\|w\|^2 = -b(u, u, w) + b(v, v, w) =$$

$$= -b(w, u, w) - b(v, w, w) = -b(w, u, w) \leqslant \sqrt{2}\|u\|\cdot|w|\cdot\|w\| \leqslant$$
$$\leqslant \sqrt{2}\, c_7\cdot|w|\cdot\|w\| = \sqrt{2}|(I - P_m)w|\cdot\|(I - P_m)w\|\cdot c_7 \leqslant$$
$$\leqslant \frac{\nu}{2}\|(I - P_m)w\|^2 + \frac{c_7^2}{\nu}|(I - P_m)w|^2,$$

whence

$$\frac{d}{dt}|w|^2 + \left(\nu\lambda_m - \frac{2c_7^2}{\nu}\right)|(I - P_m)w|^2 \leqslant 0\,.$$

Since $|w|^2 = |(I - P_m)w|^2$ we conclude that if

$$\lambda_m > 2(c_7\nu^{-1})^2 \tag{7.49}$$

then

$$|w(t)|^2 \leqslant \exp\left[-\varepsilon(t - t_0)\right]|w(t_0)|^2 \tag{7.49'}$$

for all real numbers $t_0 \leqslant t$, where $\varepsilon = \nu\lambda_m - 2c_7^2\nu^{-1} > 0$. But $|w(t_0)|^2 \leqslant$ $\leqslant 4(c'_{85})^2$ (see (6.4′)), so that letting $t_0 \to -\infty$ in (7.49′) we get $|w(t)| = 0$ for any t; for $t = 0$ it results $u_0 = v_0$. This conclusion proves that φ is injective. Let ψ be, on $A_0 = \varphi(\operatorname{supp}\mu)$, the inverse of φ, while on $X \diagdown A_0$ take ψ equal to an element fixed of $\operatorname{supp}\mu$. Since A_0 is compact in X, ψ is obviously a Borel map from X to N which, on A_0, is continuous. Taking $A = A_0$ in (7.47′) we obtain $\pi(A_0) = 1$ thus $\varphi \circ \psi(x) = x$ π - a.e. and $\psi \circ \varphi(u) = u$ μ - a.e. All other necessary relations for the « isomorphism » between $\{S(t)\}$ and $\{pr_m(t)\}$ given in Sec. 7.4.a) can now be easily verified on account of (7.47)-(7.47′). The proof finishes by the remark that, in virtue of (7.29), the relation (7.46) implies (7.49).

Remarks. 1°. A simple analysis of the constants occuring in (7.46) shows that c_{123} behaves in the same manner as the Reynolds number, that is it is an increasing function in $1/\nu$, $|f|$ and the « size » of Ω.

2°. As we already said in the beginning of this Section, Proposition 3 and Remark 1° above are coherent with the physical behaviour of a fluid. A first rigorous mathematical content to this behaviour was given in Hopf [1], who, also exhibited a mathematical model (based on other equations than the Navier-Stokes equations) in which this behaviour was valid. A more precise conjecture was given in Prodi [3], namely that any invariant (Borel) probability (in the case of plane fluids, i.e. $n=2$) has a support of finite dimension. The results above can be considered as a weakened form of this conjecture, as well as of the behaviour which is expected for turbulent flows (see again Hopf [1]).

c) An R^m-valued stochastic process $\{\xi_t\}_{0\leqslant t<\infty}$ is called a Markov-process if for any $0\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_k<\infty$ we have

$$P(\{\xi_{t_k}\in A\}|\xi_{t_1},\xi_{t_2},\ldots,\xi_{t_{k-1}})=P(\{\xi_{t_k}\in A\}|\xi_{t_{k-1}}) \tag{7.50}$$

for any Borel set A in R^m (see Doob [1], Ch. II, § 6). Let us recall that if $B\in\mathfrak{E}$ (see Sec. 7.4.*a*)), then $P(B|\xi_{t_i},\xi_{t_{i+1}},\ldots,\xi_{t_{k-1}})$ is the conditional expectation of the characteristic function of B with respect to the σ-algebra $\mathfrak{E}_{t_i,\ldots,t_{k-1}}$ generated by $\xi_{t_j}^{-1}(A_j)$ where $j=i,i+1,\ldots,k-1$ and A_j are Borel subsets in R^m; this means that $P(B|\xi_{t_i},\xi_{t_{i+1}},\ldots,\xi_{t_{k-1}})$ is a R^m-valued function on E, measurable with respect to $\mathfrak{E}_{t_i,\ldots,t_{k-1}}$ and satisfying

$$P(B\cap B')=\int_{B'}\chi_B\,dP=\int_{B'}P(B)(\xi_{t_i},\xi_{t_{i+1}},\ldots,\xi_{t_{k-1}})\,dP$$

for all $B'\in\mathfrak{E}_{t_i,t_{i+1},\ldots,t_{k-1}}$ (see Doob [1], Ch. I, § 8-9). Most stochastic processes which occur in the mathematical description of phenomena in nature (and society) are actually Markov processes. Therefore it is natural to ask whether our process $\{pr_m(t)\}_{0\leqslant t<\infty}$ is or not a Markov-one. The answer to this question is surprisingly simple:

PROPOSITION 4. *Let m satisfy*

$$m>4c_{123}\,. \tag{7.46'}$$

Then $\{pr_m(t)\}_{0\leqslant t<\infty}$ *is a Markov process if an only if there exists a Borel set* $\boldsymbol{\sigma}_\mu\subset\operatorname{supp}\mu$ *such that* $\mu(\boldsymbol{\sigma}_\mu)=1$ *and* $P_m|\boldsymbol{\sigma}_\mu$ *is injective.*

(REMARK. Since up to a μ-null set any Borel set is a σ-compact set in supp μ, if a $\boldsymbol{\sigma}_\mu$ with the specified conditions exists, then $\boldsymbol{\sigma}_\mu$ is up to a μ-null set a countable union of compact sets of dimension $\leqslant m$; indeed if $\boldsymbol{\sigma}_\mu = \bigcup_{p=1}^{\infty} \boldsymbol{K}_p$ modulo a μ-null set, where $\boldsymbol{K}_p$ are all compact, $P_m|\boldsymbol{K}_p$, being injective, is a homeomorphism from $\boldsymbol{K}_p$ onto $P_m\boldsymbol{K}_p$, so that $\boldsymbol{K}_p$ is of dimension $\leqslant m$.)

PROOF. If $\boldsymbol{\sigma}_\mu$ exists, then $pr_m(0)|\boldsymbol{\sigma}_\mu$ is a continuous injective map; since we can suppose that $\boldsymbol{\sigma}_\mu$ is a σ-compact set, the inverse map $q = (pr_m(0)|\sigma_\mu)^{-1}$ will be a Borel function (from $pr_m(0)\,\boldsymbol{\sigma}_\mu$ in $\operatorname{supp}\mu$). Therefore if $u \in S(t_{k-1})^{-1}\boldsymbol{\sigma}_\mu$ we will have

$$pr_m(t_k)\,u = pr_m(0)\,S(t_k)\,u = pr_m(0)\,S(t_k - t_{k-1})\,S(t_{k-1})\,u =$$
$$= pr_m(0)\,S(t_k - t_{k-1})\,q\cdot pr_m(0)\,S(t_{k-1})\,u = [pr_m(t_k - t_{k-1})\,q]\big(pr_m(t_{k-1})\,u\big)\,.$$

The function $\xi \mapsto \big(pr_m(t_k - t_{k-1})\,q\big)(\xi)$ is an R^m-valued Borel function defined on $pr_m(0)\,\boldsymbol{\sigma}_\mu$. Extend it by 0 on $R^m \setminus pr_m(0)\boldsymbol{\sigma}_\mu$ and denote this extension by β. Then since $\mu\big(S(t_{k-1})^{-1}\,\boldsymbol{\sigma}_\mu\big) = 1$, the above relations show that

$$pr_m(t_k)\,u = \beta\big(pr_m(t_{k-1})\,u\big) \qquad \mu\text{ - a.e.}\,.$$

In virtue of this fact it is immediate that (7.50) is satisfied for $\xi_\bullet = pr_m(\cdot)$.

For the converse statement we need the following property of invariant probabilities:

LEMMA. *Let m satisfy (7.46′). Then for all $0 < t_1 < t_2 < \ldots < t_{k-1} < \infty$ and $u, v \in \operatorname{supp}\mu$ we have*

$$(7.51) \qquad |S(t_{k-1})\,u - S(t_{k-1})\,v|^2 \leqslant |(I - P_m)\big(S(t_1)\,u - S(t_1)\,v\big)|^2 \cdot$$
$$\cdot \exp\left[-\nu \frac{c_{113}}{2}\, m(t_{k-1} - t_1)\right] + c_{124}\cdot \max_{1\leqslant j\leqslant k-2} |P_m S(t_j)\,u - P_m S(t_j)\,v|^2 +$$
$$+ c'_{124}\cdot \max_{1\leqslant j\leqslant k-2} (t_{j+1} - t_j)$$

where $c_{124} - c'_{124}$ are some constants depending only on Ω, ν, $|f|$.

We shall prove this Lemma after we have achieved the proof of the Proposition. To this aim let $u_0, v_0 \in \operatorname{supp}\mu$ such that $P_m u_0 = P_m v_0$ and $|u_0 - v_0| > 0$. As in the proof of the preceding Proposition 3, we can show that there exists two individual solutions $u(\cdot)$,

$v(\cdot)$ on $(-\infty, \infty) = R$, such that $u(t), v(t) \in \operatorname{supp} \mu$ for all $t \in R$ and $u(0) = u_0$, $v(0) = v_0$. Since they are analytic in t (as N-valued function; see again Masuda [1]) and since $P_m u(t) = P_m v(t)$ for all $t \in R$ implies $u(t) = v(t)$ for all $t \in R$ (see again the proof of the preceding Proposition 3), we can find, for any $\delta > 0$ and any $l > 0$ a system $\tau_1 < \tau_2 < \ldots < \tau_{k-2} < \tau_{k-1} = 0 < \tau_k$ such that

$$\max_{1 \leqslant j \leqslant k-2} |\tau_{j+1} - \tau_j| \leqslant \delta\,, \qquad \tau_k \in [\tfrac{1}{2}, 1]\,, \qquad \tau_1 < -l \tag{7.52}$$

and

$$P_m u(\tau_j) \neq P_m v(\tau_j)\,, \qquad \text{for all } j \neq k-1\,. \tag{7.52'}$$

Take ϱ sufficiently small that

$$\begin{cases} A(\varrho) = \{u\colon u \in \operatorname{supp}\mu,\ |u - u_0| < \varrho\}\,, \\ B(\varrho) = \{u\colon u \in \operatorname{supp}\mu,\ |u - v_0| < \varrho\}\,, \end{cases} \tag{7.53}$$

be disjoint and also

$$\big(P_m S(\tau_k) A(\varrho)\big) \cap \big(P_m S(\tau_k) B(\varrho)\big) = \emptyset\,. \tag{7.53'}$$

By (7.51) we have

$$\begin{aligned} |S(-\tau_1)u - S(-\tau_1)v|^2 &\leqslant \exp[-c_{125} l](2r_0)^2 + \\ &+ c_{124} \cdot \max_{1 \leqslant j \leqslant k-2} |P_m[S(\tau_j - \tau_1)u - S(\tau_j - \tau)v]|^2 + c'_{124}\delta \end{aligned} \tag{7.51'}$$

where

$$r_0 = \sup\{|u|\colon u \in \operatorname{supp}\mu\} \qquad \text{and} \qquad c_{125} = \nu c_{113}(m - c_1)\,.$$

Take l and δ such that

$$\exp[-c_{125} l](2r_0)^2 + c'_{124}\delta < \left(\frac{\varrho}{2}\right)^2. \tag{7.51''}$$

Then (7.51′) becomes

$$\begin{aligned} |S(-\tau_1)u - S(-\tau_1)v|^2 &\leqslant \left(\frac{\varrho}{2}\right)^2 + c_{124} \cdot \\ &\cdot \max_{1 \leqslant j \leqslant k-2} |P_m[S(\tau_j - \tau_1)u - S(\tau_j - \tau_1)v]|^2 \end{aligned} \tag{7.51'''}$$

for all $u, v \in \operatorname{supp} \mu$. Take now $\eta > 0$ sufficiently small that

$$(7.54) \qquad \begin{cases} \boldsymbol{a}_j = \{u\colon u \in \operatorname{supp}\mu,\ |P_m(u - u(\tau_j))| < \eta\}\,, \\ \boldsymbol{b}_j = \{u\colon u \in \operatorname{supp}\mu,\ |P_m(u - v(\tau_j))| < \eta\}\,, \end{cases}$$

be disjoint, for all $j = 1, 2, \ldots, k-2$, and that

$$(7.54') \qquad c_{124}\eta^2 \leqslant \left(\frac{\varrho}{2}\right)^2 .$$

The sets $\boldsymbol{a}_j$, $\boldsymbol{b}_j$ are open in supp μ. On account of the analyticity of $S(\cdot)u$ (for any $u \in \boldsymbol{N}$; see again Masuda [1]) the maps $S(t)$, $t \geqslant 0$, will be injective. Therefore $S(t)|\operatorname{supp}\mu$ is a homeomorphism of supp μ (*onto* itself). It results that

$$\boldsymbol{A} = S(-\tau_1)[\boldsymbol{a}_1 \cap S(\tau_2 - \tau_1)^{-1}\boldsymbol{a}_2 \cap \ldots \cap S(\tau_{k-2} - \tau_1)^{-1}\boldsymbol{a}_{k-2}]$$

and

$$\boldsymbol{B} = S(-\tau_1)[\boldsymbol{b}_1 \cap S(\tau_2 - \tau_1)^{-1}\boldsymbol{b}_2 \cap \ldots \cap S(\tau_{k-2} - \tau_1)^{-1}\boldsymbol{b}_{k-2}]$$

are open in supp μ. Moreover

$$(7.55) \qquad u_0 \in \boldsymbol{A}\,, \quad v_0 \in \boldsymbol{B} \quad \text{and} \quad \boldsymbol{A} \subset \boldsymbol{A}(\varrho)\,, \quad \boldsymbol{B} \subset \boldsymbol{B}(\varrho)\,.$$

Indeed since $S(\tau_j - \tau_1)u(\tau_1) = u(\tau_j)$ we deduce first that $u(\tau_1) \in \boldsymbol{a}_1 \cap \cap S(\tau_2 - \tau_1)^{-1}\boldsymbol{a}_2 \cap \ldots \cap S(\tau_{k-2} - \tau_1)^{-1}\boldsymbol{a}_{k-2}$, and consequently that $u_0 = = S(-\tau_1)u(\tau_1) \in \boldsymbol{A}$. If $u \in \boldsymbol{a}_1 \cap S(\tau_2 - \tau_1)^{-1}\boldsymbol{a}_2 \cap \ldots \cap S(\tau_{k-2} - \tau_1)^{-1}\boldsymbol{a}_{k-2}$ then, by (7.51‴) (with $v = u(\tau_1)$) and by (7.54)-(7.54′) we get

$$|S(-\tau_1)u - u_0|^2 \leqslant \left(\frac{\varrho}{2}\right)^2 + \left(\frac{\varrho}{2}\right)^2 < \varrho^2,$$

hence $S(-\tau_1)u \in \boldsymbol{A}(\varrho)$. The same argument applies for $v_0 \in \boldsymbol{B}$ and $\boldsymbol{B} \subset \boldsymbol{B}(\varrho)$. Therefore there exists $0 < \varrho_0 < \varrho$ such that

$$(7.56) \qquad \boldsymbol{A}(\varrho_0) \subset \boldsymbol{A} \quad \text{and} \quad \boldsymbol{B}(\varrho_0) \subset \boldsymbol{B}\,.$$

Set $t_1 = 0$, $t_j = \tau_j - \tau_1$, $j = 1, 2, \ldots, k$, and

$$(7.57) \qquad \begin{cases} A_j \ = pr_m(0)\boldsymbol{a}_j\,, \qquad B_j = pr_m(0)\boldsymbol{b}_j\,, \qquad j = 1, 2, \ldots, k-2\,, \\ A_{k-1} = pr_m(0)\boldsymbol{A}(\varrho)\big(= pr_m(0)\boldsymbol{B}(\varrho)\big)\,, \\ A_k \ = pr_m(0)\,S(\tau_k)\boldsymbol{A}(\varrho)\,, \qquad B_k = pr_m(0)\,S(\tau_k)\boldsymbol{B}(\varrho)\,. \end{cases}$$

Since $\operatorname{supp}\mu$ is compact and $pr_m(t)$ is (for any $t\geqslant 0$) a continuous map from $\operatorname{supp}\mu$ to R^m, and since thus any open set $\boldsymbol{\sigma}$ in $\operatorname{supp}\mu$ is σ-compact and therefore $pr_m(t)\boldsymbol{\sigma}$ is σ-compact in R^m too, we deduce that all the sets A_j, B_j, defined by (7.57), are Borel sets in R^m. Moreover for $j=1,2,\ldots,k-2$

$$(7.57')\qquad \begin{cases} pr_m(t_j)u\in A_j & \text{if and only if} \quad S(\tau_j-\tau_1)u\in \boldsymbol{a}_j\,, \\ pr_m(t_j)u\in B_j & \text{if and only if} \quad S(\tau_j-\tau_1)u\in \boldsymbol{b}_j\,, \end{cases}$$

therefore

$$\begin{aligned}\boldsymbol{A}' &= \{pr_m(t_1)\in A_1,\ldots,pr_m(t_{k-2})\in A_{k-2}\}= \\ &= \{u\colon u\in\operatorname{supp}\mu,\ pr_m(t_1)u\in A_1,\ldots,pr_m(t_{k-2})u\in A_{k-2}\}= \\ &= \boldsymbol{a}_1\cap S(\tau_2-\tau_1)^{-1}\boldsymbol{a}_2\cap\ldots\cap S(\tau_{k-2}-\tau_1)^{-1}\boldsymbol{a}_{k-2}\,,\end{aligned}$$

and similarly

$$\begin{aligned}\boldsymbol{B}' &= \{pr_m(t_1)\in B_1,\ldots,pr_m(t_{k-2})\in B_{k-2}\}= \\ &= \boldsymbol{b}_1\cap S(\tau_2-\tau_1)^{-1}\boldsymbol{b}_2\cap\ldots\cap S(\tau_{k-2}-\tau_1)^{-1}\boldsymbol{b}_{k-2}\,.\end{aligned}$$

In virtue of (7.55) and (7.57) we can now deduce that

$$(7.58)\qquad \begin{cases} pr_m(t_{k-1})u=pr_m(0)S(-\tau_1)u\in A_{k-1}\,, \\ pr_m(t_k)u=pr_m(0)S(\tau_k)S(-\tau_1)u\in pr_m(0)S(\tau_k)\boldsymbol{A}(\varrho)\subset A_k\,, \\ \hfill \text{for } u\in\boldsymbol{A}', \end{cases}$$

and

$$(7.58')\qquad \begin{cases} pr_m(t_{k-1})u=pr_m(0)S(-\tau_1)u\in A_{k-1}\,, \\ pr_m(t_k)u=pr_m(0)S(\tau_k)S(-\tau_1)u\in pr_m(0)S(\tau_k)\boldsymbol{B}(\varrho)\subset B_k \\ \hfill \text{for } u\in\boldsymbol{B}', \end{cases}$$

where (by (7.53′))

$$(7.58'')\qquad A_k\cap B_k=\emptyset\,.$$

Suppose now that $\{pr_m(t)\}_{0\leqslant t<\infty}$ is a Markov process. Then in particular we must have

$$(7.59)\qquad P(\{pr_m(t_k)\in A_k\}|pr_m(t_1),\ldots,pr_m(t_{k-1})) = \\ = P(\{pr_m(t_k)\in A_k\}|pr_m(t_{k-1}))\ .$$

This implies in an obvious way that

$$(7.59')\qquad \mu(\boldsymbol{Y}\cap\{pr_m(t_{k-1})\in A_{k-1},\, pr_m(t_k)\in A_k\}) = \\ = \int\limits_{\boldsymbol{Y}\cap\{pr_m(t_{k-1})\in A_{k-1}\}} P(\{pr_m(t_k)\in A_k\}|pr_m(t_{k-1}))\, d\mu$$

where for $\boldsymbol{Y}$ we (can and) will take either $\boldsymbol{A}'$ or $\boldsymbol{B}'$. On account of the fact that $\operatorname{supp}\mu$ is a metric compact space, for any real valued function φ on $\operatorname{supp}\mu$ measurable with respect to the σ-algebra generated by $\{pr_m(t_{k-1})^{-1}A:$ A Borel subset of $R^m\}$ there exists a Borel function β defined on R^m such that

$$\varphi(u) = \beta(pr_m(t_{k-1})u)\ ,\qquad \mu\text{ - a.e.}$$

(see Doob [1], App., Th. 1.5). Therefore for a convenient Borel function β defined on R^m we will have μ - a.e. that

$$P(\{pr_m(t_k)\in A_k\}|pr_m(t_{k-1})) = \beta\circ pr_m(t_{k-1})\ .$$

It is plain that we can choose β such that $0\leqslant\beta\leqslant 1$. By (7.58),

$$\boldsymbol{A}'\cap\{pr_m(t_{k-1})\in A_{k-1}\} = \boldsymbol{A}'\cap\{pr_m(t_{k-1})\in A_{k-1},\, pr_m(t_k)\in A_k\} = \boldsymbol{A}'$$

so that with the choice $\boldsymbol{Y}=\boldsymbol{A}'$, (7.59′) becomes

$$\mu(\boldsymbol{A}') = \int\limits_{\boldsymbol{A}'}\beta(pr_m(t_{k-1})u)\, d\mu(u)\ ,$$

whence

$$(7.60)\qquad \beta(pr_m(t_{k-1})u) = 1\ ,\qquad \mu\text{ - a.e. on } \boldsymbol{A}'.$$

On the other hand, by (7.58′)-(7.58″) we have

$$\boldsymbol{B}'\cap\{pr_m(t_{k-1})\in A_{k-1}\} = \boldsymbol{B}',$$

$$\boldsymbol{B}'\cap\{pr_m(t_{k-1})\in A_{k-1},\, pr_m(t_k)\in A_k\} = \emptyset\ ,$$

so that with the choice $\boldsymbol{Y}=\boldsymbol{B}'$, (7.59′) becomes

$$0=\int_{\boldsymbol{B}'}\beta\big(pr_m(t_{k-1})\,u\big)\,d\mu(u)\,,$$

whence

$$\beta\big(pr_m(t_{k-1})\,u\big)=0\,,\qquad \mu\text{ - a.e. on } \boldsymbol{B}'. \tag{7.60′}$$

From (7.60)-(7.60′) we infer, using the invariance of μ, that

$$\beta\big(pr_m(0)\,u\big)=1\,,\qquad \mu\text{ - a.e. on } \boldsymbol{A}\,, \tag{7.61}$$

$$\beta\big(pr_m(0)\,u\big)=0\,,\qquad \mu\text{ - a.e. on } \boldsymbol{B}\,. \tag{7.61′}$$

Set

$$\gamma_0=\left\{\xi\colon\,\xi\in R^m,\ |\xi-pr_m(0)\,u_0|<\frac{\varrho_0}{2}\right\} \tag{7.62}$$

and

$$\begin{aligned}\boldsymbol{\alpha}_0&=\left\{u\colon\,u\in\operatorname{supp}\mu,\ |(I-P_m)(u-u_0)|<\frac{\varrho_0}{2}\right\},\\ \boldsymbol{\beta}_0&=\left\{u\colon\,u\in\operatorname{supp}\mu,\ |(I-P_m)(u-v_0)|<\frac{\varrho_0}{2}\right\}.\end{aligned} \tag{7.62′}$$

Then by (7.56) we have

$$\big(pr_m(0)^{-1}\gamma_0\big)\cap\boldsymbol{\alpha}_0\subset\boldsymbol{A}\,,\qquad \big(pr_m(0)^{-1}\gamma_0\big)\cap\boldsymbol{\beta}_0\subset\boldsymbol{B}\,,$$

hence, by (7.61)-(7.61′)

$$\begin{cases}\beta\big(pr_m(0)\,u\big)=1\,, & \mu\text{ - a.e. on } \big(pr_m(0)^{-1}\gamma_0\big)\cap\boldsymbol{\alpha}_0\,,\\ \beta\big(pr_m(0)\,u\big)=0\,, & \mu\text{ - a.e. on } \big(pr_m(0)^{-1}\gamma_0\big)\cap\boldsymbol{\beta}_0\,.\end{cases} \tag{7.61″}$$

Set $\varepsilon_0=\{\xi\colon\,\xi\in\gamma_0,\,\beta(\xi)=1\}$. Then (7.61″) implies that

$$pr_m(0)^{-1}\varepsilon_0\supset\big(pr_m(0)^{-1}\gamma_0\big)\cap\boldsymbol{\alpha}_0\,,\qquad \big(pr_m(0)^{-1}\varepsilon_0\big)\cap\boldsymbol{\beta}_0=\emptyset \tag{7.62‴}$$

up to some sets of μ-measure 0. Let now μ_0 be defined by

$$\mu_0(A)=\mu\big(pr_m(0)^{-1}A\big)\qquad\text{for all Borel sets } A\subset R^m\,.$$

Then we can desintegrate μ by means of $pr_m(0)$ and μ_0 (see Dinculeanu [1], § 20, 3 and Bourbaki [1], § 3, n. 1), namely there will exist a probability valued function λ_ξ defined on $pr_m(0)(\operatorname{supp}\mu) = S_0(\subset R^m)$ such that

$$\mu(\boldsymbol{\omega}) = \int_{S_0} \lambda_\xi(\boldsymbol{\omega})\, d\mu_0(\xi) \tag{7.63}$$

for all Borel sets $\boldsymbol{\omega} \subset \operatorname{supp}\mu$; moreover we can suppose that

$$\operatorname{supp}\lambda_\xi \subset (\operatorname{supp}\mu) \cap pr_m(0)^{-1}\{\xi\} \tag{7.63'}$$

for all $\xi \in S_0$(see (Dinculeanu [1], § 20, 2 and 3)).

It results (in virtue of (7.62‴))

$$\int_{\varepsilon_0} \lambda_\xi(\boldsymbol{\alpha}_0)\, d\mu_0(\xi) = \mu\Big(\big(pr_m(0)^{-1}\varepsilon_0\big) \cap \boldsymbol{\alpha}_0\Big) =$$
$$= \mu\Big(\big(pr_m(0)^{-1}\gamma_0\big) \cap \boldsymbol{\alpha}_0\Big) = \int_{\gamma_0} \lambda_\xi(\boldsymbol{\alpha}_0)\, d\mu_0(\xi)\,,$$

hence

$$\lambda_\xi(\boldsymbol{\alpha}_0) = 0\,, \qquad \mu_0\text{ - a.e. on } \gamma_0 \setminus \varepsilon_0\,. \tag{7.64}$$

On the other side, again in virtue of (7.62‴), we have also

$$\int_{\varepsilon_0} \lambda_\xi(\boldsymbol{\beta}_0)\, d\mu_0(\xi) = \mu\Big(\big(pr_m(0)^{-1}\varepsilon_0\big) \cap \boldsymbol{\beta}_0\Big) = \mu(\emptyset) = 0$$

hence

$$\lambda_\xi(\boldsymbol{\beta}_0) = 0\,, \qquad \mu_0\text{ - a.e. on } \varepsilon_0\,. \tag{7.64'}$$

Thus we have proved that for any $u_0, v_0 \in \operatorname{supp}\mu$ satisfying $P_m u_0 = P_m v_0$ and $|u_0 - v_0| > 0$ there exists open sets $\boldsymbol{\alpha}_0$, $\boldsymbol{\beta}_0$, γ_0 of the type (7.62)-(7.62′) and a Borel set $\varepsilon_0 \subset \gamma_0$ such that (7.64)-(7.64′) be valid. Since $\big(\boldsymbol{\alpha}_0 \cap pr_m(0)^{-1}\gamma_0\big) \times \big(\boldsymbol{\beta}_0 \cap pr_m(0)^{-1}\gamma_0\big)$ is a neighbourhood of $\{u_0, v_0\}$ in $\operatorname{supp}\mu \times \operatorname{supp}\mu$ and since

$$\boldsymbol{M} = \{\{u_0, v_0\} : \{u_0, v_0\} \in \operatorname{supp}\mu \times \operatorname{supp}\mu,\ P_m u_0 = P_m v_0,\ |u_0 - v_0| > 0\}$$

is a metric separable space there exists a countable family of sets $\boldsymbol{\alpha}_0^j$, $\boldsymbol{\beta}_0^j$, γ_0^j and ε_0^j, $j=1,2,\ldots$, of the kind described above such that

$$\text{(7.65)} \qquad \boldsymbol{M} \subset \bigcup_{j=1}^{\infty}\left[\left(\boldsymbol{\alpha}_0^j \cap pr_m(0)^{-1}\gamma_0^j\right) \times \left(\boldsymbol{\beta}_0^j \cap pr_m(0)^{-1}\gamma_0^j\right)\right].$$

Let η_0^j denote the set of $\xi \in \gamma_0^j$ such that either

$$\lambda_\xi(\boldsymbol{\beta}_0^j) > 0\,, \quad \text{if } \xi \in \varepsilon_0^j$$

or

$$\lambda_\xi(\boldsymbol{\alpha}_0^j) > 0\,, \quad \text{if } \xi \in \gamma_0^j \diagdown \varepsilon_0^j\,.$$

By the relations of type (7.64)-(7.64′) we have $\mu_0(\eta_0^j) = 0$. Put $\eta_0 = \bigcup_{j=1}^{\infty} \eta_0^j$; then $\mu_0(\eta_0) = 0$. Let $\xi_0 \notin \eta_0$ be such that the support of λ_{ξ_0} contains at least two points $u_0 \neq v_0$. Obviously $\{u_0, v_0\} \in \boldsymbol{M}$, thus by (7.65), there exists a j such that

$$\{u_0, v_0\} \in \left(\boldsymbol{\alpha}_0^j \times pr_m(0)^{-1}\gamma_0^j\right) \times \left(\boldsymbol{\beta}_0^j \cap pr_m(0)^{-1}\gamma_0^j\right)\,.$$

Since $pr_m(0)u_0 = pr_m(0)v_0 = \xi_0 \notin \eta_0^j$ we have either $\lambda_{\xi_0}(\boldsymbol{\alpha}_0^j) = 0$ or $\lambda_{\xi_0}(\boldsymbol{\beta}_0^j) = 0$. In the first case $u_0 \notin \operatorname{supp} \lambda_{\xi_0}$, while in the second $v_0 \notin \operatorname{supp} \lambda_{\xi_0}$. Contradiction! Therefore we can conclude that

$$\text{(7.66)} \qquad \lambda_\xi = \delta_{h(\xi)} \qquad \text{for all } \xi \in S_0 \diagdown \eta_0\,,$$

where $\{h(\xi)\}$ is the support of λ_ξ. Since for any $v \in \boldsymbol{N}$ and $\xi \in S_0 \diagdown \eta_0$ we have $\lambda_\xi\big((\cdot, v)\big) = \big(h(\xi), v\big)$ we deduce that $h(\cdot)$ is an $\boldsymbol{N}$-valued weakly μ_0-measurable function thus (since $\boldsymbol{N}$ is separable) a strongly μ_0-measurable function. Therefore changing the definition of $h(\cdot)$ on a Borel set of μ_0-measure 0, we can suppose that $h(\cdot)$ is an $\boldsymbol{N}$-valued Borel function defined on a Borel set $T_0 \subset S_0$ such that $\mu_0(S_0 \diagdown T_0) = 0$. Let

$$\boldsymbol{\sigma}_\mu = \{h(\xi) \colon \xi \in T_0\}$$

and observe that $pr_m(0)h(\xi) = \xi$ (for $\xi \in T_0$) shows the injectivity of $h(\cdot)$. Therefore $\boldsymbol{\sigma}_\mu$ is a Borel set (see Kuratowski [1], § 35, V). Introducing $\boldsymbol{\omega} = \boldsymbol{\sigma}_\mu$ in (7.63) we obtain

$$\mu(\boldsymbol{\sigma}_\mu) = \int_{S_0} \lambda_\xi(\boldsymbol{\sigma}_\mu)\, d\mu_0(\xi) = \int_{T_0} \lambda_\xi(\boldsymbol{\sigma}_\mu)\, d\mu_0(\xi) = \int_{T_0} \delta_{h(\xi)}(\boldsymbol{\sigma}_\eta)\, d\mu_0(\xi) = \mu_0(T_0) = \mu_0(S_0) = 1\,.$$

Since it is plain that $P_m|\boldsymbol{\sigma}_\mu$ is injective, we finished the proof of Proposition 4.

PROOF OF THE LEMMA. For $w(\cdot) = u(\cdot) - v(\cdot)$ where $u(\cdot) = S(\cdot)u$, $v(\cdot) = S(\cdot)v$, $u, v \in \operatorname{supp}\mu$, we have (a.e. on $(0, \infty)$)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|(I-P_m)w|^2 + \nu\|(I-P_m)w\|^2 &\leqslant |b(u, u, (I-P_m)w) - \\ &\quad - b(v, v, (I-P_m)w)| \leqslant |b(w, u, (I-P_m)w)| + \\ &\quad + |b(v, P_m w, (I-P_m)w)| \leqslant |b((I-P_m)w, u, (I-P_m)w)| + \\ &\quad + |b(P_m w, u, (I-P_m)w)| + |b(v, P_m w, (I-P_m)w)| \leqslant \\ &\leqslant 2\sqrt{2}\cdot c_7|(I-P_m)w|\cdot\|(I-P_m)w\| + \\ &\quad + c_{126}\|(I-P_m)w\|\cdot|P_m w|^{\frac12}\cdot\|P_m w\|^{\frac12} \leqslant \frac{\nu}{2}\|(I-P_m)w\|^2 + \\ &\quad + \frac{8c_7^2}{\nu}|(I-P_m)w|^2 + c'_{126}|P_m w|\cdot\|P_m w\|\,,\end{aligned}$$

whence

$$\frac{d}{dt}|(I-P_m)w|^2 + \nu c_{113}(m - 2c_{123})|(I-P_m)w|^2 \leqslant 2c'_{126}c'_{113}m^{\frac12}|P_m w|^2.$$

By an elementary integration we have for $t \geqslant t_1 \geqslant 0$

$$\begin{aligned}(7.67)\qquad |(I-P_m)w(t)|^2 &\leqslant |(I-P_m)w(t_1)|^2\cdot\exp[-\nu c_{113}(m-2c_{123})(t-t_1)] + \\ &\quad + \int_{t_1}^{t} \exp[-\nu c_{113}(m-2c_{123})(t-\tau)]\, c''_{126} m^{\frac12}|P_m w(\tau)|^2\, d\tau \leqslant \\ &\leqslant |(I-P_m)w(t_1)|^2 \exp[-\nu c_{113}(m-2c_{123})(t-t_1)] + \\ &\quad + c'''_{126}\frac{m^{\frac12}}{m-2c_{123}}\max_{t_1\leqslant\tau\leqslant t}|P_m w(\tau)|^2 \leqslant |(I-P_m)w(t_1)|^2\cdot \\ &\quad \cdot\exp\left[-\frac{\nu c_{113} m}{2}(t-t_1)\right] + \frac{c''''_{126}}{\sqrt{m}}\max_{t_1\leqslant\tau\leqslant t}|(P_m w(\tau)|^2,\end{aligned}$$

by (7.46′); here $c''_{126}-c'''''_{126}$ are some suitable constants (depending only on Ω, ν, $|f|$). On the other hand, a.e. on $(0, \infty)$,

$$\frac{1}{2}\frac{d}{dt}|P_m w|^2+\nu\|P_m w\|^2\leqslant|b(u, u, P_m w)|+|b(v, v, P_m w)|\leqslant$$
$$\leqslant c_{127}|P_m w|^{\frac{1}{2}}\cdot\|P_m w\|^{\frac{3}{2}}\leqslant\nu\|P_m w\|^2+\frac{1}{2}c'_{127},$$

whence

$$|P_m w(t'')|^2\leqslant|P_m w(t')|^2+c'_{127}(t''-t') \tag{7.68}$$

for all $t''\geqslant t'\geqslant 0$; here again c'_{127} depends only on Ω, ν, $|f|$. From (7.67)-(7.68) we infer easily

$$|(I-P_m)w(t_{k-1})|^2\leqslant|(I-P_m)w(t_1)|^2\exp\left[-\frac{\nu c_{113}m}{2}(t-t_1)\right]+$$
$$+\frac{c'''''_{126}}{\sqrt{m}}\max_{1\leqslant j\leqslant k-2}|P_m w(t_j)|^2+\frac{c'''''_{126}c'_{127}}{\sqrt{m}}\max_{1\leqslant j\leqslant k-2}(t_{j+1}-t_j),$$

whence

$$|w(t_{k-1})|^2\leqslant|(I-P_m)w(t_1)|^2\exp\left[-\frac{\nu c_{113}m}{2}(t_{k-1}-t_1)\right]+ \tag{7.69}$$
$$+\left(1+\frac{c'''''_{126}}{\sqrt{m}}\right)\max_{1\leqslant j\leqslant k-2}|P_m w(t_j)|^2+\left(\frac{c''_{126}c'_{127}}{\sqrt{m}}+c'_{127}\right)\max_{1\leqslant j\leqslant k-2}(t_{j+1}-t_j).$$

The relation (7.51), results readily from (7.69). This concludes the proof of the Lemma used in the proof of the preceding Proposition 4.

8. Turbulence and Reynolds equations.

1. ***a***) The first mathematical attempt concerning turbulence was made by O. Reynolds [1] (see also Hinze [1], § 6.1), who replaced the individual solutions $u(t, x)=\{u_1(t, x), u_2(t, x), \ldots, u_n(t, x)\}$ of the Navier-Stokes equations (1.1)-(1.2) (which experimentally were for turbulent flows impossible to determine) by their « mean » value $\overline{u(t, x)}=$ $=\{\overline{u_1(t, x)}, \ldots, \overline{u_n(t, x)}\}$ which differs from the « real » solution by a « turbulent term » $\delta u(t, x)=\{\delta u_1(t, x), \ldots, \delta u_n(t, x)\}$; these entities are related by the Reynolds equations (1.3). As already told in the Introduction, the arguments leading from (1.1)-(1.2) to (1.3) were and still are formal. The aim of the present paragraph 8 is to show that from

the statistical version (3.13_{I}) of the Navier-Stokes equations, the passage to the Reynolds equations is absolutely natural and rigorous and moreover to show how the results of the preceding paragraphs can be subsequently applied to the study of the Reynolds equations and thus to the theory of turbulence.

To this aim, let $\{\mu_t\}_{0<t<T}$ (*) be a statistical solution of the Navier-Stokes equations (with some initial data μ satisfying (3.5)). This means that for any test functional Φ satisfying (3.8′), $\{\mu_t\}_{0<t<T}$ satisfies the equation (3.13_{I}), i.e.

$$(8.1)\qquad \int_0^T\Big\{\int_N[-\Phi'_t(t,u)+\nu((u,\Phi'_u(t,u)))+b(u,u,\Phi'_u(t,u))]\,d\mu_t(u)\Big\}\,dt = \\ =\int_N\Phi(0,u)\,d\mu(u)+\int_0^T\Big[\int_N(f(t),\Phi'_u(t,u))\,d\mu_t(u)\Big]\,dt\,.$$

Let us set

$$(8.2)\qquad \overline{u}=U^\mu=\int_N u\,d\mu(u)$$

and

$$(8.2')\qquad \overline{u(t)}=U^{\mu_t}=\int_N u\,d\mu_t(u)\,,\qquad \text{for } t\in(0,T)\,.$$

By (3.16″), the function (in $t\in(0,T)$)

$$(\overline{u(t)},v)=\int_N(u,v)\,d\mu_t(u)$$

is measurable for all $v\in N$, thus $\overline{u(\cdot)}$ is a (strongly) N-valued measurable function on $(0,T)$; actually, by

$$|\overline{u(t)}|\leqslant\int_N|u|\,d\mu_t(u)\leqslant\Big(\int_N|u|^2\,d\mu_t(u)\Big)^{\frac12},$$

$$\|\overline{u(t)}\|^2\leqslant\Big(\int_N\|u\|\,d\mu_t(u)\Big)^2\leqslant\int_N\|u\|^2\,d\mu_t(u)\,,$$

(*) Firstly we shall consider the case $T<\infty$.

we infer (in virtue of (3.16)-(3.16′)) that

$$\overline{u(\cdot)} = L^\infty(0, T; \boldsymbol{N}) \cap L^2(0, T; \boldsymbol{N}^1) . \tag{8.3}$$

We take in (8.1), the functional

$$\Phi(t, u) = \big(u, v(t)\big) , \qquad \text{for } u \in \boldsymbol{N}^1, \ t \in [0, T] , \tag{8.4}$$

where

$$v(\cdot) \in C_0([0, T); \boldsymbol{N}^1) \cap C_0^1([0, T); \boldsymbol{N}) . \tag{8.4′}$$

Then (8.1) becomes

$$\int_0^T [-\big(\overline{u(t)}, v'(t)\big) + \nu\big(\big(\overline{u(t)}, v(t)\big)\big)]\, dt + \int_0^T \Big[\int_{\boldsymbol{N}} b(u, u, v(t))\, d\mu_t(u)\Big] dt = \\ = (\bar{u}, v(0)) + \int_0^T (f(t), v(t))\, dt . \tag{8.1′}$$

But if $\int_{\boldsymbol{N}} \|u\|^2 d\mu_t(u) < \infty$ (thus if also $\overline{u(t)} \in \boldsymbol{N}^1$), which is true a.e. on $(0, T)$, then

$$\int_{\boldsymbol{N}} b\big(u, u, v(t)\big)\, d\mu_t(u) = \int_{\boldsymbol{N}} b\big(u - \overline{u(t)}, u - \overline{u(t)}, v(t)\big)\, d\mu_t(u) + \\ + \int_{\boldsymbol{N}} b\big(\overline{u(t)}, u - \overline{u(t)}, v(t)\big)\, d\mu_t(u) + \int_{\boldsymbol{N}} b\big(u - \overline{u(t)}, \overline{u(t)}, v(t)\big)\, d\mu_t(u) + \\ + b\big(\overline{u(t)}, \overline{u(t)}, v(t)\big) = \int_{\boldsymbol{N}} b\big(u - \overline{u(t)}, u - \overline{u(t)}, v(t)\big)\, d\mu_t(u) + \\ + b\Big(\int_{\boldsymbol{N}} (u - \overline{u(t)})\, d\mu_t(u), \overline{u(t)}, v(t)\Big) + b\Big(\overline{u(t)}, \int_{\boldsymbol{N}} (u - \overline{u(t)})\, d\mu_t(u), v(t)\Big) + \\ + b\big(\overline{u(t)}, \overline{u(t)}, v(t)\big) = \int_{\boldsymbol{N}} b\big(u - \overline{u(t)}, u - \overline{u(t)}, v(t)\big)\, d\mu_t(u) + \\ + b\big(\overline{u(t)}, \overline{u(t)}, v(t)\big) , \tag{8.4″}$$

where we used the fact that $b(\cdot, \cdot, \cdot)$ is a trilinear continuous functional on $N^1 \times N^1 \times N^1$. In virtue of (8.4″), the equation (8.1′) can be given the following form

$$\int_0^T [-(\overline{u(t)}, v'(t)) + \nu((\overline{u(t)}, v(t))) + b(\overline{u(t)}, \overline{u(t)}, v(t))]\, dt = \tag{8.5}$$

$$= (\bar{u}, v(0)) + \int_0^T \Big[(f(t), v(t)) - \int_N b(u - \overline{u(t)}, u - \overline{u(t)}, v(t))\, d\mu_t(u) \Big]\, dt\,.$$

Let now $v(\cdot)$ be any function satisfying the conditions (i)-(iii) in Sec. 2.3 (i.e. the conditions to which are subjected the test functions in (2.8)). Introducing in (8.5) instead of $v(\cdot)$ the function defined by

$$v_m(t) = m \int_t^{t+1/m} v(\tau)\, d\tau\,, \qquad \Big(\text{hence also } v'_m(t) = m \int_t^{t+1/m} v'(\tau)\, d\tau\Big)$$

on $[0, T-1/m)$ and $v(t) = 0$ for $t \in [T-1/m, T]$ where m is sufficiently large in order that $v(\cdot)$ satisfy (8.4′) and afterwards letting $m \to \infty$, we finally can conclude that (8.5) holds for any function $v(\cdot)$ satisfying the conditions (i)-(iii) in Sec. 2.3. Introducing the bilinear continuous map $B(\cdot, \cdot)$ (from $N^1 \times N^1$ in N^{-1}) defined by (2.4′), we will have that

$$\int_0^T [-(\overline{u(t)}, v'(t)) + \nu((\overline{u(t)}, v(t))) + b(\overline{u(t)}, \overline{u(t)}, v(t))]\, dt = \tag{8.6}$$

$$= (\bar{u}, v(0)) + \int_0^T \Big(f(t) - \overline{B(u - \overline{u(t)}, u - \overline{u(t)})}, v(t) \Big)\, dt$$

where

$$\overline{B(u - \overline{u(t)}, u - \overline{u(t)})} = \int_N B(u - \overline{u(t)}, u - \overline{u(t)})\, d\mu_t(u) \tag{8.6′}$$

is taken in N^{-1} for those values of t for which $\int_N \|u\|^2 d\mu_t(u) < \infty$ (i.e. a.e. on $(0, T)$); moreover in (8.6), $v(\cdot)$ is any function satisfying the conditions (i)-(iii) in Sec. 2.3.

DEFINITION. *A (weak) nonstationary solution of the Reynolds equations on* $(0, T)$ *is, by definition, a function* $\overline{u(\cdot)}$ *satisfying* (8.3) *and a* N*-valued stochastic process* $\{\delta u(t)\}_{0<t<T}$ *such that*

$$(8.7) \qquad \begin{cases} \overline{\delta u(t)}(=\text{Mean of } \delta u(t)) = 0 \\ \overline{\|\delta u(\cdot)\|^2}(=\text{Mean of } \|\delta u(\cdot)\|^2) \in L^1(0, T) \end{cases}$$

and that

$$(8.8) \qquad \int_0^T [-(\overline{u(t)}, v'(t)) + \nu((\overline{u(t)}. v(t))) + b(\overline{u(t)}, \overline{u(t)}, v(t))]\, dt =$$

$$= (\overline{u}_0, v(0)) + \int_0^T \left(f(t) - \overline{B(\delta u(t), \delta u(t))}, v(t)\right) dt$$

for some $\overline{u}_0 \in N$ *and all functions* $v(\cdot)$ *satisfying the conditions* (i)-(iii) *in Sec.* 2.3. *In case* $f(\cdot)$ *does not depend on* t, a weak stationary solution of the Reynolds equations is by definition a solution $\{\overline{u(\cdot)}, \delta u(\cdot)\}$ which does not depend on t, that is (*) *a stationary solution of the Reynolds equations (with right term* f *independent of* t*) is a* $u \in N^1$ *and a random variable* δu *in* N *such that*

$$(8.7') \qquad \overline{\delta u} = 0, \quad \overline{\|\delta u\|^2} < \infty,$$

and

$$(8.8') \qquad \nu((\overline{u}, v)) + b(\overline{u}, \overline{u}, v) = (f - \overline{B(\delta u, \delta u)}, v), \quad \textit{for all } v \in N^1.$$

A nonstationary solution of the Reynolds equations, resp. a stationary one, will be called real *if moreover the family of probabilities* $\{\mu_t'\}_{0<t<\infty}$, *resp. the probability* μ', *defined on* N *by*

$$(8.9) \qquad \mu_t'(\boldsymbol{\omega}) = \text{Prob}\,(\{\delta u + \overline{u(t)} \in \boldsymbol{\omega}\})$$

$$(8.9') \qquad \mu'(\boldsymbol{\omega}) = \text{Prob}\,(\{\delta u + \overline{u} \in \boldsymbol{\omega}\})$$

(*) We let to the reader the care to verify this (along the same line as for the similar fact concerning the statistical solutions; see Sec. 6.1).

(*where* $\boldsymbol{\omega}$ *is a Borel set* $\subset \boldsymbol{N}$) *is a statistical solution of the Navier-Stokes with some initial data* μ_0 *satisfying* (3.5) *and* $\int_{\boldsymbol{N}} u\, d\mu(u) = \bar{u}_0$, *resp. a stationary statistical solution of the Navier-Stokes equations.*

It is clear that physics must be interested only in real solutions of the Reynolds equations. Also it is plain that the equation (8.8) is with regard to (1.3), as (2.8) is with regard to (1.1)-(1.2). Usually $\bar{u}$ (or $\overline{u(t)}$ in the nonstationary case) is called the *mean velocity.*

We conclude this Section with the following simple but basic

THEOREM 1. (*a*) *Let* $\{\mu_t\}_{0<t<\infty}$ *be a statistical solution of the Navier-Stokes equations with initial data* μ *satisfying* (3.5). *Define* $\overline{u(t)}$ *for* $0<t<T$ *and* $\bar{u}$ *as in* (8.2)-(8.2′). *Then there exists a stochastic process* $\{\delta u(t)\}_{0<t<T}$ *such that* $\{\overline{u(\cdot)}, \delta u(\cdot)\}$ *be a real solution of the Reynolds equations such that*

$$\mu_t' = \mu_t \qquad \textit{for } 0<t<T \ \text{(see (8.9))}\,. \tag{8.10}$$

(*b*) *Let* μ *be a stationary statistical solution of the Navier-Stokes equations* (with right term $f \in \boldsymbol{N}$ independent of t), *let* $\bar{u} = U^{\mu}$ *and* $\delta u =$ *= the random variable* $u \mapsto u$ *defined on* $\boldsymbol{N}$, *endowed with the probability* $\operatorname{Prob}(\{\delta u \in \boldsymbol{\omega}\}) = \mu(\boldsymbol{\omega} + \bar{u})$, $\boldsymbol{\omega}$ *Borel set* $\subset \boldsymbol{N}$. *Then* $\{\bar{u}, \delta u\}$ *is a real stationary solution of the Reynolds equations such that*

$$\mu = \mu' \qquad \text{(see (8.9′))}\,. \tag{8.10′}$$

PROOF. We shall prove only (*a*) since (*b*) can be proved much simpler. Concerning (*a*), define on $X = \prod_{t\in(0,T)} X_t$, where $X_t = \boldsymbol{N}$ for all $t \in (0, T)$, the product probability P of the Borel probabilities μ_t on X_t (see Loève [1], Part I, Ch. I, § 4.2) and for any $t \in (0, T)$, define on X the function $\delta u(t)$ by

$$\delta u(t)\big(\{u_t\}_{0<t<T}\big) = u_t - \overline{u(t)}\,.$$

Then obviously $\{\delta u(t)\}_{0<t<T}$ satisfies (8.7) and

$$\operatorname{Prob}\big(\{\delta u(t) + \overline{u(t)} \in \boldsymbol{\omega}\}\big) = \operatorname{Prob}\big(\{\{u_t\}_{0<t<T}\colon u_t \in \boldsymbol{\omega}\}\big) = \mu_t(\boldsymbol{\omega})$$

for any Borel set $\boldsymbol{\omega} \subset \boldsymbol{N}$, this $\{\delta u(t)\}_{0<t<T}$ satisfies also (8.9)-(8.10).

Finally in virtue of

$$\overline{B(\delta u(t),\, \delta u(t))} = \text{Mean}\,\big(B(\delta u(t),\, \delta u(t))\big) = \text{Mean}\,\Big(B\big(u_t - \overline{u(t)},\, u_t - \overline{u(t)}\big)\Big) = \\ = \int_{N} B\big(u - \overline{u(t)},\, u - \overline{u(t)}\big)\, d\mu_t(u)\;,$$

the equation (8.8) follows directly from (8.6)-(8.6′); with this we conclude the proof of the theorem.

In virtue of the preceding Theorem 1 the study of real solutions of the Reynolds equations, concerns the statistical solutions of the Navier-Stokes equations. In the next Sections we shall present some properties of the statistical solutions, the meaning of which is better understood if they are considered as properties of the real solutions of the Reynolds equations (*).

b) Our definition of a real stationary solution of the Reynolds equations can be also justified by a further argument. Namely, our definition allows a rigorous proof for the deduction of the time independent Reynolds equations (and its solutions, i.e. the real stationary solutions) by the time-average method (see also Monin-Yaglom [1], §§ 4.7 and 5.1).

Let us recall this formal approach to the Reynolds equations. Let $u(\cdot)$ be an individual solution on $(0, \infty)$ of the Navier-Stokes equations with initial data $u_0 \in \boldsymbol{N}$. Set

$$(8.11) \qquad M_{t_0}u(t) = \frac{1}{t_0}\int_0^{t_0} u(\tau + t)\, d\tau\;, \qquad 0 \leqslant t < \infty\;,\; 0 < t_0 < \infty\;.$$

We are allowed to suppose that as statistical solution (i.e. $\{\delta_{u(t)}\}_{0<t<\infty}$) our solution satisfies the strengthened energy equation (see Sec. 3.4 and 4.1). Then $u(\cdot) \in L^\infty(0, \infty; \boldsymbol{N})$ thus

$$(8.11') \qquad |M_{t_0}u(t)| \leqslant c_{128} \qquad \text{for all } t \geqslant 0\;,\; t_0 > 0\;,$$

(*) From now on, throughout the present paragraph 8, we shall suppose that the right term f is time independent and belongs to $\boldsymbol{N}$; henceforth we will be concerned with solutions on $(0, \infty)$, that is, with entities defined on the whole of $(0, \infty)$ such that their restriction to any $(0, T)$, $T < \infty$, are solutions in the sense of our definitions.

and

$$(8.11'') \qquad \|M_{t_0}u(\cdot)\| \leqslant c'_{128} \qquad \text{for all } t_0 > 0 \,,$$

where $c_{128} - c'_{128}$ are some suitable large constants (depending on $u(\cdot)$) Since moreover $u(\cdot) \in L^2_{\text{loc}}(0, \infty; N^1)$, $M_t u(\cdot)$ is an N-valued absolutely continuous functions such that

$$(8.11''') \qquad (M_{t_0}u)'_{\square}(\boxed{t}) \in L^2_{\text{loc}}(0, \infty; N^1) \cap L^\infty(0, \infty; N) \,.$$

On the other hand, in N^{-2} we have

$$(8.12) \qquad (M_{t_0}u)'_{\square}(\boxed{t}) = \frac{1}{t_0}\int_0^{t_0} u'_{\square}(\boxed{t+\tau})\, d\tau =$$

$$= \frac{1}{t_0}\int_0^{t_0} [f - \nu D_e u(t+\tau) - B(u(t+\tau), u(t+\tau))]\, d\tau =$$

$$= f - \nu D_e M_t u(t) - \frac{1}{t_0}\int_0^{t_0} B(u(t+\tau), u(t+\tau))\, d\tau$$

so that if $v(\cdot) \in C^1([0, \infty); N) \cap C_0([0, \infty); N^2)$ then, by (8.12),

$$(8.13) \qquad \frac{d}{dt}(M_{t_0}u(t), v(t)) - (M_{t_0}u(t), v'(t)) + \nu((M_{t_0}u(t), v(t))) +$$

$$+ \frac{1}{t_0}\int_0^{t_0} b(u(t+\tau), u(t+\tau), v(\tau))\, d\tau = (f, v(t)) \,,$$

where the integral is equal to (see, as sample, the computations (8.4''))

$$b(M_{t_0}u(t), M_{t_0}u(t), v(t)) + \frac{1}{t_0}\int_0^{t_0} b(u(t+\tau) - M_{t_0}u(t), u(t+\tau) -$$

$$- M_{t_0}u(t), v(t))\, dt = b(M_{t_0}u(t), M_{t_0}u(t), v(t)) +$$

$$+ \left(\frac{1}{t_0}\int_0^{t_0} B(u(t+\tau) - M_{t_0}u(t), u(t+\tau) - M_{t_0}u(t))\, d\tau, v(t)\right) .$$

Therefore integrating (8.14) we finally get the equation

$$\int_0^\infty \left[-\left(M_{t_0}u(t), v'(t)\right) + \nu\left(\left(M_{t_0}u(t), v(t)\right)\right) + \right. \tag{8.14}$$

$$\left. + b\left(M_{t_0}u(t), M_{t_0}u(t), v(t)\right)\right] dt = \left(M_{t_0}u(0), v(0)\right) +$$

$$+ \int_0^{t_0} \left(f - \frac{1}{t_0}\int_0^{t_0} B\left(u(t+\tau) - M_{t_0}(t), u(t+\tau) - M_{t_0}u(t)\right) d\tau, v(t)\right) dt$$

where $v(\cdot)$ is an arbitrary function satisfying

$$v(\cdot) \in C^1([0, \infty); N) \cap C_0([0, \infty); N^2) . \tag{8.14'}$$

(One should not forget that $N^2 = D_D$; see Sec. 2.4). At this point, in any treatise on hydrodynamics where this approach is presented (of course with different notations, etc.; see for instance: Schlichting [1], Ch. XVIII, §§ 2-4) the equation (8.8′) is formally obtained from (8.14) simply letting $t_0 \to \infty$. As far as we know there is no proof in the literature that this passage from (8.14) to (8.8′) is permitted. Our more careful approach to the Reynolds equations will allow us to prove the permissibility of the passage at least if the space dimension n is $= 2$.

To this aim note firstly that (8.11″) implies that there exists a sequence $0 < t_1 < t_2 < \ldots < t_j \to \infty$ such that, for some $u^* \in N^1$,

$$M_{t_j}u(0) \to u^*, \quad \text{in } N^1 \text{ weakly}, \tag{8.15}$$

that is,

$$\left(\left(M_{t_j}u(0), v\right)\right) \to \left(\left(u^*, v\right)\right), \quad \text{for all } v \in N^1. \tag{8.15'}$$

Since the imbedding $N^1 \subset N$ is compact, (8.15) implies that

$$M_{t_j}u(0) \to u^*, \quad \text{in } N \text{ (strongly)}. \tag{8.15''}$$

Note now that

$$\left\| \frac{1}{t_j}\int_0^{t_j} B\left(u(\tau) - u^*, u(\tau) - u^*\right) d\tau \right\|_{N^{-1}} \leqslant$$

$$\leqslant \frac{1}{t_j}\int_0^{t_j} \|B(u(\tau)-u^*, u(\tau)-u^*)\|_{N^{-1}} d\tau \leqslant$$

$$\leqslant c_1 \frac{1}{t_j}\int_0^{t_j} \|u(\tau)-u^*\|^2 d\tau \leqslant 2c_1\left(\frac{1}{t_j}\int_0^{t_j}\|u(\tau)\|^2 d\tau + \|u^*\|^2\right) \leqslant c_{129}$$

where c_{129} is a convenient constant; the last boundness is a consequence of the fact that $u(\cdot)$ satisfies the energy inequality. In virtue of the established boundness, we can suppose (passing to a subsequence if necessary) that

$$(8.16) \qquad \frac{1}{t_j}\int_0^{t_j} B(u(\tau)-u^*, u(\tau)-u^*)\, d\tau \to b\,, \qquad \text{in } N^{-1} \text{ weakly}$$

for some $b \in N^{-1}$, that is

$$(8.16') \qquad \frac{1}{t_j}\int_0^{t_j} b(u(\tau)-u^*, u(\tau)-u^*, v)\, d\tau \to (b, v)\,, \qquad \text{for all } v \in N^1.$$

Since

$$(8.17) \qquad M_{t_j}u(t) = M_{t_j}u(0) + \frac{1}{t_j}\int_{t_j}^{t_j+t} u(\tau)\, d\tau - \frac{1}{t_j}\int_0^t u(\tau)\, d\tau$$

we have for all $j = 1, 2, \ldots$ and $t \geqslant 0$

$$(8.17') \qquad |M_{t_j}u(t) - M_{t_j}u(0)| \leqslant \frac{tc_{130}}{t_j}$$

thus, by (8.15″), for $j \to \infty$

$$(8.18) \qquad M_{t_j}u(t) \to u^* \text{ in } N, \qquad \text{uniformly on } [0, T]\,,$$

for any $T \in (0, \infty)$.

In virtue of the Remark 3° in Sec. 6.1.*b*), there exists $\mathfrak{M} \in \mathcal{M}(\{\delta_{u(t)}\}_{0<t<\infty})$ such that

$$(8.19) \qquad \mathfrak{M}(\Phi) = \lim_{j\to\infty} \frac{1}{t_j}\int_0^{t_j} \Phi(u(\tau))\, d\tau$$

for all $\Phi \in \mathcal{C}_{1,1}$ for which the last limit exists. Fix such an $\mathfrak{M}$ and let μ denote the stationary statistical solution of the Navier-Stokes equations corresponding to $\mathfrak{M}$ (i.e. related to $\mathfrak{M}$ by the relation

$$\mathfrak{M}(\Phi) = \int_N \Phi(u)\, d\mu(u)\,, \qquad \text{for all } \Phi \in \mathcal{C}_2;$$

see Sec. 6.1.b)). Then for, any $v \in \boldsymbol{N}$,

$$(u^*, v) = \lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} (u(\tau), v)\, d\tau = \mathfrak{M}((\cdot, v)) = \int_N (u, v)\, d\mu(u) = (U^\mu, v)\,,$$

so that

$$u^* = U^\mu = \int_N u\, d\mu(u)\,. \tag{8.20}$$

Using (8.16), (8.19) and (6.17) (with the choice $\mathfrak{M}^* = \mathfrak{M}$, $\mu^* = \mu$ and $\Phi(\cdot) = (\cdot, v)$, $v \in \boldsymbol{N}^2$) we infer that

$$(b, v) = \mathfrak{M}(b(\cdot, \cdot, v)) = \int_N b(u, u, v)\, d\mu(u)$$

for all $v \in \boldsymbol{N}^1$, that is

$$\frac{1}{t_j} \int_0^{t_j} B(u(\tau) - u^*, u(\tau) - u^*)\, d\tau \to \to \overline{B(u - \overline{u}, u - \overline{u})}\,, \quad \text{in } \boldsymbol{N}^{-1} \text{ weakly}\,, \tag{8.20$'$}$$

where

$$\overline{u} = U = u^* \quad \text{and } \overline{B(u - \overline{u}, u - \overline{u})} = \int_N B(u - \overline{u}, u - \overline{u})\, d\mu(u)\,. \tag{8.20$''$}$$

In this manner, *any limit point for the time average of the velocities* (see (8.11)) *is the mean velocity* $\overline{u}$ $(= U^\mu)$ *of a real stationary solution of the Reynolds equations* (namely that corresponding to μ). *This solution can be choosen in such a way that* (8.20′) *shall hold.* Plainly these conclusions hold for both cases $n = 2$ and $n = 3$, i.e. for the plane fluids as well as for the three-dimensional fluids.

In the case $n = 2$ we shall prove now *that taking in* (8.14) *the values* t_j *for* t_0 *and afterwards letting* $j \to \infty$, *one obtains the relations* (8.7′)-(8.8′), with the solution $\{\overline{u}, \delta u\}$ corresponding to the stationary statistical solution μ. Thus let $n = 2$. Then by (2.16) we will have here

$$\|M_{t_0} u(t)\| \leqslant c_{131} \qquad \text{for } t_0 > 0 \tag{8.21}$$

and also, by (8.17),

$$\|M_{t_j} u(t) - M_{t_j} u(0)\| \leqslant \frac{c'_{131}(1+t)}{t_j}, \qquad \text{for } j = 1, 2, . \tag{8.21'}$$

and all $t \geqslant 0$. It is plain that by (8.15) and (8.21′) we have, for $j \to \infty$,

$$-\int_0^\infty (M_{t_j} u(t), v'(t))\,dt - (M_{t_j} u(0), v(0)) \to -\int_0^\infty (u^*, v'(t))\,dt + (u^*, v(0)) = 0\,,$$

$$\int_0^\infty ((M_{t_j} u(t), v(t)))\,dt \to \int_0^\infty ((u^*, v(t)))\,dt\,,$$

and moreover, using also (8.17′),

$$\left|\int_0^\infty b(M_{t_j} u(t), M_{t_j} u(t), v(t))\,dt - \int_0^\infty b(u^*, u^*, v(t))\,dt\right| \leqslant$$

$$\leqslant \int_0^\infty [|b(M_{t_j} u(t) - u^*, M_{t_j} u(t), v(t))| + |b(u^*, M_{t_j} u(t) - u^*, v(t))|]\,dt \leqslant$$

$$\leqslant c_{132} \int_0^\infty \|v(t)\| \cdot |M_{t_j} u(t) - u^*|^{\frac12} dt \leqslant c'_{132} \int_0^\infty \|v(t)\| \left(\frac{1+t}{t_j} + |M_{t_j} u(0) - u^*|\right)^{\frac12} dt \to 0$$

where $c_{132} - c'_{132}$ are some convenient constants. Thus taking $v(\cdot) = \varrho(\cdot) v$, with $\varrho \in C_0^1([0, \infty))$, $v \in \boldsymbol{D}_D$, and $t_0 = t_j$ in (8.14) and letting $j \to \infty$ we arrive to

$$\int_0^\infty \varrho(t)[\nu((\overline{u}, v)) + b(\overline{u}, \overline{u}, v) - (f, v)]\,dt = \tag{8.22}$$

$$= \lim_{j\to\infty} \int_0^\infty \varrho(t) \left(\frac{1}{t_j} \int_0^{t_j} B(u(t+\tau) - M_{t_j} u(t), u(t+\tau) - M_{t_j} u(t))\,d\tau, v\right) dt$$

where $\varrho(\cdot)\in C_0^1([0,\infty))$ and $v\in \boldsymbol{D}_D$ are arbitrary and $\bar{u}=u^*$. But

$$\left|\left(\frac{1}{t_j}\int_0^{t_j} B\big(u(t+\tau)-\bar{u},\, u(t+\tau)-\bar{u}\big)\,d\tau,\, v\right)-\right.$$
$$\left.-\left(\frac{1}{t_j}\int_0^{t_j} B\big(u(t+\tau)-M_{t_j}u(t),\, u(t+\tau)-M_{t_j}u(t)\big)\,d\tau,\, v\right)\right|\leqslant$$
$$\leqslant\frac{1}{t_j}\int_0^{t_j}\big[|b\big(u^*-M_{t_j}u(t),\, u(t+\tau)-M_{t_j}u(t),\, v\big)|+$$
$$+\,|b\big(u(t+\tau)-u^*,\, u^*-M_{t_j}u(t),\, v\big)|\big]\,d\tau\leqslant$$
$$\leqslant\frac{1}{t_j}\int_0^{t_j} c_{133}|u^*-M_{t_j}u(t)|^{\frac{1}{2}}\big[\|u(t+\tau)\|+c_{131}\big]\,d\tau\leqslant c'_{133}|u^*-M_{t_j}u(t)|$$

and

$$\left|\left(\frac{1}{t_j}\int_0^{t_j} B\big(u(t+\tau)-\bar{u},\, u(t+\tau)-\bar{u}\big)\,d\tau,\, v\right)\right|-$$
$$-\left(\frac{1}{t_j}\int_0^{t_j} B\big(u(\tau)-\bar{u},\, u(\tau)-\bar{u}\big)\,d\tau,\, v\right)\Bigg|\leqslant\frac{1}{t_j}\int_0^{t}|b\big(u(\tau)-u^*,\, u(\tau)-u^*,\, v\big)|\,d\tau+$$
$$+\frac{1}{t_j}\int_{t_j}^{t_j+t}|b\big(u(\tau)-u^*,\, u(\tau)-u^*,\, v\big)\,dt\leqslant\frac{c_{134}(1+t)}{t_j}$$

(for some convenient constant c_{134}), so that, by (8.17′) and (8.20′), we can give to (8.22) the form

$$\text{(8.22}'\text{)}\qquad \int_0^{\infty}\varrho(t)\big[\nu(\!(\bar{u}, v)\!)+b(\bar{u},\bar{u},v)-(f,v)+\big(\overline{B(u-\bar{u},\, u-\bar{u})},\, v\big)\big]\,dt=0\,.$$

Since (8.22′) is true for all $\varrho\in C_0^1([0,\infty))$ we deduce finally that

$$\nu(\!(\bar{u}, v)\!)+b(\bar{u},\bar{u},v)=\big(f-\overline{B(u-\bar{u},\, u-\bar{u})},\, v\big)$$

is valid for all $v \in \boldsymbol{D}_D$ thus also for all $v \in \boldsymbol{N}^1$. This finishes the proof of our assertion.

2. ***a***) In this Section we shall give some properties concerning the stationary solutions of the Reynolds equations. We begin with the following

PROPOSITION 1. *Let $\{\bar{u}, \delta u\}$ be a real stationary solution of the Reynolds equations. Then*

$$\overline{\|\delta u\|^2} \leqslant \nu^{-1}\left(\overline{B(\delta u, \delta u)}, \bar{u}\right) . \tag{8.23}$$

PROOF. Let μ be the stationary solution of the Navier-Stokes equations to which corresponds $\bar{u}$ and δu. Then

$$\begin{cases} \bar{u} = \int\limits_{\boldsymbol{N}} u \, d\mu(u) , \\ \overline{\|\delta u\|^2} = \int\limits_{\boldsymbol{N}} \|u - \bar{u}\|^2 d\mu(u) = \int\limits_{\boldsymbol{N}} \|u\|^2 d\mu(u) - \|\bar{u}\|^2. \end{cases} \tag{8.24}$$

On the other side, since μ satisfies the energy inequality (namely (6.3) with $\psi(\xi) \equiv \xi$) we have

$$\nu \int\limits_{\boldsymbol{N}} \|u\|^2 d\mu(u) \leqslant \int\limits_{\boldsymbol{N}} (f, u) \, d\mu(u) . \tag{8.24$'$}$$

Replacing in (8.8′), v by $\bar{u}$ (which is permitted since $\bar{u} \in \boldsymbol{N}^1$) we obtain

$$\nu \|\bar{u}\|^2 = (f, \bar{u}) - \left(\overline{B(\delta u, \delta u)}, \bar{u}\right) , \tag{8.25}$$

whence, by (8.24)-(8.24′), it results (8.23).

REMARK. 1°. In spite of its simplicity, the relation (8.23) has a certain physical meaning. Indeed $\left(\overline{B(\delta u, \delta u)}, \bar{u}\right)$ represents the amount of the energy transfer from the mean velocity $\bar{u}$ to the turbulent fluctuations by work done agains the so called Reynolds stresses (see Hunt [1], Sec. 6.2.). Thus (8.23) shows that *if there is no energy transfer from the mean flow to the turbulent fluctuations then* the stationary solution is a Dirac measure (namely $\delta_{\bar{u}}$) that is *there is no turbulent fluctuation.*

We pass now to out main result in this Section. It seems reasonable to measure the degree of turbulence by the amount of the energy transfered from the mean velocity $\bar{u}$ to the turbulent fluctuations, i.e. by $(\overline{B(\delta u, \delta u)}, \bar{u})$. There are physicists who believe that to determine the turbulent flow of a viscous incompressible fluid one has to search for that one, among all possible ones, which maximizes the dissipation of energy (see Malkus [1], Sec. 3 and Nihoul [1]).

In our case this principle would ask to determine a real solution of the Reynolds equation which maximizes $(\overline{B(\delta u, \delta u)}, \bar{u})$. Our next theorem yields a definitive rigorous mathematical background to this principle:

THEOREM 2. *Among all real stationary solutions $\{u, \delta u\}$ of the Reynolds equations there exists at least one $\{u_{\max}, \delta u_{\max}\}$ such that*

$$(\overline{B(\delta u_{\max}, \delta u_{\max})}, \bar{u}_{\max}) \geqslant (\overline{B(\delta u, \delta u)}, \bar{u}) \tag{8.26}$$

for all $\{\bar{u}, \delta u\}$. Moreover $\bar{u}_{\max}$ and $\overline{B(\delta u_{\max}, \delta u_{\max})}$ are uniquely determined (though the corresponding stationary statistical solution yielding $\{u_{\max}, \delta u_{\max}\}$ might not be uniquely determined).

The proof of this theorem is a direct consequence of the following

LEMMA. *The set $\boldsymbol{MV}$ of all mean velocities $\bar{u}$ occurring in all real stationaty solutions $\{\bar{u}, \delta u\}$ of the Reynolds equations is a convex compact set in $\boldsymbol{N}$.*

PROOF. Obviously $\boldsymbol{MV} = \{U^{\mu}\}$ where μ runs over the set $\mathcal{S}$ of all stationary statistical solutions of the Navier-Stokes equations, so that, again obviously, $\boldsymbol{MV}$ is convex. Let now $u_0 \in \overline{\boldsymbol{MV}}$. Then there exists a sequence $\{\mu_j\}_{j=1}^{\infty}$ of stationary statistical solutions such that $U^{\mu_j} \to u_0$, in $\boldsymbol{N}$, for $j \to \infty$. Let $\mathfrak{M}_{\mu}$ be a cluster in $\mathcal{S}$ (considered in $\mathcal{C}^*_{2:1,1}$; see Sec. 6.4.*a*)) of the sequence $\{\mathfrak{M}_{\mu_j}\}_{j=1}^{\infty}$. Since $(\cdot, v) \in \mathcal{C}_2$ and (for $j \to \infty$)

$$\mathfrak{M}_{\mu_j}((\cdot, v)) \to (u_0, v)\,, \qquad \text{for all } v \in \boldsymbol{N},$$

we infer that

$$(u_0, v) = \mathfrak{M}_\mu((\cdot, v)) = \int_N (u, v)\, d\mu(u) , \qquad \text{for all } v \in N;$$

thus $u_0 = U^\mu \in MV$. This finishes the proof of the Lemma.

PROOF OF THEOREM 2. Set

$$\lambda = \sup\, (\overline{B(\delta u, \delta u)}, \overline{u}) \tag{8.27}$$

where the supremum is taken with respect to all real stationary solutions $\{\overline{u}, \delta u\}$ of the Reynolds equations. By (8.25)

$$-\lambda = \inf[\nu\|\overline{u}\|^2 - (f, \overline{u})] \tag{8.27'}$$

where $\overline{u}$ runs over the whole of MV (see the Lemma above). Let now $\{\overline{u}_j\}_{j=1}^\infty \subset MV$ be chosen such that

$$\nu\|\overline{u}_j\|^2 - (f, \overline{u}_j) \to -\lambda , \qquad \text{for } j \to \infty . \tag{8.27''}$$

Passing if necessary to a subsequence we can suppose also (in virtue of the Lemma) that $\overline{u}_j \to \overline{u}_0$ in N, for some $\overline{u}_0 \in MV$, that is, for some real stationary solution $\{\overline{u}_0, \delta u_0\}$. Consequently

$$\nu\|P_m\overline{u}_0\|^2 - (f, \overline{u}_0) = \lim_{j\to\infty} [\nu\|P_m\overline{u}_j\|^2 - (f, \overline{u}_j)] \leqslant$$
$$\leqslant \lim_{j\to\infty} [\nu\|\overline{u}_j\|^2 - (f, \overline{u}_j)] = -\lambda ,$$

for all $m = 1, 2, \ldots$. Letting $m \to \infty$ we obtain $\nu\|\overline{u}_0\|^2 - (f, \overline{u}_0) \leqslant -\lambda$ hence $\nu\|\overline{u}_0\|^2 - (f, \overline{u}_0) = -\lambda$. By (8.25) we have

$$(\overline{B(\delta u_0, \delta u_0)}, \overline{u}_0) = \lambda . \tag{8.27''}$$

This establishes the existence of $\{u_{\max}, \delta u_{\max}\}$. It remains to prove only the uniqueness of $\overline{u}_{\max}$. This results readily from the fact that $\|\cdot\|$ is a strict convex function on MV, thus the proof is complete.

In the same manner one can prove the following theorem similar to Theorem 2:

THEOREM 2'. *Among all real stationary solutions $\{\bar{u}, \delta u\}$ of the Reynolds equations there exists at least one $\{\bar{u}_{\min}, \delta u_{\min}\}$ such that*

$$\|\bar{u}_{\min}\| \leqslant \|\bar{u}\| \tag{8.28'}$$

for all $\{\bar{u}, \delta u\}$. Moreover $\bar{u}_{\min}$ and $\overline{B(\delta u_{\min}, \delta u_{\min})}$ are uniquely determined (though the corresponding stationary statistical solution yielding $\{u_{\min}, \delta u_{\min}\}$ might not be uniquely determined).

REMARKS. 2°. A simple (and perhaps strange) connection between $\bar{u}_{\min}$ and $\bar{u}_{\max}$ is the following:

$$(f, \bar{u}_{\max} - \bar{u}_{\min}) \geqslant 0 \,, \tag{8.28}$$

the equality holding if and only if $\bar{u}_{\max} = \bar{u}_{\min}$. Indeed (8.28) results readily from

$$\nu\|\bar{u}_{\min}\|^2 \leqslant \nu\|\bar{u}_{\max}\|^2 \,, \qquad \nu\|\bar{u}_{\max}\|^2 - (f, \bar{u}_{\max}) \leqslant \nu\|\bar{u}_{\min}\|^2 - (f, \bar{u}_{\min}) \,,$$

moreover if in (8.28), we have equality, then $\|\bar{u}_{\max}\| = \|\bar{u}_{\min}\|$ thus by the uniqueness of $\bar{u}_{\min}$, we will have $\bar{u}_{\min} = \bar{u}_{\max}$.

3°. One should notice that from a physical point of view, the mean velocity $\bar{u}_{\min}$ is also distinguished since it minimizes the total dissipated energy; indeed for $u \in \boldsymbol{N}^1$ we have

$$\|u\|^2 = \sum_{i,j=1}^{n} \frac{1}{2} \int_{\Omega} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 dx = \int_{\Omega} |\operatorname{rot} u(x)|^2 dx \tag{8.29}$$

and the right term in (8.29) measures (up to a constant factor) this energy (see Landau-Lifschitz [1], § 16). For (8.29) we send the reader to Serrin [1], formula (72.1').

b) The aim of this Section is to show that to some extent the preceding results concerning the stationary case have analogues in the nonstationary case. In order to avoid some artificial difficulties we shall still suppose that the right term is time-independent.

THEOREM 3. *Let μ be a (Borel) probability on $\boldsymbol{N}$ satisfying* (3.5) *and let $T > 0$. Among all statistical solutions $\{\mu_t\}_{0<t<T}$ of the Navier-*

Stokes equations satisfying the strengthened energy inequality, with initial data μ, *there exists at least one* $\{\mu_t^{\min}\}_{0<t<T}$ *minimizing*

$$(8.30)\qquad \int_0^T \Big\| \int_N u\, d\mu_t(u) \Big\|^2 dt\,.$$

If $\{\overline{u(\cdot)^{\min}}, \delta u(\cdot)^{\min}\}$ *denotes the solution of the Reynolds corresponding to* $\{\mu_t^{\min}\}_{0<t<T}$ *then* $u^{\min}(t)$ *and* $\overline{B(\delta u(t)^{\min}, \delta u(t)^{\min})}$ *are uniquely determined a.e. on* $(0, T)$.

PROOF. The proof will follow the same line as that of Theorem 2, Sec. 8.2.*a*). We start by choosing a sequence $\big\{\{\mu_t^{(j)}\}_{0<t<T}\big\}_{j=1}^{\infty} \subset \mathcal{S}^\mu$ (see Sec. 4.2.*a*)) such that

$$\left\{ \int_0^T \Big\| \int_N u\, d\mu_t(u) \Big\|^2 dt \right\}_{j=1}^{\infty}$$

converges to the infimum λ^2 of the integrals (8.30) where $\{\mu_t\}$ runs over $\mathcal{S}^\mu$. In virtue of Theorem 1 in Sec. 4.2.*a*), $\mathcal{S}^\mu$ considered as a subset of $\mathfrak{L}^*_{1,1;2}$ is a compact convex set, thus there exists a $\{\mu_t\}_{0<t<T} \in \mathcal{S}^\mu$ which is a w^*-cluster point in $\mathfrak{L}^*_{1,1}$ of the sequence $\big\{\{\mu_t^{(j)}\}_{0<t<T}\big\}_{j=1}^{\infty}$. Since for $v(\cdot) \in L^2(0, T; N^1)$, the function $\Psi(t, \cdot) = ((\cdot, v(t)))$ in t, belongs to $L^2(0, T; \mathcal{C}_{1,1}) = \mathfrak{L}_{1,1}$ it results without difficulty that

$$(8.31)\qquad \int_N u\, d\mu_\bullet(u) = \overline{u(\cdot)}$$

is a cluster point of $\{\overline{u^{(j)}(\cdot)} = \int_N u\, d\mu_\bullet^{(j)}(u)\}_{j=1}^{\infty}$ in the weak topology of $L^2(0, T; N^1)$. Since weak convergence in a Hilbert space does not increase the norms we will have

$$\|\overline{u(\cdot)}\|_{L^2(0,T;N^1)} \leqslant \lim \|\overline{u^{(j)}(\cdot)}\|_{L^2(0,T;N^1)} = \lambda\,.$$

so that in virtue of the definition of λ,

$$\lambda^2 \leqslant \int_0^T \|\overline{u(t)}\|^2 dt = \|\overline{u(\cdot)}\|^2_{L^2(0,T;N^1)} \leqslant \lambda^2,$$

thus

$$\int_0^T \overline{\|u(t)\|^2}\, dt = \lambda^2. \tag{8.31'}$$

It remains to prove the uniqueness part of the theorem. For $\overline{u_{\min}(\cdot)}$, this results readily since the norm is strictly convex in $L^2(0, T; N^1)$ and the set

$$\left\{\int_N u\, d\mu_\bullet(u) : \{\mu_t\}_{0<t<T} \in S^\mu\right\}$$

is convex in $L^2(0, T; N^1)$, while for $\overline{B(\delta u(t)^{\min}, \delta u(t)^{\min})}$ this follows directly from (8.8) and the uniqueness of $\overline{u^{\min}(\cdot)}$.

This concludes the proof of Theorem 3.

REMARK. 1°. Similarly to the stationary case, one can consider other extremum problems for the mean velocities $\overline{u(\cdot)}$. For instance in Foias [1] was considered instead the integrals (8.30) the integrals (see Foias [1], § 6)

$$\int_0^T \left[\frac{\overline{|u(t)|^2}}{2} + \nu \int_0^t \|\overline{u(\tau)}\|^2 d\tau\right] dt \tag{8.30'}$$

where

$$\overline{u(t)} = \int_N u\, d\mu_t(t), \qquad 0 < t < T \tag{8.30''}$$

Plainly the above proof also works for (8.30')-(8.30'') instead of (8.30).

2°. Theorem 3 has some obvious weak features. The main one is the following: Take $\tilde{T} > T$ and denote by $\overline{u^{\min}(\cdot)}$ and $\overline{\tilde{u}^{\min}(\cdot)}$ the two means velocity functions given by Theorem 3 applied on $(0, T)$ resp. $(0, \tilde{T})$. One should expect that if the extremum principle used to peak up $\overline{u^{\min}(\cdot)}$ on $(0, T)$ and $\overline{\tilde{u}^{\min}(\cdot)}$ on $(0, \tilde{T})$ has some real significance, then $\overline{\tilde{u}^{\min}(\cdot)}$ has to agree with $\overline{u^{\min}(\cdot)}$ on $(0, T)$, this by causality reasons. We don't know if this « causality » is respected.

9. Hopf's equation for turbulence.

1. In the researches of E. Hopf ([3]; see also Monin-Yaglom [2], Ch. X, § 28, Sec. 28.1) concerning turbulence, as a statistical mechanics based on the Navier-Stokes equations, the fundamental equation is expressed in terms of a functional of positive type on $\boldsymbol{N}$, supposed to be the characteristic function of some hypothetical probability on $\boldsymbol{N}$. In this paragraph we shall firstly establish in a rigorous way Hopf's equation for turbulence (see equation (9.6) below) as a consequence of our basic equation (3.13_{I}); this approach to Hopf's functional equation automatically solves the initial value problem for Hopf's equation. We shall conclude this paragraph with some immediate properties on the solutions of Hopf's equations which are direct consequences of properties of the statistical solutions of the Navier-Stokes equations.

We begin our discussion by some simple definitions. By a *Fourier-Stieltjes transform* of a bounded Borel measure ν on $\boldsymbol{N}$ we mean the function $\hat{\nu}$ defined on $\boldsymbol{N}$ by the formula

$$\hat{\nu}(y) = \int_{\boldsymbol{N}} \exp[i(u, y)]\, d\nu(u)\,, \qquad y \in \boldsymbol{N}. \tag{9.1}$$

If ν is a probability on $\boldsymbol{N}$ then, $\hat{\nu}$ will be called the *characteristic function* of ν. In this case $\hat{\nu}$ is *positive definite* that is

$$\sum_{j,k=1}^{p} \xi_j \bar{\xi}_k \hat{\nu}(y_j - y_k) \geqslant 0 \tag{9.2}$$

for all complex numbers ξ_j and elements $y_j \in \boldsymbol{N}$, $j = 1, 2, \ldots, p$, and all $p = 1, 2, \ldots$. Note also that $\hat{\nu}$ is always a continuous function on $\boldsymbol{N}$.

For a bounded Borel measure in $\boldsymbol{N}$ let $\nu^{(m)}$ denote the Borel measure in $P_m\boldsymbol{N}$ defined by

$$\nu^{(m)}(\boldsymbol{\omega}) = \nu(P_m^{-1}\boldsymbol{\omega})\,, \qquad \boldsymbol{\omega} \text{ Borel set} \subset P_m\boldsymbol{N}, \tag{9.3}$$

$m = 1, 2, \ldots$. Let now $\hat{\nu}(y) = 0$ for all $y \in \boldsymbol{N}$. Then for $y \in P_m\boldsymbol{N}$ we will have

$$\int_{P_m\boldsymbol{N}} \exp[i(u, y)]\, d\nu^{(m)}(u) = \int_{\boldsymbol{N}} \exp[i(P_m u, y)]\, d\nu(u) = \hat{\nu}(P_m y) = 0\,,$$

so that the uniqueness theorem for the Fourier-Stieltjes transform on R^m (see Rudin [1], p. 17) we deduce $\nu^{(m)} = 0$. But for any $\Phi \in \mathfrak{C}_0$ we have

$$\int_N \Phi(u)\, d\nu(u) = \lim_{m\to\infty} \int_N \Phi(P_m u)\, d\nu(u) = \lim_{m\to\infty} \int_{P_m N} \Phi(u)\, d\nu^{(m)}(u) = 0$$

so that $\nu = 0$. Therefore as in the classical case, $\hat{\nu}$ determines uniquely ν. This simple remark will permit us to introduce one of the main operators occuring in Hopf's equation, namely: For any ν satisfying

$$\int_N \|u\|^2 d|\nu|(u) < \infty \tag{9.4}$$

(where $|\nu|$ denotes the modulus of ν; see Dinculeanu [1], § 3, Sec. 4) and any $v \in N^1$ the function $b(u, u, v)$ in u is, plainly, belonging to $L^1(\nu)$ (since $b(P_m u, P_m u, v) \to b(u, u, v)$ ν - a.e. and both are dominated by $c_{135}\|u\|^2$ with some convenient constant c_{135}), thus

$$\int_N b(u, u, v) \exp[i(u, y)]\, d\nu(u)$$

makes sense for all $y \in N$; moreover this integral is a continuous linear functional (in v) on N^1 therefore it defines an element $-i(L\hat{\nu})(y)$ of N^{-1}. In this manner we have obtained an operator L applying all Fourier-Stieltjes transforms ν of Borel measures on N, satisfying (9.4), in N^{-1}-valued functions defined on N; let us emphasize that L is defined by the formula

$$\big((L\hat{\nu})(y), v\big) = i \cdot \int_N b(u, u, v) \exp[i(u, y)]\, d\nu(u) \tag{9.5}$$

for

$$\text{all } v \in N^1, \quad y \in N. \tag{9.5'}$$

With these preliminaries, we can state the following

THEOREM 1. *Let $\{\mu_t\}_{0<t<T}$ be a statistical solution of the Navier-Stokes equations with initial data μ satisfying* (3.5). *Let $\chi(\cdot, t)$ denote the characteristic function of μ_t and χ that of μ. Then for any $y \in N^1$*

and any $\varrho(\cdot)\in C_0^1([0, T))$ *we have*

$$(9.6)\qquad -\int_0^T \varrho'(t)\,\chi(y, t)\,dt + \int_0^T \varrho(t)\Big[\nu\big((\chi_v'(y, t), y)\big) + \big((L\chi(\cdot, t))(y), y\big)\Big]\,dt = \varrho(0)\,\chi(y) + i\int_0^T \varrho(t)\big(f(t), y\big)\,\chi(y, t)\,dt\,.$$

DEFINITION. A functional $\chi(y, t)$ defined on $N\times(0, T)$ satisfying (9.6) for all $\varrho(\cdot)\in C_0^1([0, T))$, $y\in N^1$ and for some functional $\chi(y)$ defined on N is by definition *a solution of Hopf's functional equation for turbulence*. This is justified by the fact that (9.6) is for Hopf's functional equation (see Monin-Yaglom [2], formulae (28.18)-(28.19)) what (2.8) is for the Navier-Stokes equations, namely a more elaborate form in the framework of Functional Analysis.

The function χ occuring in (9.6) is called the initial data of $\chi(\cdot, \cdot)$. With these definition we can reformulate shortly Theorem 1 in the following manner: *Let* $\{\mu_t\}_{0<t<T}$ *be a statistical solution of the Navier-Stokes equations with initial data* μ *satisfying* (3.5) *then the characteristic function of* μ_t, $0<t<T$, *yields a solution of Hopf's functional equation for turbulence with initial data the characteristic function of* μ.

PROOF OF THEOREM 1. Fix $y\in N^1$ and $\varrho(\cdot)\in C_0^1([0, T))$, and define

$$(9.7)\qquad \Phi(t, u) = \varrho(t)\cdot\exp\,[i(u, y)]$$

for $(t, u)\in(0, T)\times N$. Clearly

$$(9.7')\qquad \Phi_t'(t, u) = \varrho'(t)\cdot\exp\,[i(u, y)]$$

and

$$(9.7'')\qquad \Phi_u'(t, u) = i\varrho(t)\cdot y\cdot\exp\,[i(u, y)]\;;$$

(9.7)-(9.7″) show that $\Phi(\cdot, \cdot)\in\mathcal{G}$ and satisfies (3.8′) (see Sec. 3.1.a)). Therefore we can replace this Φ in (3.13_{I}) obtaining

$$-\int_0^T \varrho'(t)\,\chi(y, t)\,dt + \int_0^T \varrho(t)\Big\{\int_N [i((u, y\cdot\exp\,[i(u, y)]\,]) +$$

$$+ ib\bigl(u, u, y\cdot \exp[i(u, y)]\bigr)\bigr] d\mu_t(u)\Bigr\} dt =$$

$$= \varrho(0)\cdot\chi(y) + i\int_0^T \varrho(t)\Bigl\{\int_N (f(t), y\cdot\exp[i(u, y)])\, d\mu_t(u)\Bigr\} dt =$$

$$= \varrho(0)\cdot\chi(y) + i\int_0^T \varrho(t)(f(t), y)\cdot\chi(y, t)\, dt\,,$$

so that taking into account also (9.5)-(9.5′) it remains only to show that a.e. on $(0, T)$ we have

$$i\int_N ((u, y))\exp[i(u, y)]\, d\mu_t(u) = ((\chi_v'(y, t), y))\,. \tag{9.8}$$

Taking into account that $\int_N \|u\|^2 d\mu_t(u) < \infty$ a.e. on $(0, T)$ it will sufficient to prove that if ν is any (Borel) probability on N satisfying

$$\int_N \|u\|^2 d\nu(u) < \infty \tag{9.4′}$$

(hence also (9.4)) then $\hat\nu(\cdot)$ is differentiable in $\boldsymbol{N}$ and its Frechet differential $\hat\nu'_{\square}(\boxdot)$ satisfies

$$(\hat\nu'(y), v) = i\int_N (u, v)\exp[i(u, y)]\, d\nu(u) \qquad \text{if } v \in \boldsymbol{N}, \tag{9.9}$$

$$((\hat\nu'(y), v)) = i\int_N ((u, u))\exp[i(u, y)]\, d\nu(u) \quad \text{if } v \in \boldsymbol{N}^1. \tag{9.9′}$$

Let now $y, v \in \boldsymbol{N}$, y being fixed. Then

$$\frac{1}{|v|}\left|\hat\nu(y + v) - \hat\nu(y) - i\int_N (u, v)\exp[i(u, y)]\, d\nu(u)\right| =$$

$$= \frac{1}{|v|}\left|\int_N \exp[i(y, u)][\exp[i(u, v)] - 1 - i(u, v)]\, d\nu(u)\right| \leqslant$$

$$\leqslant \int_N \frac{|\exp[i(u, v)] - 1 - i(u, v)|}{|v|}\, d\nu(u) \leqslant$$

$$\leqslant \int_N \frac{|(u, v)|^2}{|v|}\, d\nu(u) \leqslant |v|\int_N |u|^2 d\nu(u) \to 0$$

for $v \to 0$ (in $\boldsymbol{N}$). Thus $\hat{\nu}$ is differentiable and (9.9) holds. By (9.9) we have

$$\hat{\nu}'(y) = i\int_{\boldsymbol{N}} u \exp[i(u, y)]\, d\nu(u) \tag{9.9''}$$

thus for any $m = 1, 2, \dots$

$$\|P_m \hat{\nu}'(y)\|^2 = \left\| i \cdot \int_{\boldsymbol{N}} P_m u \exp[i(u, y)]\, d\nu(u) \right\|^2 \leqslant$$
$$\leqslant \left(\int_{\boldsymbol{N}} \|P_m u\| d\nu(u) \right)^2 \leqslant \int_{\boldsymbol{N}} \|P_m u\|^2 d\nu(u) \leqslant \int_{\boldsymbol{N}} \|u\|^2 d\nu(u)\,,$$

so that $\lim_{m\to\infty} \|P_m \hat{\nu}'(y)\| < \infty$. It is plain that this implies that $\hat{\nu}'(y) \in \boldsymbol{N}^1$. But then, since in virtue of (9.4′), the integral in (9.9″) is actually taken only over $\boldsymbol{N}^1$, by scalar multiplication in $\boldsymbol{N}^1$ of (9.9″) we obtain directly (9.9′). This finishes the proof of Theorem 1.

REMARK. One should note that in our form (9.6) of Hopf's functional equation for turbulence all entities involved have a consistent mathematical meaning.

2. ***a***) We shall give, as samples, existence and uniqueness theorems for the solutions of Hopf's equation (9.6) which are direct consequence of the corresponding facts for the statistical solutions of the Navier-Stokes equations.

THEOREM 2. *Let $\chi(\cdot)$ be a positive definite functional on $\boldsymbol{N}$ such that $\chi(0) = 1$ and $\chi''(y)$ exists* (*) *and satisfies*

$$\mathrm{Trace}_{\boldsymbol{N}}[-\chi''(0)] < \infty \; (**)\,. \tag{9.10}$$

(*) For a functional χ on $\boldsymbol{N}$, $\chi''(y_0)$ denotes the second Frechet differential of χ in y_0, that is a linear continuous operator $\chi''(y_0)$ from $\boldsymbol{N}$ into $\boldsymbol{N}$ such that

$$\frac{1}{|v|}\left|\chi'_{\square}(\boxed{y_0 + v}) - \chi'_{\square}(\boxed{y_0}) - \chi''(y_0)v\right| \to 0\,, \qquad \text{for } |v| \to 0\,.$$

Thus for the existence of $\chi''(y_0)$ it is necessary first that $\chi'(y)$ exists in a neighbourhood of y_0.

(**) For a linear continuous operator $A \geqslant 0$ from $\boldsymbol{N}$ into $\boldsymbol{N}$, by $\mathrm{Trace}_{\boldsymbol{N}} A$ we will denote the usual operator-trace, namely $\sum_{m=1}^{\infty}(Ae_m, e_m)$ (where $\{e_m\}_{m=1}^{\infty}$, is

Then there exists a solution $\chi(\cdot, t)$ on $(0, T)$ of the Hopf functional equation (9.6) *with initial data $\chi(\cdot)$ such that for any $t \in (0, T)$, $\chi(\cdot, t)$ is a positive definite functional on $\boldsymbol{N}$ for which $\chi(0, t) = 1$, $\chi''_{\square}(\square, t)$ exist and satisfies*

$$\operatorname{Trace}_{\boldsymbol{N}}\left[-\chi''_{\square}(\boxed{0}, \cdot)\right] \in L^1(0, T) . \tag{9.11}$$

$$\operatorname{Trace}_{\boldsymbol{N}^1}\left[-\chi''_{\square}(\boxed{0}, \cdot)\right] \in L^\infty(0, T) , \tag{9.11'}$$

Proof. In virtue of the next Lemma 1 (see below) there exists a Borel probability μ on $\boldsymbol{N}$ satisfying (3.5) such that $\chi = \hat{\mu}$. Let $\{\mu_t\}_{0<t<T}$ be the statistical solution of the Navier-Stokes equations (constrocted in Sec. 3.2) with initial data μ and let $\chi(\cdot, t) = \hat{\mu}_t(\cdot)$. By Theorem 1, Sec. 9.1, $\chi(\cdot, t)$ is a solution on $(0, T)$ of Hopf's functional equation, with initial data χ. The relations (9.11)-(9.11′) follow readily from the relations (9.12)-(9.12′) below (for μ_t instead of μ). The proof is complete modulo that of the following

Lemma 1. *Let μ be a (Borel) probability on $\boldsymbol{N}$ satisfying* (3.5). *Then $\hat{\mu}$ is a positive definite functional on $\boldsymbol{N}$, $\hat{\mu}''$ exists and*

$$\operatorname{Trace}_{\boldsymbol{N}}\left[-\hat{\mu}''(0)\right] = \int_{\boldsymbol{N}} |u|^2 d\mu(u) ; \tag{9.12}$$

moreover

$$\operatorname{Trace}_{\boldsymbol{N}^1}\left[-\hat{\mu}''(0)\right] = \int_{\boldsymbol{N}} \|u\|^2 d\mu(u) . \tag{9.12'}$$

Conversely if χ is a positive definite functional on $\boldsymbol{N}$ such that $\chi(0) = 1$, χ'' exists and

$$\operatorname{Trace}_{\boldsymbol{N}}\left[-\chi''(0)\right] < \infty \tag{9.13}$$

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an orthogonal basis in $\boldsymbol{N}$), which does not depend on $\{e_m\}_{m=1}^{\infty}$; we shall also use the notation

$$\operatorname{Trace}_{\boldsymbol{N}^1}(A) = \sum_{m=1}^{\infty} \lambda_m (A w_m, w_m) .$$

We must also remark that for a positive definite functional χ on $\boldsymbol{N}$, if χ'' exists then $-\chi''(0) \geqslant 0$. Indeed fixing a $v \in \boldsymbol{N}$ and defining $\varphi(s) = \chi(sv)$ for $s \in R$, we will obtain a positive definite function $\varphi \in C^2(R)$ such that $\varphi''(0) = (\chi''(0)v, v)$. The positive definitess of φ implies $-[\varphi(s) + \varphi(-s) - 2\varphi(0)] \geqslant 0$ for any $s \geqslant 0$ which, by $s \to +0$, yields $\varphi''(0) \leqslant 0$. Thus $\big(((-\chi''(0))v, v\big) \geqslant 0$ for all $v \in \boldsymbol{N}$, i.e. $-\chi''(0) \geqslant 0$.

then $\chi = \mu$ *for some* (*uniquely determined*) *Borel probability* μ *on* $\boldsymbol{N}$ *satisfying* (3.5).

PROOF. We have already verified (see (9.9″)) that μ' exists and

$$\hat{\mu}'(y) = i\int_{\boldsymbol{N}} u \exp[i(u, y)]\, d\mu(u)\,, \qquad y \in \boldsymbol{N}\,.$$

Now note that for $z \neq 0$

$$\frac{1}{|z|}\left|\hat{\mu}'(y+z) - \hat{\mu}'(y) + \int_{\boldsymbol{N}} (u, z)\, u \exp[i(u, y)]\, d\mu(u)\right| \leqslant$$

$$\leqslant \frac{1}{|z|}\int_{\boldsymbol{N}} |u|\cdot|\exp[i(u, z)] - 1 - i(u, z)|\, d\mu(u) \leqslant$$

$$\leqslant \frac{|z|}{2}\int_{\{u:\, u\in \boldsymbol{N},\ |u|\leqslant r\}} |u|^3 d\mu(u) + 2\int_{\{u:\, u\in \boldsymbol{N},\ |u|>r\}} |u|^2 d\mu(u) = \delta(|z|, r)$$

for any $r \in (0, \infty)$. Let $\varepsilon > 0$. By (3.5) there exists an r_ε such that

$$2\int_{\{u:\, u\in \boldsymbol{N},\ |u|>r_\varepsilon\}} |u|^2 d\mu(u) < \frac{\varepsilon}{2}\,.$$

Thus letting $|z|$ be small enough we will have $\delta(|z|, r_\varepsilon) < \varepsilon$. This fact shows that $\hat{\mu}''(y)$ exists and

(9.14) $$\hat{\mu}''(y)\, z = -\int_{\boldsymbol{N}} (u, z)\, u \exp[i(u, y)]\, d\mu(u)$$

for

(9.14′) $$\text{all } y, z \in \boldsymbol{N}\,.$$

Now it is clear that

$$(\mu''(0)\, w_m, w_m) = -\int_{\boldsymbol{N}} |(u, w_m)|^2 d\mu(u)$$

hence

$$\mathrm{Trace}_{\boldsymbol{N}}(-\hat{\mu}''(0)) = \sum_{m=1}^{\infty} ((-\mu''(0))\, w_m, w_m) =$$

$$= \int_{\boldsymbol{N}} \sum_{m=1}^{\infty} |(u, w_m)|^2 d\mu(u) = \int_{\boldsymbol{N}} \|u\|^2\, d\mu(u)\,,$$

which, by (3.5), is finite. This establishes (9.12). For (9.12′), observe that

$$\sum_{m=1}^{\infty} \lambda_m |(u, w_m)|^2 = \|u\|^2 \qquad \text{for all } u \in N$$

(where as usual we put $\|u\| = \infty$ if $u \in N \setminus N^1$), and then proceed exactly as for (9.12).

It remains to prove the converse part of Lemma 1. By Bochner's theorem for the abelian locally compact group R^m (see Rudin [1], Ch. 1), we can find a Borel probability $\mu^{(m)}$ in N carried by $P_m N$ $(m = 1, 2, \ldots)$ such that

$$\chi(y) = \int_{P_m N} \exp[i(u, y)]\, d\mu^{(m)}(u) \tag{9.15}$$

for

$$\text{all } y \in P_m N, \qquad m = 1, 2, \ldots . \tag{9.15′}$$

Now for any $s \in R$ and $j = 1, 2, \ldots, m$, we have

$$\begin{aligned} -\chi(sw_j) - \chi(-sw_j) + 2\chi(0) &= \\ = \int_{P_m N} [-\exp[is(u, w_j)] - \exp[-is(u, w_j)] + 2]\, d\mu^{(m)}(\mu) &= \\ = \int_{P_m N} 4\left(\sin \frac{s(u, w_j)}{2}\right)^2 d\mu^{(m)}(u)\,, \end{aligned}$$

whence

$$\begin{aligned} \int_{P_m N} \left(\frac{\sin s(u, w_j)/2}{s(u, w_j)/2}\right)^2 |(u, w_j)|^2 d\mu^m(u) &= \\ = -\frac{1}{s^2}[\chi(sw_j) + \chi(-sw_j) - 2\chi(0)] &\to -(\chi''(0) w_j, w_j) \end{aligned}$$

for $s \neq 0$, $s \to 0$. By Fatou's theorem (see Riesz–Sz.-Nagy [1], Ch. II, § 20), we conclude that

$$\int_{P_m N} |(u, w_j)|^2 d\mu^{(m)}(u) \leqslant -(\chi''(0) w_j, w_j)$$

for all $j = 1, 2, \dots, m$. It results

$$(9.16)\qquad \int_{N} |u|^2 d\mu^{(m)}(u) = \int_{P_m N} |u|^2 d\mu^{(m)}(u) =$$

$$= \int_{P_m N} \sum_{j=1}^{m} |(u, w_j)|^2 d\mu^{(m)}(u) \leqslant \sum_{j=1}^{m} (-\chi''(0) w_j, w_j) \leqslant \mathrm{Trace}_N[-\chi''(0)] = c_{135}$$

for all $m = 1, 2, \dots$. Let $\boldsymbol{B}_k = \{u\colon u \in \boldsymbol{N}, |u| \leqslant k\}$, be endowed with the weak topology of $\boldsymbol{N}$, for all $k = 1, 2, \dots$. Since $\mathcal{C}(\boldsymbol{B}_k)$ is separable, for all $k = 1, 2, \dots$, by Cantor's diagonal method we can choose a subsequence $\{\mu^{(m_j)}\}_{j=1}^{\infty}$ such that for any k, there exists a Borel measure μ_k on B_k satisfying

$$(9.17)\qquad \int_{\boldsymbol{B}_k} \varphi(u)\, d\mu^{(m_j)}(u) \to \int_{\boldsymbol{B}_k} \varphi(u)\, d\mu_k(u)$$

for all $\varphi \in \mathcal{C}(\boldsymbol{B}_k)$. As we already observed in Sec. 3.3, μ_k is actually a Borel measure on $\boldsymbol{N}$ such that $\operatorname{supp} \mu_k \subset \boldsymbol{B}_k$. On the other side since for any $h, k \geqslant 1$

$$\{\varphi | \boldsymbol{B}_k \colon \varphi \in \mathcal{C}(\boldsymbol{B}_{k+h})\}$$

is, by the Weierstrass-Stone theorem, dense in $\mathcal{C}(\boldsymbol{B}_k)$, it results that for Borel subsets $\boldsymbol{\omega}$ of $\boldsymbol{B}_k$ the measure μ_{k+h} agrees with μ_k. Thus if for any Borel subset $\boldsymbol{\omega}$ of $\boldsymbol{N}$ we set

$$(9.18)\qquad \mu(\boldsymbol{\omega}) = \mu_1(\boldsymbol{\omega} \cap \boldsymbol{B}_1) + \sum_{k=1}^{\infty} \mu_{k+1}(\boldsymbol{\omega} \cap (\boldsymbol{B}_{k+1} \setminus \boldsymbol{B}_k))$$

we will have $\mu(\boldsymbol{\omega}) \geqslant 0$ and

$$(9.18)\qquad \mu(\boldsymbol{\omega}) = \mu_k(\boldsymbol{\omega}) \quad \text{if } \boldsymbol{\omega} \subset \boldsymbol{B}_k\,.$$

Moreover, since by (9.16),

$$(9.19)\qquad \mu_k(\boldsymbol{B}_k) = \int_{\boldsymbol{B}_k} 1\, d\mu_k(u) = \lim_{j\to\infty} \int_{\boldsymbol{B}_k} 1\, d\mu^{(m_j)}(u) = \lim_{j\to\infty} \mu^{(m_j)}(\boldsymbol{B}_k) \left\{ \begin{array}{l} \leqslant 1 \\ \geqslant 1 - \dfrac{c_{135}}{k^2} \end{array} \right.$$

we have

$$\mu_{k+1}(\boldsymbol{B}_{k+1} \setminus \boldsymbol{B}_k) = \mu(\boldsymbol{B}_{k+1}) - \mu(\boldsymbol{B}_k) \leqslant \frac{c_{131}}{k^2}\,,$$

so that, in (9.18) we will also have

$$\mu(\boldsymbol{\omega} \cap (\boldsymbol{B}_{k+1} \setminus \boldsymbol{B}_k)) \leqslant \frac{c_{131}}{k^2}, \qquad \text{for all } k = 1, 2, \ldots$$

and all Borel subset $\boldsymbol{\omega}$ of $\boldsymbol{N}$. This fact easily implies that μ is a bounded Borel measure on $\boldsymbol{N}$. By (9.18)-(9.19) we have

$$1 - \frac{c_{131}}{k^2} \leqslant \mu(\boldsymbol{B}_k) \leqslant 1 \tag{9.19'}$$

whence letting $k \to \infty$ we deduce $\mu(\boldsymbol{N}) = 1$, thus μ is a Borel probability on $\boldsymbol{N}$. Let now m $(= 1, 2, \ldots)$ be fixed and let j be sufficiently large so that $m_j \geqslant m$. Then

$$c_{135} \geqslant \int_{\boldsymbol{N}} |u|^2 d\mu^{(m_j)}(u) = \int_{P_{m_j}\boldsymbol{N}} |u|^2 d\mu^{(m_j)}(u) =$$
$$= \int_{P_{m_j}\boldsymbol{N}} \left(\sum_{i=1}^{m_j} |(u, w_i)|^2 \right) d\mu^{(m_j)}(u) \geqslant \int_{P_{m_j}\boldsymbol{N}} \left(\sum_{i=1}^{m} |(u, w_i)|^2 \right) d\mu^{(m_j)}(u) =$$
$$= \int_{\boldsymbol{N}} \left(\sum_{i=1}^{m} |(u, w_i)|^2 \right) d\mu^{(m_j)}(u) \geqslant \int_{\boldsymbol{B}_k} \left(\sum_{i=1}^{m} |(u, w_i)|^2 \right) d\mu^{(m_j)}(u) \to$$
$$\to \int_{\boldsymbol{B}_k} \left(\sum_{i=1}^{m} |(u, w_j)|^2 \right) d\mu_k(u) = \int_{\boldsymbol{B}_k} |P_m u|^2 d\mu(u)$$

for $j \to \infty$ and any $k = 1, 2, \ldots$. It results (letting firstly $k \to \infty$ and then $m \to \infty$) that

$$\int_{\boldsymbol{N}} |u|^2 d\mu(u) \leqslant c_{135}$$

Thus μ satisfies (3.5). It remains to show that $\hat{\mu} = \chi$. To this aim take $y \in P_m\boldsymbol{N}$, fix m and let j be large enough (that is, such that $m_j \geqslant m$). Then since $y \in P_{m_j}\boldsymbol{N}$, we will have, by (9.15),

$$|\chi(y) - \hat{\mu}(y)| = \left| \int_{\boldsymbol{N}} \exp[i(ym, u)]\, d\mu^{(m_j)}(u) - \int_{\boldsymbol{N}} \exp[i(y, u)]\, d\mu(u) \right| \leqslant$$
$$\leqslant \left| \int_{\boldsymbol{B}_k} \exp[i(y, u)]\, d\mu^{(m_j)}(u) - \int_{\boldsymbol{B}_k} \exp[i(y, u)]\, d\mu(u) \right| + \frac{2c_{135}}{k^2}$$

where we used (9.19)-(9.19′) and where $k = 1, 2, \ldots$ is arbitrary. Letting $j \to \infty$ we deduce

$$|\chi(y) - \hat{\mu}(y)| \leqslant \frac{2c_{135}}{k^2},$$

whence letting $k \to \infty$ we obtain $\chi(y) = \hat{\mu}(y)$ for all $y \in \bigcup_{m=1}^{\infty} P_m N$, which is dense in N, hence $\chi = \hat{\mu}$. With this the proof of Lemma 1 is achieved.

b) A positive definite function χ on N will be called of exponential type if for any $y \in N$, the function $s \mapsto \chi(sy)$ defined on R can be exrended analytically to an entire function $\chi(\zeta; y)$ on the complex plane C, satisfying

$$(9.20) \qquad |\chi(\zeta; y)| \leqslant c_{136} \exp\big((c_{137}|\operatorname{Im}\zeta| \cdot |y|)\big), \qquad \text{for all } \zeta \in C,\ y \in N,$$

where c_{136} and c_{137} are some constants depending on χ (but independent on ζ and y).

LEMMA 2. *Let μ be a Borel probability on N. Then $\hat{\mu}$ is of exponential type if and only if μ is with bounded support in N.*

PROOF. The only non immediate implication is that if $\hat{\mu}$ is of exponential type then μ is with bounded support in N. Let thus $\hat{\mu}(\zeta; y)$, $\zeta \in C$, be the analytic extension on the whole C of $\hat{\mu}(ty)$, $t \in R$. We have

$$\hat{\mu}(t, y) = \hat{\mu}(ty) = \int_N \exp[it(u, y)]\, d\mu(u) = \int_R \exp[its]\, d\nu_y(s)$$

where for any Borel set $\sigma \subset R$ we set

$$\nu_y(\sigma) = \mu(\{u\colon u \in N,\ (u, y) \in \sigma\}).$$

By the Paley-Wiener theorem (see for instance Hörmander [1], Ch. 1, where the theorem is given for distributions, in particular for measures) we deduce that

$$\operatorname{supp} \nu_y \subset \{t\colon t \in R,\ |t| \leqslant c_{137}(\hat{\mu})|y|\}.$$

Consequently

$$(9.20') \qquad \operatorname{supp}\mu \subset \bigcap_{y\in N} \{u\colon u \in \boldsymbol{N},\ |(u, y)| \leqslant c_{137}(\hat{\mu})|y|\} = \\ = \{u\colon u \in \boldsymbol{N},\ |u| \leqslant c_{137}(\hat{\mu})\}\,.$$

This finishes the proof of Lemma 2.

THEOREM 3. (*a*) *If in Theorem 2, Sec.* 9.2.*a*), *we suppose moreover that the initial data* χ *is of exponential type then the solution of the Hopf functional solution* $\chi(\cdot, t)$ *on* $(0, T)$ *given by Theorem* 2, *Sec.* 9.2.*a*) *can be chosen such that*

(i) $\chi(\cdot, t)$ *is for any* $t \in (0, T)$ *also of exponential type uniformly in* t (that is (9.20) will be valid for all $\chi(\cdot, t)$ with a same constant c_{137});

(ii) *for any* $y \in \boldsymbol{N}$, *the function* $\chi(y, \cdot)$ *will be continuous on* $[0, T]$:

(*b*) *If* $n = 2$, *the solution* $\chi(\cdot, t)$ *on* $(0, T)$ *with initial data* χ *given by Theorem* 2, *Sec.* 9.2.*a*) *and the statement* (*a*) *of the present theorem is uniquely determined.*

The proof of this theorem is straightforward. Indeed Part (*a*) follows directly from Theorem 1.2, Lemma 2 (in this paragraph) and Theorem 2 in Sec. 3.3. Concerning Part (*b*), by the preceding Lemmae 1-2 (in this paragraph) a solution $\chi(\cdot, t)$ fulfilling the properties stated in Theorems 2 and 3(*a*) above, will be of the form $\chi(\cdot, t) = \hat{\mu}_t(\cdot)$, where $\{\mu_t\}_{0<t<T}$ is a statistical solution of the Navier-Stokes equations with initial data μ such that (on account of the preceding Lemma 2; see (9.20')) has a bounded support in N and μ_t have their support uniformly bounded in N, for $t \in (0, T)$. In virtue of Sec. 5.1.*a*)-*b*), this is sufficient to assure the uniqueness of μ_t a.e. on $(0, T)$. Thus $\chi(\cdot, t)$ is uniquely determined a.e. on $(0, T)$. By the continuity property (ii) in Part (*a*) of the statement, it results readily that $\chi(\cdot, t)$ is uniquely determined for all $t \in (0, T)$. We can conclude here the sketch of the proof of Theorem 3.

We hope that *the content of this Sec.* 9.2 *is suffcient to illustrate the manner by which one can obtain significant results on Hopf's functional equation for turbulence as direct consequences of our previous results on statistical solutions of the Navier-Stokes equations.* Therefore we conclude this Section with the following

REMARK. The time independent Hopf's functional equation has plainly the form

$$(9.21) \qquad \nu\big((\chi'(y), y)\big) + \big((L\chi)(y), y\big) = i(f, y)\,\chi(y)\,, \qquad \text{for all } y \in \boldsymbol{N}\,,$$

where on χ one has to suppose that it is positive definite, $\chi(0)=1$ and

$$\text{(9.21')} \qquad \mathrm{Trace}_{N}(-\chi''(0)) < \infty\,, \qquad \mathrm{Trace}_{N^1}(-\chi''(0)) < \infty;$$

plainly by our preceding discussion the only solutions of (9.21)-(9.21′) are the Fourier-Stieltjes transforms $\hat{\mu}$ of the stationary statistical solutions μ of the Navier-Stokes equations. Again some of our results on these last solutions can be restated for the time independent solutions of Hopf's functional equation. We don't go into any further detail concluding here our comments.

3. The aim of this short last Section is to give to the operator L, involved in Hopf's functional equation for turbulence (see (9.5)-(9.6) and (9.21)) a « concrete » form which essentially coincides with that of Hopf [3] and is more adequate for computations. However this form will be valid only under some additional regularity conditions, thus we will suppose that $n=2$ or 3 and that the Borel measure occuring in (9.5) satisfies a stronger condition than (9.4), namely

$$\text{(9.22)} \qquad \int_{N} (\|u\|_2)^2 \, d\nu(u) < \infty$$

(where we put $\|u\|_2=\infty$ for $u\in N\setminus N^2$). By Sobolev's theorem we have $H^2(\Omega)\subset C(\Omega)$ (see Agmon [1], Lemma 13.1), where the imbedding is a continuous map, so that the Dirac functional $\delta_x\colon \varphi\mapsto\varphi(x)$ is continuous on $H^2(\Omega)$ for all $x\in\Omega$. Consequently for any $v\in N^1$ we will have

$$\text{(9.23)} \qquad (L\hat{\nu}(y), v) = i\int_{N} b(u, u, v)\exp[i(u, v)]\, d\nu(y) =$$

$$= -i\int b(u, v, u)\exp[i(u, y)]\, d\nu(u) =$$

$$= -i\int_{N}\left[\int_{\Omega}\left(\sum_{j,k=1}^{n} u_j(x)\,\frac{\partial v_k(x)}{\partial x_j}\,u_k(x)\right)dx\right]\exp[i(u, y)]\, d\nu(u) =$$

$$= -i\sum_{j,k=1}^{n}\int_{\Omega}\frac{\partial v_k(x)}{\partial x_j}\left[\int_{N}\delta(u_j)\,\delta(u_k)\exp[i(u, y)]\, d\nu(u)\right]dx$$

where we have used Fubini's theorem and the fact that actually the integrals on N can be restricted to N^2, by (9.22). By the same proof as that of (9.14) we can establish that $\hat{\nu}''$ exists and is given by the formula

$$\hat{\nu}''(y)z = -\int_N (u, z)u \exp[i(u, y)]\,d\nu(u)\,, \qquad y, z \in N\,. \tag{9.24}$$

But

$$|(\hat{\nu}''(y)z_1, z_2)| \leqslant \Big(\int_N \|u\|_2^2\,d\nu(u)\Big)\|z_1\|_{N^{-2}}\cdot\|z_2\|_{N^{-2}}$$

so that the bilinear functional

$$z_1, z_2 \mapsto (\hat{\nu}''(y)z_1, z_2)$$

extends by continuity to a bilinear continuous functional on the whole N^{-2} (since N is dense in N^{-2}). We shall denote the value of this extension in the poinr $\{z_1, z_2\} \in N^{-2}\times N^{-2}$ by $\hat{\nu}''(y)(z_1, z_2)$. Let now $\delta_x e_j$ denote the functional on N^2 defined by $(\delta_x e_j)(u) = \delta_x(u_j) = u_j(x)$ for all $u = \{u_1, \ldots, u_n\} \in N^2$ and $x \in \Omega$. Then from (9.23)-(9.24) we can readily infer that

$$\big((L\hat{\nu})(y), v\big) = i\sum_{j,k=1}^{n}\int_\Omega \frac{\partial v_k(x)}{\partial x_j}\,[\hat{\nu}''(y)(\delta_x e_j, \delta_x e_k)]\,dx \tag{9.25}$$

for

$$\text{all } y \in N \quad \text{and} \quad v = \{v_1, \ldots, v_n\} \in N^1\,. \tag{9.25$'$}$$

The formulae (9.13)-(9.13′) give a way to compute $L\hat{\nu}$ directly without passing to the representation (9.1) which was explicitely involved in the initial definition (9.5) of L.

Concluding remark. We hope that it is clear that our statistical version (1.8) (or (3.13_I)) of the Navier-Stokes equations can be consistently handled in the framework of the nowadays linear functional analysis (§§ 3-7) and that it permits a rigorous mathematical approach to turbulence (§§ 8-9). We intend to study in the future in more detail some typical concrete cases, following the line developed in this paper.

Connections of our paper with those of Bass [1], [2], Malkus [1], Nihoul [1], Kolmogorov, Monin, Obukov, Yaglom (see Monin-Yaglom [1], [2]), Rosen [1], Vo-Khan [1] will also be given in some subsequent papers.

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