

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

TITO A. CIRIANI

**The free parametrization for linear programming
with bounded variables**

Rendiconti del Seminario Matematico della Università di Padova,
tome 47 (1972), p. 77-107

[<http://www.numdam.org/item?id=RSMUP_1972__47__77_0>](http://www.numdam.org/item?id=RSMUP_1972__47__77_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1972, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

*Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques*
<http://www.numdam.org/>

THE FREE PARAMETRIZATION FOR LINEAR PROGRAMMING WITH BOUNDED VARIABLES

TITO A. CIRIANI *)

SUMMARY - « Free parametrization » is the study of the free variations of the coefficients in a row or column, in the neighborhood of the optimal solution of linear programming problems. The coefficients are not related by any particular function.

After a brief introduction of the optimization algorithm, free parametrization is described in theoretic detail. This is followed by an example which emphasises the uniformity of the techniques of post-optimal analysis and parametrization from the point of view of free parametrization.

In the appendix the quasi-null matrices are described with their properties and some related theorems which are employed in free parametrization.

1. Introduction to parametrization.

The importance of sensitivity analysis comes to light in those linear programming (LP) problems in which the coefficients of the model may vary.

Due to the simple approach of the problem, once we have reached the optimum, it is easy to obtain information which is then used for post-optimal analysis.

For example, considering the objective function (OF), it is interesting in many cases to know the limits within a single coefficient at a time may vary (the rest being constants) without changing either the structure or the value of the optimal solution, excepting the OF itself. This analysis is usually known as cost-ranging.

*) Indirizzo dell'A.: IBM Italia, Centro Scientifico di Pisa, Lungarno Mediceo 55, Pisa.

However, very often more than one coefficient at a time must be changed for the following reasons:

1) The model, representing a dynamic event, must be adapted to the new equilibrium reached by the phenomenon. For example, the components cost of an alloy adjusts itself to the trend of the raw materials market.

2) The OF and the constraints of model may be not completely linear: the optimal solution is such within the limits of the linearization forced into the problem. It is necessary, therefore, to make a check by changing the coefficients on the optimal solution.

3) The coefficients of the model are roughly known and, therefore, they can vary within a certain range rather than assume a fixed value.

Taking into consideration these requirements the post-optimal analysis related to the variation of a single coefficient at a time is insufficient. Generally, the parametrization technique is used. This technique studies the simultaneous changes of more than one coefficient under the hypothesis that they are connected by a linear function. For instance, let us indicate the coefficients of the OF c_j , then their variations are introduced by the following relationship:

$$\Delta c_j = \lambda \cdot p_j$$

where p_j is a constant and where λ is a scalar parameter.

Parametrization seeks the two λ values, corresponding to the values of the c_j coefficients, which do not change the optimal solution except for the OF.

These research tools are considered as an integral section of an LP system since, as we have already mentioned, they give essential information for the sensitivity analysis.

We think that a contribution to this type of analysis may come from the study of the free variation (indicated as « free parametrization », FP) not only of the OF coefficients but also of any row or column. In the following sections after the setting up of the FP theory, we give an algorithm for the solution of this problem and a two-dimensional example which gives a clear picture of the results.

2. Set up of a linear programming model.

Before considering the FP it is useful to introduce the symbols used in this paper for LP and briefly describe the optimization criteria.

An LP model may always be set up as follows:

$$\begin{array}{lcl}
 2.1 & \left\{ \begin{array}{l} \text{Under the conditions:} \\ \lambda^1 \leq EY_c \leq \Lambda^1 \\ \lambda^2 \leq Y_c \leq \Lambda^2 \\ \text{optimize the function:} \\ I^f EY_c \end{array} \right. & \begin{array}{l} \begin{array}{l} \text{(bounds of the con-} \\ \text{straints)} \\ \text{(bounds of the var-} \\ \text{iables)} \end{array} \\ \begin{array}{l} \text{[the bounds} \\ \text{can be posi-} \\ \text{tive or neg-} \\ \text{ative, finite} \\ \text{or infinite]} \end{array} \\ \text{(objective function)} \end{array}
 \end{array}$$

where:

- E is the coefficient matrix ($m \times n$),
- Y_c is the unknowns vector (n),
- λ^1, Λ^1 are the bound vectors of the constraints (m),
- λ^2, Λ^2 are the bound vectors of the variables (n),
- I^f is the f th row of the unit matrix I ($m \times m$).

By introducing the new variables:

$$Y_r = EY_c$$

we have the following equivalent formulation of the problem:

$$\begin{array}{lcl}
 2.2 & \left\{ \begin{array}{l} \text{Under the conditions:} \\ AY = \Phi \\ l \leq Y \leq L \\ \text{optimize the variable:} \\ Y_f (= I^f EY_c) \end{array} \right. & \text{with: } Y = \begin{bmatrix} Y_c \\ Y_r \end{bmatrix}
 \end{array}$$

where matrix A [with $m \times (n+m)$ dimensions] is obtained from the original matrix E and from the unit matrix $-I$ ($m \times m$). In particular, the vector of matrix A , corresponding to the OF Y_f , is the following:

$$A_f = -I_f.$$

The limits l and L are relative both to variables Y_c and Y_r of vector Y ($n+m$).

By changing the origin of the coordinates in the space of the Y 's, in order to make one of the bounds of the variables equal to zero we can write:

$$X = Y - R^1)$$

where R is the reference vector of the Y variables. In the case of an unbounded variable, the change is not effected and the corresponding reference value is therefore null.

We can finally have the following problem:

$$2.3 \quad \left| \begin{array}{l} \text{Under the conditions:} \\ AX = B \\ \gamma \leq X \leq \Gamma \\ \text{optimize the variable } X_f. \end{array} \right.$$

The new vectors are given by the relationship:

$$X = Y - R$$

and from the formula 2.2. Therefore, we have:

$$B = -AR$$

$$\gamma = l - R$$

$$\Gamma = L - R.$$

¹⁾ It is assumed that if the lower bound is finite then it is used as a reference.

With respect to 2.1, the formula 2.3 simplifies the theoretic algorithm of optimization and is better suited for data processing.

3. Generalized simplex algorithm set up.

Since FP is based on optimization concepts, we think it is better to introduce them in their natural context, and to use them later for different purposes.

The simplex algorithm is based on some simple relationships which can be deduced from formula 2.3.

Since the set of the X 's of the space with $(n+m)$ dimensions defined by:

$$AX=B$$

$$\gamma \leq X \leq \Gamma$$

is a convex hyperpolygon, the optimum of X_f , if it exists, is on a vertex of the set. This means that the optimum, if it exists, is obtained for « at least » n variables at one of their bounds and « at most » m variables within their bounds.

Given a « base » solution, in which $k \geq n$ variables are at their bounds, we indicate with $\underline{X}$ the vector of the solution corresponding to only the m variables within their bounds, and with $\underline{A}$ the corresponding base matrix, taken from the original matrix A . The generalized simplex algorithm seeks at each step, which variable among those out of the basis may improve of OF X_f .

Let us differentiate the relationship:

$$AX=B$$

considering only one variable X_s out of the basis at a time and its corresponding column vector A_s , we have:

$$\underline{A}d\underline{X} + A_s dX_s = \Phi$$

that is:

$$3.1 \quad d\underline{X} = -\underline{A}^{-1}A_s dX_s .$$

For the LP model 2.3, there is always a matrix $\underline{A}$ which has the inverse matrix. Certainly, such a matrix is that one formed by the m columns, corresponding to vector Y_r equal to the unit matrix $-I$.

From 3.1 for the OF we have:

$$dX_f = I^f dX = -I^f \underline{A}^{-1} A_s dX_s$$

and finally the variation of X_f , with respect to the variable X_s out of the basis, is equal to:

$$3.2 \quad \frac{dX_f}{dX_s} = -I^f \underline{A}^{-1} A_s .$$

This relationship is useful for studying FP.

By means of relationship 3.2 the simplex seeks a variable X_s (if it exists) among those out of the basis, that improves the OF X_f . If this variable is found, we go on to consider the constraints:

$$3.3 \quad \gamma \leq \underline{X} + dX \leq \Gamma$$

where the dX is given by 3.1. In this way we find the variable in the basis which may leave room for the X_s and which, without violating 3.3, is put at one of its bounds. This procedure goes on until we find an X_s or until we discover an unbounded solution or an infeasible solution of the problem itself.

If no X_s out of the basis improves the OF and the bounds 3.3 are not violated, then the solution found for the 2.3 is the optimum, provided that X_f is finite.

4. Free parametrization of a row.

When we use the expression free parametrization of a row A^r of matrix A , we mean the study of the free variations of the coefficients of the row A^r , in the neighborhood of the optimal solution of LP problem. The coefficients are not related by any particular function.

Let us introduce some definitions which will be useful later on. We indicate with:

$$A[\alpha]^i$$

a matrix A whose i th row has been substituted by vector α . Thus also with:

$$A[\beta]_j$$

we indicate the matrix A whose j th column is substituted by vector β . The operator $[\]$ is called « substitution ».

Vectors α and β may also be substituted by a matrix expression, which gives as a results a row vector and a column vector respectively. Furthermore, we indicate the null matrix with Z . By applying the substitution to Z we obtain the so-called « quasi-null » matrices. In Appendix A, to which we shall occasionally refer in the following sections, we give some properties of these matrices with some theorems.

4.1. *Search for the limits of the stability region (boundary region).*

After what has said above, we can state in analytical expressions the FP problem.

Let us consider matrix F so defined:

$$F = A + Z[k]^r$$

where k is a vector with $(n+m)$ dimensions and the quasi-null matrix is $m \times (n+m)$.

The elements of vector k are, therefore, the variations of the coefficients of the r th row of matrix A .

The problem of FP consists in finding in the Euclidean space of the parameters k_j the region (called « stability region ») in which the k_j 's may vary without changing the structure of the optimal (in the sense that the n variables at one of their limits are always the same).

We define « boundary region » that region in which a variable out of the basis may be exchanged with a variable in the basis without changing the optimum, as specified above.

To determine the stability region we must, first of all, find the hyperplanes which define the boundary region. Then we consider the intersection among those of the semi-spaces found by the hyperplanes which contain the $n+m$ of the a_i^r coefficients (of row A^r) to which the optimum corresponds. Finally, we must check the limits of the X 's, as in formula 2.3.

The boundary region is obtained by setting to zero the derivate of the OF with respect to the variable out of the basis which is being considered. Therefore, we will as many hyperplanes as there are variables out of the basis (n). By putting to zero this derivative with respect to any variable out of the basis X_s , from 3.2 we have:

$$\frac{dX_s}{dX_f} = -I^f(A + Z[\underline{k}]^r)^{-1}(A_s + k_s I_r) = 0$$

where $\underline{k}$ is the vector of the variations corresponding to the variables in the basis at the optimum. From Theorem 2 and from the properties of the quasi-null matrices (see Appendix A) we have:

$$I^f \underline{A}^{-1}(A_s + I_r k_s - I_r \underline{\bar{K}} \underline{A}^{-1} A_s - I_r \underline{\bar{K}} \underline{A}^{-1} I_r k_s) = 0$$

where

$$\underline{\bar{K}} = \frac{1}{1 + \underline{k} \underline{A}^{-1} I_r} \cdot \underline{k}.$$

Therefore we must have:

$$1 + \underline{k} \underline{A}^{-1} I_r \neq 0.$$

By multiplying by the denominator of $\underline{\bar{K}}$ and using:

$$\alpha = \underline{A}^{-1} A_s$$

we get the relationship:

$$4.1.1 \quad \alpha_f + (I^f \underline{A}^{-1} I_r) k_s + \underline{k} [\alpha_f \underline{A}^{-1} I_r - (I^f \underline{A}^{-1} I_r) \alpha] = 0$$

which defines the boundary region for the variable out of the basis X_s . It represents one of the n limits of the stability region.

We rewrite 4.1.1 substituting the symbol $\odot$ for the equals sign and putting α_f left hand, keeping in mind that $\odot$ may assume the value $\geq$, $\leq$ or $=$.

By introducing the notations:

$$\sigma_r = I^f \underline{A}^{-1} I_r$$

and

$$\omega = \alpha_f \underline{A}^{-1} \underline{I}_r - \sigma_r \alpha$$

we have:

$$4.1.2 \quad \sigma_r k_s + \underline{k} \omega \odot - \alpha_f .$$

Since the origin of the coordinates, in the space of the k_j 's, belongs to the stability region, the symbol $\odot$ depends on the sign of α_f .

Therefore, we may conclude that:

$$4.1.3 \quad \odot = \begin{cases} \leq & \text{if } \alpha_f < 0 \\ \geq & \text{if } \alpha_f > 0 \\ = & \text{if } \alpha_f = 0. \end{cases}$$

4.2. *Limits of the stability region corresponding to the constraints of the variables.*

Since the variation of the coefficients of any row A^r changes the value of the optimal solution, we must take into consideration the limits on the variables, as results from 2.3.

At the optimum the base solution is:

$$\underline{X} = \underline{A}^{-1}(-\underline{A}\underline{R} - \underline{A}\underline{Y}) = \underline{A}^{-1}\underline{B}$$

where:

$\underline{A}^{-1}$	inverse matrix of the matrix	$(m \times m)$
$\underline{A}$	base matrix	$(m \times m)$
$\underline{R}$	reference vector of the variables in the basis	(m)
$\underline{A}$	matrix of the variables out of the basis	$(m \times n)$
$\underline{Y}$	solution in 2.2 of the variables out of the basis	(n) :

$$\underline{Y} = \underline{X} + \underline{R}.$$

By applying the FP to the r th row of matrix A , we may introduce the matrix:

$$F = A + Z[k]^r \quad [\text{with } m \times (m+n) \text{ dimensions}]$$

so that the new base solution is:

$$4.2.1 \quad \underline{\underline{X}}^* = \underline{\underline{F}}^{-1}(-\underline{\underline{F}}\underline{\underline{R}} - \underline{\underline{F}}\underline{\underline{Y}}).$$

From Theorem 2 we have:

$$\underline{\underline{F}}^{-1} = \underline{\underline{A}}^{-1}(\underline{\underline{I}} - \underline{\underline{Z}}[\underline{\underline{K}}\underline{\underline{A}}^{-1}]^r)$$

with:

$$\underline{\underline{K}} = \frac{1}{1 + \underline{\underline{k}}\underline{\underline{A}}^{-1}\underline{\underline{I}}_r} \cdot \underline{\underline{k}}$$

while from the definition of $\underline{\underline{F}}$ we can deduce:

$$\underline{\underline{F}} = \underline{\underline{A}} + \underline{\underline{Z}}[\underline{\underline{k}}]^r$$

$$\underline{\underline{F}} = \underline{\underline{A}} + \underline{\underline{Z}}[\underline{\underline{k}}]^r$$

where $\underline{\underline{k}}$'s are the parameters corresponding to the m variables in the basis and $\underline{\underline{k}}$'s those corresponding to the n variables out of the basis.

By substituting in 4.2.1 the expressions found, we have:

$$4.2.2 \quad \underline{\underline{X}}^* = \underline{\underline{X}} - \underline{\underline{A}}^{-1}\underline{\underline{I}}_r\underline{\underline{K}}\underline{\underline{X}} - (\underline{\underline{k}}\underline{\underline{R}} + \underline{\underline{k}}\underline{\underline{Y}})\underline{\underline{A}}^{-1}\underline{\underline{I}}_r \cdot \frac{1}{1 + \underline{\underline{k}}\underline{\underline{A}}^{-1}\underline{\underline{I}}_r}.$$

By multiplying the expression by the denominator of $\underline{\underline{K}}$, we have, for the components of vector $\underline{\underline{X}}^*$, the following:

$$4.2.3 \quad \underline{\underline{X}}^*_i(1 + \underline{\underline{k}}\underline{\underline{A}}^{-1}\underline{\underline{I}}_r) = \underline{\underline{X}}_i + \underline{\underline{k}}\underline{\underline{X}}_i\underline{\underline{A}}^{-1}\underline{\underline{I}}_r - \underline{\underline{k}}\underline{\underline{Y}}\underline{\underline{I}}^i\underline{\underline{A}}^{-1}\underline{\underline{I}}_r$$

for the $i \in J_b$, the set of the indices of the variables in the basis, being

$$\underline{\underline{k}} = |\underline{\underline{k}}\underline{\underline{k}}| \quad \text{and} \quad \underline{\underline{Y}} = \left| \frac{\underline{\underline{Y}}}{\underline{\underline{Y}}} \right| = \left| \frac{\underline{\underline{X}} + \underline{\underline{R}}}{\underline{\underline{X}} + \underline{\underline{R}}} \right|$$

both $(m+n)$.

In 2.3 the only finite values of the limits for the variables may be:

$$\gamma_i = 0 \quad \text{and} \quad \Gamma_i = \begin{cases} 0 \\ \underline{\underline{U}}_i = \underline{\underline{L}}_i - \underline{\underline{l}}_i. \end{cases}$$

Finally, from 4.2.1 we have:

$$4.2.4 \quad \begin{cases} -X_i \leq \underline{k} X_i \underline{A}^{-1} I_r - k Y I^i \underline{A}^{-1} I_r & (\text{for } \gamma_i = 0) \\ U_i - X_i \geq \underline{k} (X_i - U_i) \underline{A}^{-1} I_r - k Y I^i \underline{A}^{-1} I_r & (\text{for } \Gamma_i \neq \infty) \end{cases} \quad [\text{with } i \in J_b]$$

The 4.1.2's, with the choice 4.1.3 with 4.2.4, define the stability region for the FP of any row A^r .

These relationships are easily used for data processing since the parameters k_j are explicit and their coefficients are known at the optimum.

5. Free parametrization of a column.

When we use the expression free parametrization of a column A_c of the matrix A , we mean the study of the free variations of the coefficients of the column A_c , in the neighborhood of the optimal solution of LP problems. The coefficients are not related by any particular function.

As for one of the rows of matrix A , also in the case of a column it is interesting to find the limits of the stability region, in the space of the parameters k_j , defined by the relationship:

$$F = A + Z[k]_c$$

where A is the matrix of the optimization model and k is a column vector having m components.

In the case of the FP of the coefficients of any column A_c , we may have two cases. The corresponding variable X_c is in the basis at the optimum or it is out of the basis at the optimum.

5.1. Search for the limits of the stability region, in the case of variables in the basis at the optimum.

By means of the definition of the boundary region, we can find the expressions of the limits for the stability region.

As we did for the FP of a row, we set to zero the derivative of the

OF with respect to any variable out of the basis X_s . From 3.2 we have:

$$\frac{dX_f}{dX_s} = -I^f(\underline{A} + Z[k]_c)^{-1}A_s = 0$$

From corollary 1 of Theorem 2 (see Appendix A) we have:

$$-I^f(I - Z[\underline{A}^{-1}\underline{K}]_c)A^{-1}A_s = 0$$

where:

$$\underline{K} = \frac{1}{1 + I^c \underline{A}^{-1} \underline{k}} \cdot \underline{k}$$

therefore, we must have:

$$1 + I^c \underline{A}^{-1} \underline{k} \neq 0$$

Indicating in the same way:

$$\alpha = \underline{A}^{-1}A_s$$

from the properties of the quasi-null matrices we have:

$$5.1.1 \quad \alpha_f + (\alpha_f I^c - \alpha_c I^f) \underline{A}^{-1} \underline{k} = 0$$

which represents the boundary region for the variable out of the basis X_s . As in the case of 4.1.1 from this relationship and from the sign of α_f the limits for the stability region are:

$$5.1.2 \quad \left\{ \begin{array}{l} (\alpha_f I^c - \alpha_c I^f) \underline{A}^{-1} \underline{k} \odot -\alpha_f \\ \text{where:} \\ \odot = \left\{ \begin{array}{l} \leq \text{ if } \alpha_f < 0 \\ \geq \text{ if } \alpha_f > 0 \\ = \text{ if } \alpha_f = 0 \end{array} \right. \end{array} \right.$$

for the n variables out of the basis.

5.2. *Limits of the stability region corresponding to the constraints of the variables (FP for a variable in the basis at the optimum).*

Considering the new value of the solution, when the parameters k_j vary, we impose the limits on the variables by means of 2.3.

We must keep in mind that in the case of the coefficients of a variable in the basis we have:

$$\underline{F} = \underline{A} + Z[k]_c \quad \text{and} \quad \underline{\underline{F}} = \underline{\underline{A}}$$

where $\underline{A}$ is the base matrix at the optimum ($m \times m$) and $\underline{\underline{A}}$ the matrix out of the basis ($m \times n$).

It follows that the new solution in the neighborhood of the optimum is:

$$\underline{X}^* = \underline{F}^{-1}(-\underline{F}\underline{R} - \underline{\underline{F}}\underline{\underline{Y}})$$

according to the symbols used in section 4.

From the hypotheses made and from corollary 1 of Theorem 2 (Appendix A) we have:

$$\underline{X}^* = (I - Z[\underline{A}^{-1}\underline{k}]_c)\underline{A}^{-1}[-(\underline{A} + Z[k]_c)\underline{R} - \underline{\underline{A}}\underline{\underline{Y}}]$$

From the properties of the quasi-null matrices we reach the relationship:

$$\underline{X}^*(1 + I^c \underline{A}^{-1} \underline{k}) = \underline{X} + (\underline{X} I^c - Y_c I) \underline{A}^{-1} \underline{k}$$

For the components and taking into consideration that the finite values for the limits may be:

$$\gamma_i = 0 \quad \text{and} \quad \Gamma_i = \begin{cases} 0 \\ U_i = L_i - l_i \end{cases}$$

we conclude that:

$$5.2.1 \quad \begin{cases} -X_i \leq (X_i I^c - Y_c I_i) \underline{A}^{-1} \underline{k} & (\text{for } \gamma_i = 0) \\ U_i - X_i \geq [(X_i - U_i) I^c - Y_c I^i] \underline{A}^{-1} \underline{k} & (\text{for } \Gamma_i \neq \infty) \end{cases}$$

[Y_c is the c th component of the vector Y]

The 5.2.1's, extended to all the m variables in the basis, give the limits for the stability region, as a consequence of the limits on the variables.

5.3. *Stability region for the free parametrization of the coefficients of a variable out of the basis at the optimum.*

By means of similar considerations, we reach the relationships, which follow, taking into consideration that for a variable out of the basis we must have:

$$\underline{F} = \underline{A}$$

$$\underline{\underline{F}} = \underline{\underline{A}} + Z[k]_c$$

For the single variable X_c out of the basis, we must have:

$$5.3.1 \quad \left\{ \begin{array}{l} \alpha_f + I^f \underline{A}^{-1} k \odot 0 \\ \text{with:} \quad \odot = \left\{ \begin{array}{l} \leq \text{ for } \alpha_f < 0 \\ \geq \text{ for } \alpha_f > 0 \\ = \text{ for } \alpha_f = 0 \end{array} \right. \end{array} \right. \quad [\alpha = \underline{A}^{-1} A_c]$$

and for the m variables in the basis:

$$5.3.2 \quad \gamma_i = X_i \leq -I^i \underline{A}^{-1} k Y_c \leq \Gamma_i - X_i$$

6. Notes on the stability region.

First of all, we can see that the stability region of the FP of either a row or a column is a set which contains at least the point $k = \Phi$.

In fact it is easy to check through the 4.2.4's, the 5.2.1's and the 5.3.2's that the point $k = \Phi$ satisfies the inequalities, since the solution X is the optimum.

Therefore these inequalities do not contradict those which follow from the definition of the boundary region.

Particular attention should be given to the conditions:

$$1 + \underline{k} \underline{A}^{-1} I_r \neq 0 \quad \text{and} \quad 1 + I^c \underline{A}^{-1} k \neq 0$$

for the FP of the r th row and the c th column (in the basis) respectively.

In fact, if the stability region contains a segment of the hyperplane defined by:

$$6.1 \quad 1 + \underline{k} \underline{A}^{-1} \underline{I}_r = 0 \quad \text{or} \quad 1 + \underline{I}^c \underline{A}^{-1} \underline{k} = 0$$

the computational algorithm must seek the alternative optimal solution, if it exists.

It is well known that the main reason for truncation errors is due to the representation of the numbers in the computer, made through a finite number of digits. Therefore, we may conclude that the stability region, for a programming algorithm, is defined by the inequalities found, except for changes in a stratum, having the hyperplane defined by one of the 6.1's as the median element and limited width, as a consequence of the effects produced by truncation errors (provided that the 6.1's pass through the stability region).

7. Free parametrization of the objective function.

From 2.3 it is evident that the objective function corresponds to the f th row of matrix A , with the property that:

$$7.1 \quad A_f = -I_f$$

Furthermore, the corresponding variable X_f is always in the basis at the optimum.

After what has been said above, we can state in analytical terms the problem of the FP of the OF.

Let us consider matrix F so defined:

$$F = A + Z[k]^f$$

where k is a vector with $(n+m)$ dimensions having the component $k_f = 0$, and the null matrix Z is $m \times (n+m)$.

The elements of vector k are, therefore, the variations of the coefficients of the OF, corresponding to the f th row of matrix A .

The Problem of the FP of the OF consists in finding in the Euclidean space of the parameters k_j , the region (called stability region) in which the k_j 's may vary without changing the structure (in the sense that the n variables at one of their limits are always the same), furthermore, the value of the optimal solution does not vary, except for the OF.

By taking advantage of the result obtained for any row and from 7.1 and the fact that component k_f is considered null, we come to the relationship:

$$7.2. \quad \alpha_f - k_s + \underline{k}\alpha = 0$$

which represents the boundary region for the variable out of the basis X_s . It defines one of the n limits of the stability region.

From 7.2 it is easy to reach the representation of the stability region:

$$7.3 \quad \left\{ \begin{array}{l} -k_s + \underline{k}\alpha \odot -\alpha_f \\ \text{with} \end{array} \right. \odot = \left\{ \begin{array}{l} \leq \text{ for } \alpha_f < 0 \\ \geq \text{ for } \alpha_f > 0 \\ = \text{ for } \alpha_f = 0 \end{array} \right.$$

It is to be noted that:

$$\underline{k}\underline{A}^{-1}I_f = 0$$

and therefore the problem of checking the inequality:

$$1 + \underline{k}\underline{A}^{-1}I_f \neq 0$$

as in the case of a row does not exist.

It is easy to check that for the OF the conditions 4.2.4 do not arise, since the coefficients of the k_j are zero due to the condition $k_f = 0$ and 7.1. Therefore, when the k_j coefficients vary in the stability region defined by the 7.3's, the optimal solution does not change (except for the OF). This means that the 7.3's define the stability region for the OF.

7.1. *Properties of the boundary region for the Objective Function.*

What fallows is of help finding of a particular property of the hyperplanes which define the boundary region.

We decompose the base matrix at the optimum A as follows:

$$7.1.1 \quad A = \begin{vmatrix} S & \Phi \\ \underline{c} & -1 \end{vmatrix}$$

where with $\underline{c}$ we indicate the row vector $(m-1)$ of the OF coefficients corresponding to the variables in the basis, except the OF variable itself, whose coefficient (-1) is given separately; Φ is the null vector $(m-1)$ and finally the matrix S $(m-1) \times (m-1)$ contains the remaining coefficients.

It is to be noted that, for easy computation, we put row f and column f last, without, however, losing the generality of the proofs.

We may easily check that:

$$A^{-1} = \begin{vmatrix} S^{-1} & \Phi \\ \underline{c}S^{-1} & -1 \end{vmatrix}$$

As for the decomposition 7.1.1, for a vector corresponding to a variable X_s out of the basis, we can write:

$$A_s = \begin{vmatrix} S_s \\ \underline{c}_s \end{vmatrix}$$

and therefore the scalar:

$$\alpha_f = I^f \alpha = I^f A^{-1} A_s$$

becomes:

$$\alpha_f = I^f \begin{vmatrix} S^{-1} & \Phi \\ \underline{c}S^{-1} & -1 \end{vmatrix} \cdot \begin{vmatrix} S_s \\ \underline{c}_s \end{vmatrix}$$

That is:

$$7.1.2 \quad \alpha_f = \underline{c}S^{-1}S_s - \underline{c}_s$$

Likewise:

$$7.1.3 \quad \alpha_j = I^j S^{-1} S_s$$

By substituting 7.1.2 and 7.1.3 in 7.2 we obtain:

$$7.1.4 \quad cS^{-1}S_s - c_s + \sum_{j \in J_b} I^j S^{-1} S_s k_j - k_s = 0$$

where J_b is the set of the indices of the variables in the basis.

There are n of these hyperplanes, as many as there are variables out of the basis. We may conclude that the projection of these hyperplanes into the spaces of the variables in the basis and of only k_s out of the basis (corresponding to the hyperplanes considered) covers the point:

$$7.1.5 \quad P \equiv \{-c_j, -c_s\}$$

with: $j \in J_b$ (the set of the variables in the basis)

7.2. Stability straight line and its properties.

Let us consider the straight line:

$$7.2.1 \quad \left\{ \begin{array}{l} \frac{k_i}{c_i} = \frac{k_j}{c_j} \quad \text{for: } i \neq j; c_i \neq 0; c_j \neq 0 \text{ with } i \text{ and } j \\ \text{extended to all the variables} \\ \text{or:} \\ k_i = 0 \quad \text{for } c_i = 0 \end{array} \right.$$

The points belonging to this straight line have the property that they do not make either the value of the optimal solution (except the OF) or the slope of the OF hyperplane vary.

It follows that 7.2.1 belongs to the stability region and, therefore, is called « stability straight line ».

The projection of the stability straight line into the space of the k_j 's corresponding to the variables in the basis, and of any k_s corresponding to a variable out of basis, covers the point P , defined by 7.1.5.

We can see that the segment of the stability straight line which starts from point P and, passing through zero goes on towards the infinite, belongs to the stability region which leaves unchanged the direction of the optimum. It is easy to check that the other segment reverses this direction. Furthermore, all the projections of hyperplanes 7.2, into the space of only the k_j 's corresponding to variables in the basis, cover the point:

$$P' \equiv \{-c_j\} \quad \text{with} \quad j \in J_b$$

through which also the projection of the stability straight line 7.2.1 into the same space passes.

7.3. *Notes on the stability region for the objective function.*

On the basis of the 7.3's it is possible to sep up an algorithm to give information about the values of the assigned parameters k_j .

In fact, we can check to see whether simultaneous variations of the OF coefficients (equal to $c_j + k_j$) change the base and it is also possible to determine the limits of the stability region within given ranges of the excepted variations of the coefficients. Finally, if we fix the range of the parameters k_j or of their linear combinations, it is possible, with the LP criterion, to optimize the OF which, from 4.2.3 is:

$$X_f^* = X_f + kY$$

where X_f^* is the new value and X_f the value at the optimum (constant) of the OF.

We consider the 7.3 endogenous constraints together with the exogenous constraints for the k_j 's. In this way, we obtain the optimum of the OF X_f^* when the coefficients of the OF itself (without any alteration of the basis) vary.

8. **Coefficients as random variables.**

Besides making a contribution to sensitivity analysis, this paper is particularly interesting when we give a probability distribution to the

coefficients of the vectors concernend in the FP, considered as variables. In this situation a problem of decision arises which often presupposes the knowledge of the probability distribution of the optimum of the OF and of the optimal solution, which in this case appear as random variables obtained from the coefficients in question. Under this light, many interesting problems arise, for example, the study of these transformed random variables, when we know the moments of the coefficients in question. We must note the simplifications which may be introduced on the basis of the relationships found, if we make particular hypotheses of the distribution of the said coefficients.

9. Post-optimal analysis and parametrization as a particular case of free parametrization.

If the parameters k_j are tied by the relationship:

$$9.1 \quad k_j = \lambda p_j$$

where: λ is the parameter common to all the variables
 j is extended to all the variables
 p_j is a constant characteristic of the j th variable

we obtain the parametrization of the OF.

By substituting 9.1, in 7.2 we have in fact:

$$\alpha_f + \lambda \sum_{j \in J_b}^m \alpha_j p_j - \lambda p_s = 0$$

that is:

$$9.2 \quad \lambda = \frac{\alpha_f}{p_s - \sum_{j \in J_b}^m \alpha_j p_j}$$

Since we consider only parameter λ , we have to calculate 9.2 for all the n variables out of the basis in order to find the minimum positive

value and the maximum negative value. In fact, within such limits of λ the basis does not change.

If we want to find also the relationships of the post-optimal analysis for the OF (cost-ranging) it is sufficient to consider just one variable at a time.

In fact, with only $p_s \neq 0$ we obtain from 9.2:

$$\lambda p_s = k_s = \alpha_f$$

which is a relationship giving the cost-ranging for the variables out of the basis.

While, by putting $p_s = 0$ and only $p_i \neq 0$ among all the variables in the basis, still from 9.2 we have:

$$\lambda p_i = k_i = - \frac{\alpha_f}{\alpha_i}$$

where the vector α corresponds to any variable X_s out of the basis.

This relationship gives the cost-ranging values for the variable in the basis X_i , choosing among all the possible ratios the positive minimum and the negative maximum.

10. Example of a two-dimensional stability region.

Let us consider the following LP problem:

maximize the function: $X_1 + X_2 = X_f$

under the conditions:

$$0 \leq 2X_1 + X_2 \leq 2 \quad (=X_3)$$

$$0 \leq X_1 + 2X_2 \leq 2 \quad (=X_4)$$

$$0 \leq X_1 \leq \infty$$

$$0 \leq X_2 \leq \infty$$

This problem may be represented geometrically as in fig. 1.

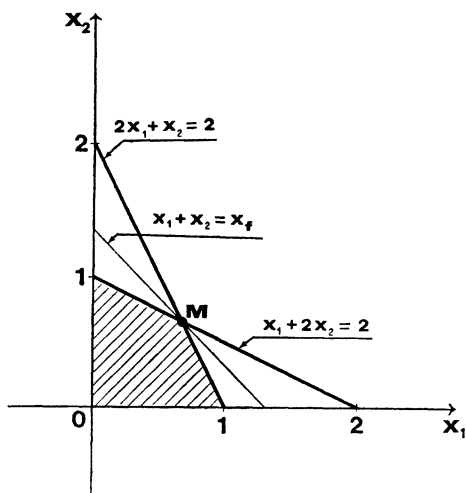


Fig. 1 - Graphic representation of an LP problem.

With the notations of the generalized simplex algorithm it follows that the matrix A (3×5) is:

$$A = \begin{vmatrix} 1 & 1 & -1 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 \\ 2 & 2 & 0 & 0 & -1 \end{vmatrix}$$

$$\begin{matrix} X_1 & X_2 & X_f & X_3 & X_4 \end{matrix}$$

At point M the function X_f assumes the maximum value.

Therefore, the matrix in the basis at the optimum is:

$$A = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

whose inverse matrix is:

$$A^{-1} = \begin{vmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ -1 & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$$

Let us consider the variables out of the basis X_3 and X_4 . If we want to analyse the FP for only the coefficients of OF X_f corresponding to the structural variables X_1 and X_2 ²⁾ (in the basis at the optimum) we must calculate the vectors corresponding to the variables out of the basis, i.e.

$$\alpha_{(X_3)} = \begin{vmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{vmatrix} \begin{matrix} X_1 \\ X_2 \\ X_f \end{matrix} \qquad \alpha_{(X_4)} = \begin{vmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{vmatrix} \begin{matrix} X_1 \\ X_f \\ X_2 \end{matrix}$$

Making use of the criteria in section 4, we obtain the following two-dimensional stability region:

$$\begin{cases} -2k_1 + k_2 \leq 1 \\ k_1 - 2k_2 \leq 1 \end{cases}$$

represented in fig. 2.

²⁾ In this case it is possible to check all the properties of the stability region, since the point P and the straight line r are on the plane:

$$k_3 = 0$$

$$k_4 = 0.$$

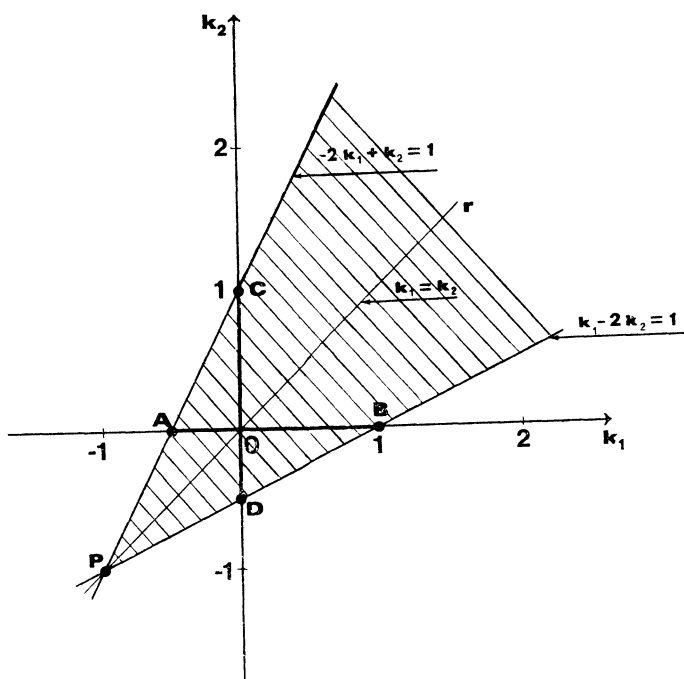


Fig. 2 - Cost ranging and free parametrization of the objective function.

It is easy to note that segments $\overline{AO}$, $\overline{OB}$ and $\overline{CO}$, $\overline{OD}$ represent the cost-ranging values for the variables X_1 and X_2 respectively. Besides, it is easy to check that the straight line r , passing through P , defined by the equation:

$$k_1 = k_2$$

is the stability straight line.

If the linear relationship exists between the variables k_1 and k_2 defined by:

$$\begin{cases} k_1 = \lambda p_1 \\ k_2 = \lambda p_2 \end{cases}$$

where p_1 and p_2 are constants, we obtain the parametrization.

By putting:

$$p_1=3 \quad \text{and} \quad p_2=1$$

and zero values for the variables out of the basis, from 9.2 we have:

$$\lambda = \begin{cases} 1 & (k_1=3, k_2=1) \\ -\frac{1}{5} \left(k_1 = -\frac{3}{5}, k_2 = -\frac{1}{5} \right) \end{cases}$$

Since in the hypothesis made:

$$k_1=3k_2$$

from fig. 3 we have the picture of parametrization with respect to the FP of the OF.

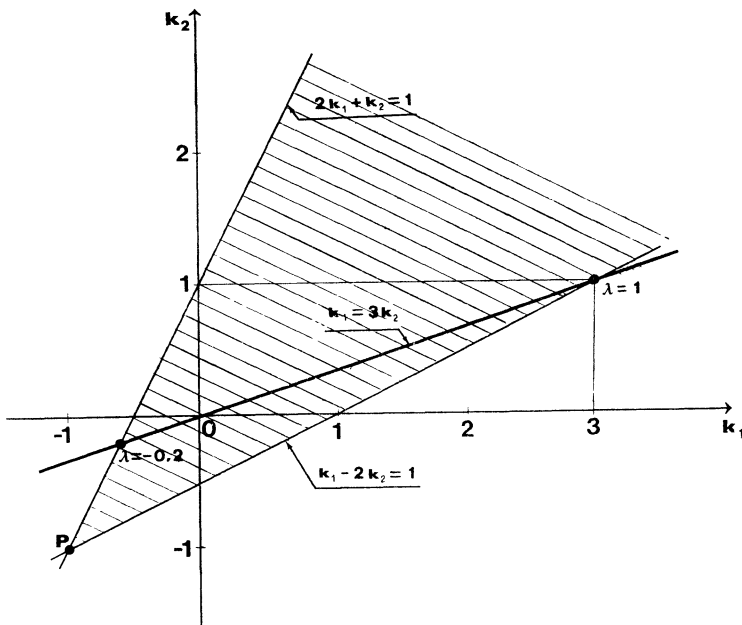


Fig 3 - Parametrization and free parametrization of the objective function.

11. Conclusions.

Free parametrization offers the possibility of having a complete picture of the problems with the variation of the coefficients of a row or column of the matrix of an LP problem.

The theoretic presentation has taken advantage of the use of the quasi-null matrices, for their property of simplifying some steps of the matrix algebra.

At present a study is in progress for the setting up of a program, based on the FP algorithm, to establish the capabilities at a data processing level.

It is a pleasure to thank Prof. M. Volpato and Prof. U. Richard for their comments and criticisms.

I am indebted to Prof. F. Giannessi, who read the paper and made interesting suggestions: this helped me in giving a better revised form of the paper.

APPENDIX A

The quasi-null matrices.

A.1. General properties.

Given the null matrix Z ($m \times m$), we define quasi-null ($q-n$) matrix the matrix Z in which only one row or only one column are substituted.

By using the substitution, already introduced in section 4, we can write that:

$Z[\varphi]^l$: substitution of the vector φ in the l th row

$Z[\theta]_l$: substitution of the vector θ in the l th column.

By using the following symbols:

$A_j \theta \beta I_j$: column vectors m

$A^i \varphi \alpha I^i$: row vectors m

A: matrix $m \times m$

I: unit matrix m

we may easily check the following properties of the $q-n$:

$$\text{a.1} \quad A \cdot Z[\theta]_i = Z[A\theta]_i$$

$$\text{a.2} \quad Z[\varphi]^j \cdot A = Z[\varphi A]^j$$

$$\text{b.1} \quad Z[\theta]_i \cdot A = \theta A^i$$

$$\text{b.2} \quad A \cdot Z[\varphi]^j = A_j \varphi$$

$$\text{c.1} \quad \alpha \cdot Z[\theta]_i = I^i \alpha \theta$$

$$\text{c.2} \quad Z[\varphi]^j \beta = I_j \varphi \beta$$

$$\text{d.1} \quad Z[\theta]_i \beta = \theta \beta_i$$

$$\text{d.2} \quad \alpha \cdot Z[\varphi]^j = \alpha_j \varphi$$

where β_i and α_j are scalar.

Given these relationships, we have the following:

$$\text{e.1} \quad Z[\theta]_i \cdot Z[\beta]_k = Z[\theta \beta_i]_k$$

$$\text{e.2} \quad Z[\varphi]^j \cdot Z[\alpha]^i = Z[\varphi_i \alpha]^i$$

$$\text{e.3} \quad Z[\varphi]^j \cdot Z[\theta]_i = Z[I_j \varphi \theta]_i = Z[I^i \varphi \theta]^j$$

$$\text{e.4} \quad Z[\theta]_i \cdot Z[\varphi]^k = \begin{cases} \varphi \theta & \text{for } i=k \\ Z & \text{for } i \neq k. \end{cases}$$

A.2. Theorems on the substitution.

THEOREM 1. Given the square matrix A , with the inverse matrix A^{-1} and given a vector column A_s different from the null vector Φ , and putting:

$$\eta = (I[A^{-1}A_s]_r)^{-1}$$

called elementary matrix, we may say that:

if by definition: $F = A[A_s]_r$

then:

$$F^{-1} = \eta A^{-1}.$$

PROOF. By the definition of $q-n$ we have:

$$F = A[A_s]_r = A - Z[A_r]_r + Z[A_s]_r.$$

For the property a.1 and for the property of the inverse matrix, we may write:

$$A^{-1} \cdot F = I - Z[I_r]_r + Z[A^{-1}A_s]_r = I[A^{-1}A_s]_r.$$

From this expression, by the definition of the elementary matrix η , we have:

$$A.1 \quad F^{-1} = \eta A^{-1}.$$

COROLLARY TO THEOREM 1. With similar steps we have:

if by definition $F = A[A^s]_r$

then:

$$F^{-1} = A^{-1}\eta$$

where the elementary matrix in this case is:

$$\eta = (I[A^s A^{-1}]_r)^{-1}.$$

THEOREM 2. Let A be a square matrix having the inverse matrix A^{-1} and let k be a row vector with the same dimensions as a vector of matrix A , then we have:

$$(A + Z[k^r])^{-1} = A^{-1}(I - Z[\bar{k}A^{-1}]_r)$$

where:

$$\bar{k} = \frac{1}{1 + kA^{-1}I_r} k$$

provided that:

$$A.2 \quad 1 + kA^{-1}I_r \neq 0.$$

PROOF. From the corollary to Theorem 1 we have:

$$A.3 \quad (A + Z[k]^r)^{-1} = A^{-1}\eta$$

where:

$$\eta = (I[(A^r + k)A^{-1}]^r)^{-1}$$

Let us examine the matrix:

$$\mu = \eta^{-1} = I[(A^r + k)A^{-1}]^r = I + Z[kA^{-1}]^r.$$

The derminant of μ is:

$$|\mu| = 1 + kA^{-1}I_r.$$

The inverse matrix of μ exists only for $|\mu| \neq 0$, from which we have the condition A.2.

This condition is the same as:

$$A^r + k \neq \sum_{\substack{i=1 \\ i \neq k}}^m \gamma_i A^i$$

where the γ_i 's are linear coefficients. Therefore, in the linear transformation of the r th row of matrix A we must avoid those values of the parametres k_j which make A_r a linear combination of the other rows.

The matrix M of the μ cofactors has the principal diagonal equal to $|\mu|$, except for the pivot element which is unit. The other non-zero elements of M are in the r th column and are:

$$-kA^{-1}I_i$$

except for the pivot which, as already stated, is unit.

By indicating with M' the transposed matrix of M , by definition of the inverse matrix we have:

$$A.4 \quad \mu^{-1} = \eta = \frac{M'}{|\mu|} = I - Z[kA^{-1}]^r$$

where by definition:

$$K = \frac{1}{|\mu|} k.$$

From the relationships A.3 and A.4 we have the proof of the theorem.

COROLLARY 1. Let A be a square matrix having the inverse matrix A^{-1} and let k be a column vector with the same dimensions as a vector of A , then with similar steps we have the relationship:

$$(A + Z[k]_c)^{-1} = (I - Z[A^{-1}\bar{k}]_c)A^{-1}$$

where:

$$\bar{k} = \frac{1}{1 + I^c A^{-1} k} \cdot k$$

provided that:

$$1 + I^c A^{-1} k \neq 0.$$

COROLLARY 2. With conditions similar to those of Theorem 2, if we suppose that:

$$A_f = -I_f \quad \text{and} \quad k_f = 0$$

then:

$$(A + Z[k]_f)^{-1} = A^{-1}(I - Z[kA^{-1}]_f).$$

In fact, it is easy to check that:

$$kA^{-1}I_f = 0$$

due to conditions A.5.

COROLLARY 3. With conditions similar to Corollary 1, if we suppose that:

$$A^f = -I^f \quad \text{and} \quad k_f = 0$$

then:

$$(A + Z[k]_f)^{-1} = (I - Z[A^{-1}k]_f)A^{-1}.$$

BIBLIOGRAPHY

- DANTZIG, G. B.: *Linear Programming and Extensions*, Princeton University Press, Princeton 1963.
- FLETCHER, R.: *Optimization*, Academic press, London 1969.
- GANTMACHER, F. R.: *Théorie des Matrixes* (2 vol.), Dunod, Paris 1966.
- GASS, S. I.: *Linear Programming - Methods and Applications*, McGraw Hill, New York 1969.
- GIANNESI, F.: *Sulla programmazione lineare stocastica*, Calcolo, Vol. IV, Fascicolo 3, Roma 1967.
- ORCHARD-HAYS, W.: *Advanced Linear Programming Computing Techniques*, McGraw Hill, New York 1968.
- PIERRE, D. A.: *Optimization Theory with Applications*, John Wiley & Sons, 1969.
- SIMONNARD, M.: *Programmation Linéaire*, Dunod, Paris 1962.
- VAJDA, S.: *Mathematical Programming*, Addison-Wesley, Reading (Mass.) 1961.
- VOLPATO, M.: *Sulla stabilità dei programmi lineari. Nuovi studi e modelli di ricerca operativa*, UTET, Torino 1970.

Manoscritto pervenuto in redazione il 4 ottobre 1971.