Rendiconti
del
Seminario Matematico
della
Università di Padova

Antonio Ambrosetti

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Rendiconti del Seminario Matematico della Università di Padova, tome 49 (1973), p. 195-204

<http://www.numdam.org/item?id=RSMUP_1973__49__195_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 49 (1973)

On the Existence of Multiple Solutions for a Class of Nonlinear Boundary Value Problems.

ANTONIO AMBROSETTI (*)

1. In a recent paper J. A. Hempel [3] has proved the existence of multiple (proper) solutions for the following nonlinear, variational boundary value problem

$$(1) \qquad \begin{cases} \sum_{i,k} \dfrac{\partial}{\partial x_i}\left(a_{i,k}(x)\dfrac{\partial u}{\partial x_k}\right) + c(x)u(x) - b\big(x, u(x)\big) = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded open set with boundary $\partial\Omega$ and $a_{i,k}(x)$, $c(x)$ and $b(x, t)$ are functions which satisfy suitable hypothesis.

The main purpose of this paper is to prove that Hempel's result is true if (under the same assumptions on $a_{i,k}(x)$ and $c(x)$) $b(x, t)$ satisfies only parity and asymptotic conditions. No monotony and positivity hypothesis are required, and moreover the existence of the second derivative is not supposed. The method used in the proof is completely different from Hempel's one and appears natural enough: we remark that the solutions of (1) are the *free* critical points of a suitable functional f on the Hilbert space $\mathring{W}^1(\Omega)$, and we prove that these critical points belong to an open subset of $\mathring{W}^1(\Omega)$, which is bounded and homotopically nontrivial. The critical points of f

(*) Indirizzo dell'A.: Istituto Matematico « L. Tonelli » - Via Derna, 1 - 56100 Pisa.

Lavoro eseguito con contributo del C.N.R. nell'ambito del Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni.

on this open set are studied by means of Lusternik-Schnirelman theory ([2], [4], [5], [1]).

2. Let Ω be a bounded open subset of $\mathbb{R}^n$; we denote by $\partial\Omega$ its boundary and by $x = (x_1, \dots, x_n)$ a point in Ω. Consider the boundary value problem

$$\text{(1)} \qquad \begin{cases} \sum_{i,k} \dfrac{\partial}{\partial x_i}\left(a_{i,k}(x)\dfrac{\partial u}{\partial x_k}\right) + c(x)\,u(x) - b\big(x, u(x)\big) = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$$

and suppose that the functions $a_{i,k}(x)$, $c(x)$ and $b(x, t)$ satisfy the following assumptions:

a) $a_{i,k}(x) = a_{k,i}(x)$ are bounded measurable functions, such that $\exists \Lambda_1, \Lambda_2, \Lambda_2 \geqslant \Lambda_1 > 0$, for which:

$$\Lambda_1|\xi|^2 \leqslant \sum_{i,k} a_{i,k}(x)\,\xi_i\,\xi_k \leqslant \Lambda_2|\xi|^2$$

b) $c(x) \in L^\varrho(\Omega)$ with $\varrho > n/2$; $c(x) \geqslant 0$ on Ω and $c(x) > 0$ on a set of positive measure on Ω;

c) for all t $b(x, t)$ is measurable in x; for almost all $x \in \Omega$ $b(x, t)$ is of class C^1 in t;

d) for almost all $x \in \Omega$, we have: $b(x, -t) = -b(x, t)$;

e) setting $\alpha(x, t) = t^{-1}b(x, t)$, we have (for almost all $x \in \Omega$):

$$\lim_{t\to 0} \alpha(x, t) = 0 \qquad \lim_{t\to+\infty} \alpha(x, t) \geqslant c(x);$$

f) $|b(x, t)| \leqslant m_1(x) + k_1|t|^r$, $|b_t(x, t)| \leqslant m_2(x) + k_2|t|^{r-1}$ (*), with $m_1(x) \in L^{2n/(2+n)}(\Omega)$, $m_2(x) \in L^{n/2}(\Omega)$ and $r < \big((n+2)/(n-2)\big)$ for $n > 2$ and r arbitrary for $n \leqslant 2$.

Let $\overset{\circ}{W}{}^1(\Omega)$ the Hilbert space obtained as closure of the class $\mathcal{D}(\Omega)$ of infinitely differentiable functions with compact support in Ω, under

(*) $b_t(x, t) = \dfrac{\partial b(x, t)}{\partial t}$.

the norm

$$\int_\Omega u^2(x)\,dx + \int_\Omega \sum_i \left(\frac{\partial u}{\partial x_i}\right)^2 dx$$

for all $u, v \in \mathring{W}^1(\Omega)$, we set

$$((u, v)) = \int_\Omega \sum_{i,k} a_{i,k}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_k}\, dx \tag{2}$$

$((\ldots))$ is a scalar product in $\mathring{W}^1(\Omega)$ and the norm $\|u\|^2 = ((u, u))$ is equivalent to the usual norm. We denote by B the operator of $\mathring{W}^1(\Omega)$ in itself defined by:

$$((Bu, v)) = \int_\Omega b(x, u(x))\, v(x)\, dx\,, \qquad \forall v \in \mathring{W}^1(\Omega)$$

and by C the (linear) operator

$$((Cu, v)) = \int_\Omega c(x)\, u(x)\, v(x)\, dx\,, \qquad \forall v \in \mathring{W}^1(\Omega)\,.$$

Then the generalized solutions of (1) are the elements u of $\mathring{W}^1(\Omega)$ such that

$$u - Cu + Bu = 0 \tag{3}$$

First of all we observe that $Bu = \operatorname{grad} h(u)$ with

$$h(u) = \int_0^1 ds \int_\Omega b(x, su(x))\, u(x)\, dx$$

and therefore that the solutions of (3) are the critical points of the func tional $f\colon \mathring{W}^1\ \Omega) \to \mathbb{R}$, defined by

$$f(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2}((Cu, u)) + h(u)\,. \tag{4}$$

From the conditions on the functions $c(x)$ and $b(x, t)$ we may deduce the following properties for the functional f and its gradient:

— f is of class C^2;

— f is even;

— C is a compact, positive self-adjoint operator and B is compact;

— f is weakly continuous and bounded on every bounded set of $\overset{\circ}{W}{}^1(\Omega)$.

Moreover f satisfies on every bounded set of $\overset{\circ}{W}{}^1(\Omega)$ the following condition (given by Palais-Smale):

CONDITION (P-S). *If u_n is a sequence such that $f(u_n)$ is bounded and* $\operatorname{grad} f(u_n) \to 0$, *then u_n has a converging subsequence.*

In fact, if u_n is a bounded sequence such that $\operatorname{grad} f(u_n) \to 0$, since B and C are compact, there exists a subsequence (which we shall still denote by u_n) such that $Cu_n - Bu_n$ is converging. Then $u_n = \operatorname{grad} f(u_n) + Cu_n - Bu_n$ is converging too.

Together with (P-S) condition, also the notion of Lusternik-Schnirelman category plays a fundamental role in the study of critical points of a functional. We use a modification of this notion: the category over compact sets, defined by F. E. Browder [2]:

DEFINITION 2.1. *Let X be a topological space. If K is a subset of X,* $\operatorname{cat}(K; X)$ *is the least integer n such that $K \subseteq \bigcup_{i=1}^{n} K_i$, with K_i closed and contractible to a point over X. If no such integers exist, we pose* $\operatorname{cat}(K; X) = +\infty$.

We define $\operatorname{cat}_K(X)$ *by* $\operatorname{cat}_K(X) = \sup\{\operatorname{cat}(K; X)$: *$K$ compact subset of X*$\}$.

For some properties of category over compact sets, see e.g. [2], [1].

We state now a lemma and a theorem, which will be useful in the following.

LEMMA 2.2. *Let f be a functional of class C^2 defined on a Hilbert space H. Let M be an open subset of H such that f is bounded from below on $\overline{M}$ (closure of M) and satisfies condition (P-S) on $\overline{M}$. Consider the Cauchy problem*

$$\begin{cases} w' = -\operatorname{grad} f(w) \\ w(0) = p \end{cases} \tag{5}$$

and denote by $w(t, p)$ *the solution of* (5) *and by* $\big(t^-(p), t^+(p)\big)$ *the maximal interval on which* $w(t, p)$ *is defined. Suppose that* $\forall p \in \overline{M}$ *and* $\forall t \in [0, t^+(p))$ $w(t, p) \in \overline{M}$. *Under these conditions we have that*:

i) $\forall p \in M$ $t^+(p) = +\infty$;

ii) $\forall p \in M$ *the* $\lim w(t, p)$ *exists and is equal to* p_0, *with* $\operatorname{grad} f(p_0) = 0$.

PROOF. The proof is standard.

THEOREM 2.3. *Let* M *be an open subset of the Hilbert space* H *and let* f *be a functional which satisfies the hypothesis of lemma 2.2. Moreover we suppose that if* $p_0 = \lim w(t, p)$ *with* $p \in M$, *then* $p_0 \in M$. *Under these conditions, denoted by* M_0 *the set of critical points of* f *in* M *we have that*: $\operatorname{card}(M_0) \geqslant \operatorname{cat}_k(M)$.

PROOF. It is similar to the proof of theorem 1.3, in [2]. Such arguments can be repeated since (i) and (ii) of lemma 2.2 are true and since the critical points to which converge the gradient lines of f, belong to M for hypothesis. Q.E.D.

3. We study now in particular our problem. First we observe that, since C is a compact, positive self-adjoint operator, then the problem

$$u = \lambda C u \qquad u \in \mathring{W}^1(\Omega)$$

has countably infinite many eigenvalues λ_n such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$. We denote by v_n the corresponding eigensolutions (normalized). It is our main purpose to prove the following theorem:

THEOREM 3.1. *Suppose that conditions* (*a*)-(*f*) *are satisfied. If* $\lambda_m < 1$, *then boundary value problem* (1) *has at least m pairs of nonzero solutions.*

The proof of the theorem is based on two lemmas:

LEMMA 3.2. *If conditions* (*a*)-(*f*) *are satisfied, then the set*

$$M = \{u \in \mathring{W}^1(\Omega) : f(u) < 0\}$$

is bounded.

LEMMA 3.3. *If the conditions of theorem 3.1 are satisfied, then the set M of the previous lemma has the following properties*:

i) *M is symmetric respect to* 0;

ii) *denoted by $\tilde{M}$ the identification space under the reflection $j: u \to -u$, we have that* $\operatorname{cat}_k(\tilde{M}) \geqslant m$.

PROOF OF LEMMA 3.2. Suppose M is not bounded. Then there exists a sequence $u_n \in \mathring{W}^1(\Omega)$ such that

$$\|u_n\| \to +\infty \tag{6}$$

$$f(u_n) < 0 . \tag{7}$$

From (7) we obtain:

$$\|u_n\|^2 < ((Cu_n, u_n)) - 2h(u_n) = \int_\Omega c(x)\,u_n^2(x)\,dx - 2\int_0^1 ds \int_\Omega b(x, su_n(x))\,u_n(x)\,dx =$$

$$= \int_\Omega c(x)\,u_n^2(x)\,dx - \int_0^1 ds \int_\Omega \alpha(x, su_n(x))\,u_n^2(x)\,dx .$$

We divide by $\|u_n\|^2$ and set $z_n(x) = \|u_n\|^{-1} u_n(x)$. We have

$$1 < \int_\Omega c(x)\,z_n^2(x)\,dx - \int_0^1 ds \int_\Omega \alpha(x, su_n(x))\,z_n^2(x)\,dx = \tag{8}$$

$$= \int_0^1 ds \int_\Omega z_n^2(x)\big(c(x) - \alpha(x, su_n(x))\big)\,dx .$$

Since $\|z_n\| = 1$, we may assume — passing to a subsequence — that $z_n(x) \to \bar{z}(x)$ almost everywhere in Ω. We denote by Ω' the set $\{x \in \Omega : \bar{z}(x) \neq 0\}$ and by Ω'' the set $\Omega \setminus \Omega'$; let $\beta_n(x, s)$ be the function given by

$$\beta_n(x, s) = \min\{c(x), \alpha(x, su_n(x))\} .$$

Since $\forall x \in \Omega$ and $\forall s \in [0, 1]$ we have that $\beta_n(x, s) \leqslant \alpha(x, su_n(x))$, from (8) we obtain

$$1 < \int_0^1 \int_\Omega z_n^2(x)[c(x) - \beta_n(x, s)]\,dx\,ds .$$

We prove that the integral in this inequality converges to zero. In fact if $x \in \Omega'$ then by (6) we have that $u_n(x) = \|u_n\| z_n(x) \to \pm \infty$; on the other hand by hypothesis for almost all $x \in \Omega$ and for all $s \in (0,1)$ we have that $\lim_{t \to \infty} \alpha(x, st) \geqslant c(x)$. Thus we conclude that, if $x \in \Omega'$ then $c(x) - \beta_n(x, s) \to 0$. If $x \in \Omega''$ we have that $|c(x) - \beta_n(x, s)| \leqslant \leqslant 2c(x)$ and $z_n(x) \to 0$. Thus we obtain a contradiction and lemma 3.2 is proved. Q.E.D.

PROOF OF LEMMA 3.3. (i) is an immediate consequence of the fact that f is even. We prove (ii). From the hypothesis we made, with easy computations we have that

$$f''(u)[v][v] = \|v\|^2 - ((Cv, v)) + \int_\Omega b_t(x, u(x))\, v^2(x)\, dx \quad (*)\,. \tag{9}$$

Since for hypothesis $b_t(x, 0) = 0$, then from (9) we obtain

$$f''(0)[v][v] = \|v\|^2 - ((Cv, v)). \tag{10}$$

Let V_m be the linear manifold spanned by $v_1, \ldots, v_m$. If $v \in V_m$, $v \neq 0$, since $\lambda_m < 1$, then we have that $\|v\|^2 < ((Cv, v))$; thus from (10) we obtain:

$$f''(0)[v][v] < 0 \qquad \forall v \in V_m,\ v \neq 0\,. \tag{11}$$

Since $f(0) = 0$, (11) implies that there exists a neighborhood U of 0 in V_m such that $\forall u \in U \setminus \{0\}$ is $f(u) < 0$. Thus there exists a sphere $K = \{u \in V_m \colon \|u\| = \varepsilon\}$ such that $K \subset M$. On the other hand, by lemma 3.2, there exists a ball Σ such that $M \subset \Sigma' = \Sigma \setminus \{0\}$. We denote by $\tilde{K}$ and $\tilde{\Sigma}'$ the identification spaces of K and Σ' respectively, under reflection j. $\tilde{\Sigma}'$ is homotopically equivalent to an infinite dimensional projective space; $\tilde{K}$ is a $m-1$ dimensional projective space canonically imbedded in this projective space. Then it is known that $\operatorname{cat}(\tilde{K}; \tilde{\Sigma}') = m$. Since $\tilde{K} \subset \tilde{M} \subset \tilde{\Sigma}'$, thus we have: $\operatorname{cat}(\tilde{K}\ \tilde{M}) \geqslant \geqslant \operatorname{cat}(\tilde{K}; \tilde{\Sigma}') = m$. Since $\tilde{K}$ is compact, this enables us to prove that $\operatorname{cat}_k(\tilde{M}) \geqslant m$. Q.E.D.

We can now prove theorem 3.1.

(*) $f''(u)[v][v]$ denotes the value that the bilinear mapping $f''(u)$ assumes when it is evaluated on the point (v, v).

PROOF OF THEOREM 3.1. We have shown in Sect. 2 that the solutions of (1) are the critical points of the functional $f(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}((Cu, u)) + h(u)$. We shall prove that f has on M $2m$ critical points (M being the set defined in lemma 3.2). Since M is bounded, then f is a functional of class C^2 bounded on $\bar{M}$, satisfying condition (P-S) on $\bar{M}$(see Sect. 2). Let $w(t, p)$ be the solution curve of Cauchy problem (5). Since

$$\frac{d}{dt} f(w(t, p)) = ((\operatorname{grad} f(w(t, p)), w'(t, p))) = -\|\operatorname{grad} f(w(t, p))\|^2$$

then the function $f(w(t, p))$ is a non-increasing function of t (for every p). Thus if $p \in M$, we have that $f(w(t, p)) \leqslant f(p) < 0$ and so $w(t, p) \in M$ $\forall t \in [0, t^+(p))$. Then by lemma 2.2 we can state that: (i) $\forall p \in M$ $w(t, p)$ is defined for $t \in [0, +\infty)$; (ii) $\forall p \in M$ the $\lim_{t \to +\infty} w(t, p)$ exists and is equal to p_0 with $\operatorname{grad} f(p_0) = 0$. We observe also that this limit p_0 belongs to M, since $f(p_0) = \lim_{t \to +\infty} f(w(t, p)) < 0$.

Let $\tilde{M}$ be the identification space of M under the reflection j. Since f is even, it induces a function $\tilde{f}$ on $\tilde{M}$, and obviously $\tilde{f}$ has the same properties of f. Then we can use theorem 2.3; hence, denoted by $\tilde{M}_0$ the set of critical points of $\tilde{f}$ on $\tilde{M}$, we have that $\operatorname{card}(\tilde{M}_0) \geqslant \operatorname{cat}_k(\tilde{M})$. Since by lemma 3.3 we have that $\operatorname{cat}_k(\tilde{M}) \geqslant m$, then it results: $\operatorname{card}(\tilde{M}_0) \geqslant m$. To complete the proof it suffices to observe that if u is a critical point for $\tilde{f}$ then $\pm u$ are critical points for f. Q.E.D.

Suppose now that no parity conditions on $tb(x, t)$ is assumed, while all the other hypothesis are satisfied. Then lemma 3.2 still holds, and lemma 3.3 shall state only that if $\lambda_1 < 1$, then $M \neq \emptyset$. Let $p \in M$; repeating the arguments used in the proof of theorem 3.1, we have that $w(t, p)$ is defined for $t \in [0, +\infty)$ and $w(t, p)$ converges to a critical point p_0. Since $f(w(t, p))$ is nonincreasing, then we have that $f(p_0) < 0$ and thus $p_0 \neq 0$. Namely:

THEOREM 3.4. *Suppose conditions (a)-(b)-(c)-(e)-(f) are verified. If $\lambda_1 < 1$ then (1) has at least a proper solution.*

The following example shows that if $\lambda_1 \geqslant 1$, then problem (1) can have only the trivial solution.

Example 3.5. In the interval $[0, \pi]$ consider the problem

$$\begin{cases} y'' + y - b(y) = 0\,, & \left(y'' = \dfrac{d^2y}{dt^2}\right), \\ y(0) = y(\pi) = 0\,, \end{cases} \tag{12}$$

where $b(y)$ is such that $yb(y) > 0$. If (12) has a proper solution $\bar{y}$, we have

$$\int_0^\pi [\bar{y}(t)\bar{y}''(t) + \bar{y}^2(t) - \bar{y}(t)\, b(\bar{y}(t))]\, dt =$$
$$= \int_0^\pi \{-(\bar{y}'(t))^2 + \bar{y}^2(t) - \bar{y}(t)\, b(\bar{y}(t))\}\, dt = 0\,.$$

But, by Poincaré inequality, we have that

$$\int_0^\pi \bar{y}^2(t)\, dt \leqslant \int_0^\pi (y'(t))^2\, dt\,,$$

and so the preceding equality cannot hold.

The example before is a particular case of a general one. In fact suppose $b(x,t)$ satisfies the condition $tb(x,t) > 0$ $(t \neq 0)$, and let $\bar{u}$ be a non-trivial solution of (1); then:

$$\|\bar{u}\|^2 - ((C\bar{u}, \bar{u})) = -((B\bar{u}, \bar{u})) < 0\,. \tag{13}$$

On the other hand, if $\lambda_1 \geqslant 1$, is also $\|u\|^2 \geqslant ((Cu, u))$ for every $u \in \mathring{W}^1(\Omega)$, wich is in contradiction with (13).

Added in proof. While the present work was in printing, we got to know that D. C. Clark has obtained some abstract results closely related to ours. (Cf. D. C. CLARK, *A variant of the Lusternik-Schnirelman theory*, Ind. Univ. Math. J., **22** (1972)).

We wish to point out that the results obtained in a previous paper (cf. A. AMBROSETTI, *Esistenza di infinite soluzioni per problemi non lineari in assenza di parametro*, Atti Acc. Naz. Lincei, **52** (5) (1972)) concerning the equation (1) with $-b(x,t)$ replaced by $+b(x,t)$, were contained, with small variants in the hypothesis, in Hempel's Thesis (cf. J. A. HEMPEL, *Superlinear variational boundary value problems and nonuniqueness*, Ph. D. Thesis, University of New England) which we did not know.

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Manoscritto pervenuto in redazione il 14 luglio 1972.