Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 47 (1972), p. 65-75

<http://www.numdam.org/item?id=RSMUP_1972__47__65_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 47 (1972)

FC-GROUPS AND RELATED CLASSES

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Herrn Professor W. Specht zum 65. Geburtstag gewidmet

The present paper deals with classes of groups centered around the notion of finite conjugacy. The main result is a linear chain of inclusions, some of which are wellknown; in a second section we consider some ramifications of this chain. As an application a rather simple proof of Černikov's theorem ([57], theorem 1) is given.

Organization of the paper: For theorems of the type « *A* implies *B*, but not conversely », we prove the « but not consersely » part by giving a counterxample, indicated as such. Notations are essentially those of P. Hall.

In particular F, G, P, T, A denote the class of finite, finitely-generated, periodic, torsion-free, abelian groups, respectively. If X is a class of groups, $\mathrm{Q}X$, $\mathrm{S}X$, $\mathrm{N}X$, N_0X, $\mathrm{L}X$, L^*X, $\mathrm{Z}X$, $\mathrm{K}X$ are defined, respectively, by:

$G \in \mathrm{Q}X$	iff G is an epimorphic image of some X-group
$G \in \mathrm{S}X$	iff G is a subgroup of some X-group
$G \in \mathrm{N}X$	iff G is a product of X-groups each of which is a normal in G
$G \in \mathrm{N}_0X$	iff G is a finite product of X-groups each of which is normal in G

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The author gratefully acknowledges support from the Deutsche Forschungsgemeinschaft.

$G \in \mathrm{L}X$	iff G is locally an X-group
$G \in \mathrm{L}^*X$	iff G is locally-normal an X-group, i.e. any finite subset of G is contained in a normal X-subgroup of G
$G \in \mathrm{Z}X$	iff G is centre-by-X, i.e. $G/Z(G) \in X$
$G \in \mathrm{K}X$	iff G has its commutator G' in X.

Furthemore, standard notations are:

$U \leq G$	U is a subgroup of G
$N \trianglelefteq G$	N is a normal subgroup of G
$C \trianglelefteq \mid G$	C is a characteristic subgroup of G
$\mid G : U \mid$	index of the subgroup U in G
$C(S)$	centralizer of the subset S in G
$Z(G)$	centre of G
G'	commutator of G

$[x, y] = x^{-1}y^{-1}xy$

$\langle S \rangle$	subgroup generated by S in G
$gp\{S \mid R\}$	group generated by the set S with relations R
Aut (G)	automorphism-group of G
$G \times H$	direct product of two groups
$D \prod_{i \in I} G_i$	direct (restricted) product of groups
$C \prod_{i \in I} G_i$	cartesian product of groups
$C(n)$	cyclic group of order n
$C(\infty)$	cyclic group of infinite order
$C(p^\infty)$	Prüfer group of type p
$GwrH$	restricted wreath product.

For help and encouragement, my thanks go to H. Heineken, W. Specht, S. E. Stonehewer, M. J. Tomkinson and E. Wittmann.

1. A linear chain of inclusions is established.

1.1. DEFINITION. A group G is said to be an FC-group, $G \in FC$, if each element $x \in G$ has only a finite number of conjugates in G, i.e. $|G : C(x)| < \infty$ for all $x \in G$.

It follows immediately that subgroups and homomorphic images of FC-groups are again FC-groups. The following propositions are easily proved.

1.2. Finitely generated FC-groups are centre-by-finite, but not conversely:

$$G \cap FC < \mathrm{z}F.$$

For a counterexample take any abelian, not finitely generated group, for instance $C(p^\infty)$.

1.3. Any centre-by-finite group G is FC, but not conversely:

$$\mathrm{z}F < FC.$$

For there exist, for each cardinal $\aleph$, FC-groups G such that $\mathrm{card}\,(G/Z(G)) > \aleph$.

1.4. Any finite-by-abelian group G. i.e. $G' \in F$, is FC, but not conversely:

$$\kappa F < FC.$$

For there exist, for each cardinal $\aleph$, FC-groups G such that $\mathrm{card}\,(G') > \aleph$.

The next theorem by I. Schur [07] shows that 1.3. is a consequence of 1.4.:

1.5. Any centre-by-finite group G is finite-by-abelian, but not conversely:

$$\mathrm{z}F < \kappa F.$$

For a counterexample take the group given by B. H. Neumann [54], a direct product of countably many copies of the quaternion group Q_8 with amalgamated centres.

A list of various proofs of 1.5. is given by K. W. Gruenberg in [70], p. 191.

The following lemma proves to be useful when dealing with *FC*-groups:

1.6. If G is FC, then for any subgroup $U \leq G$:

$$|G : C(U)| < \infty \Leftrightarrow |U : U \cap Z(G)| < \infty.$$

PROOF. « $\Rightarrow$ » Let $G = \sum_{i=1}^{n} x_i C(U)$ a coset representation; then

$$Z(G) = C(C(U)) \cap \bigcap_{i=1}^{n} C(x_i),$$

therefore

$$|U : U \cap Z(G)| = |U : U \cap C(C(U)) \cap \bigcap_{i=1}^{n} C(x_i)| = |U : U \cap \bigcap_{i=1}^{n} C(x_i)|$$
$$< \prod_{i=1}^{n} |G : C(x_i)| < \infty.$$

« $\Leftarrow$ » Let $U = \sum_{i=1}^{n} y_i (U \cap Z(G))$ a coset representation; then $C(U) = \bigcap_{i=1}^{n} C(y_i)$ and thus $|G : C(U)| < \prod_{i=1}^{n} |G : C(y_i)| < \infty$.

1.7. DEFINITION. Call a group locally normal (strictly speaking locally finite-normal), if every finite subset of G is contained in a finite normal subgroup of G, $G \in \mathrm{L}^*F$.

REMARK. This condition is evidently equivalent to the following: G is a product of finite normal subgroups:

$$G \in \mathrm{N}F.$$

Furthemore, by the lemma of Dietzmann, G is locally normal iff G is periodic FC:

$$FC \cap P = \mathrm{L}^*F = \mathrm{N}F.$$

1.8. For any FC-group G, $G/Z(G)$ is locally normal, but not conversely:

$$FC < \mathrm{zN}F$$

(cf. R. Baer [48], p. 1026).

PROOF. First of all, $G/Z(G)$ is an FC-group, because of $\mathrm{Q}FC = FC$. For $x \in G$, $C(\langle x \rangle) = C(x)$, i.e. $|G : C(\langle x \rangle)| < \infty$. Using 1.6., we get: $|\langle x \rangle : \langle x \rangle \cap Z(G)| < \infty$, hence there is a natural n such that $x^n \in Z(G)$, which means that $G/Z(G)$ is periodic. By the remark above, $G/Z(G)$ is locally normal.

For a counterexample, take any free nilpotent group, of class 2, infinite rank and finite exponent $e \neq 2$.

1.9. Centre-by-locally normal implies centre-by-locally finite, but not conversely:

$$\mathrm{zN}F < \mathrm{zL}F.$$

A counterexample is S_∞, the group of finite permutations of a countable set, which has trivial centre and is by no means FC.

1.10. If G is centre-by-locally finite, then G is locally centre-by-finite, but not conversely:

$$\mathrm{zL}F < \mathrm{Lz}F.$$

PROOF. We have to show that each finite subset is contained in a $\mathrm{z}F$-subgroup of G. Because of the inclusions $\mathrm{z}F < FC$, $\mathrm{s}FC = FC$ and $G \cap FC < \mathrm{z}F$, it is sufficient to prove that each finitely generated subgroup of G is centre-by-finite.

Let $U \leq G$ be finitely generated; then $UZ(G)/Z(G) \leq G/Z(G)$ is finite, i.e. $|U : U \cap Z(G)| < \infty$. But $|U : Z(U)| \leq |U : U \cap Z(G)|$, hence $U \in \mathrm{z}F$.

For a counterexample take the relative holomorph

$$G = \mathrm{Hol}\,(C(p^\infty), \langle \alpha \rangle),$$

where $\alpha \in \mathrm{Aut}\,(C(p^\infty))$ is defined by $x \mapsto x^\alpha = x^{1+p}$, for all $x \in C(p^\infty)$. Here $Z(G) \cong C(p)$, the socle of $C(p^\infty)$, whence $G/Z(G) \cong G$. G. is locally a $\mathrm{z}F$-group, because for every finite subset S of $C(p^\infty)$ there exists a power α^s which leaves fixed every element of S. On the other hand, G is not locally finite, since α has infinite order.

REMARK. The classes $\mathrm{LZ}F$, $\mathrm{LK}F$ and $\mathrm{L}FC$ are equal, since the theorems 1.2., 1.4., 1.5. imply $G \cap \mathrm{Z}F = G \cap \mathrm{K}F = G \cap FC$, thus we are concerned with the class of locally-FC groups.

1.11. G locally-FC implies G' locally finite, but not conversely:

$$\mathrm{L}FC < \mathrm{KL}F.$$

PROOF. Let $S \subset G'$ be a finite subset. Write the elements of S as finite products of commutators and consider the subset T of G made up by the elements occuring in these commutators. By assumption, T is contained in a $\mathrm{K}F$-subgroup of G. Again, as $\mathrm{K}F < FC$, $\mathrm{S}FC = FC$ and $G \cap FC < \mathrm{K}F$, also $\langle T \rangle \in \mathrm{K}F$. Evindently $S \leq \langle T \rangle'$, i.e. S is contained in a finite subgroup of G', whence $G \in \mathrm{KL}F$.

For a counterexample, take $G = A_5 wr C(\infty)$. Here G' is the direct product of infinitely many copies of the A_5, and therefore, G' is locally finite. G is finitely generated and not an FC-group.

We conclude this section with two trivial implications:

1.12. If G' is locally finite, then G' is periodic, but not conversely:

$$\mathrm{KL}F < \mathrm{K}P.$$

E. S. Golod and I. R. Shafarevitch have proved (cf. Herstein [68], p. 193): For every prime p there exists an infinite group G generated by three elements such that every element of G has finite order a power of p. These groups have finite factor G/G', and therefore serve as counterxamples in this case, too.

1.13. If G' is periodic, then the elements of finite order form a (characteristic) subgroup of G, but not conversely:

$$\mathrm{K}P < PT.$$

PROOF. $\mathrm{K}P = PA < P(PT) = (PP)T = PT$.

Counterexample: any non-abelian torsion-free group.

To sum up, we proved the following chain of strict inclusions:

1.14.
$$G \cap FC < \mathrm{z}F < \mathrm{K}F < FC < \mathrm{ZN}F < \mathrm{ZL}F < \mathrm{L}FC < \mathrm{KL}F < \mathrm{K}P < PT.$$

2. The chain is ramified.

We now consider some ramifications of the basic chain given above.

2.1. DEFINITION. *A group G is called a group with finite layers, $G \in FL$, if the number of elements of any given order is finite.*

Obviously, FL-groups are periodic and FC, so that they constitute a subclass of the class of locally normal groups.

Furthemore, $\mathrm{s}FL = FL = \mathrm{Q}FL$. (For the more difficult equality $FL = \mathrm{Q}FL$, see R. Baer [48], p. 1030).

The following generalization of 1.3. considerably sharpens a result of I. I. Eremin [59], p. 52 (cf. also Cernikov [63], §10).

2.2. If $G/Z(G)$ is FL, is FC, but not conversely:

$$\mathrm{z}F < \mathrm{z}FL < FC.$$

PROOF. For $x \in G$, consider the sequence $C(x) \leq C_Z(x) \leq G$, where $C_Z(x)/Z := C_{G/Z}(xZ)$. $C(x)$ is easily seen to be normal in $C_Z(x)$, and $C_Z(x)/C(x) \cong \dfrac{C_Z(x)/Z}{C(x)/Z}$. Since $\mathrm{s}FL = \mathrm{Q}FL = FL$, $C_Z(x)/C(x)$ is an FL-group. G/Z being periodic, there exists and r such that $x^r \in Z$. Now, for every $y \in C_Z(x)$, $[y, x]^r = [y^r, x] = [y, x^r] = 1$, since $[y, x] \in Z$. Therefore $y^r \in C(x)$. The FL-group $C_Z(x)/C(x)$ having finite exponent dividing r, the index $|C_Z(x) : C(x)|$ is finite. Furthermore, because of $FL < FC$ we have $|G : C_Z(x)| < \infty$; this proves that $|G : C(x)|$ is finite, and therefore $G \in FC$.

FL-groups being necessarily countable, we see that $\mathrm{z}FL \neq FC$. On the other hand, $\mathrm{z}F \neq \mathrm{z}FL$, because there are infinite FL-groups with trivial centre which can be constructed as follows:

Let τ be an injection of the set P of prime numbers into P such that $\tau(p) \equiv 1 \bmod p$ for every $p \in P$; for each $p \in P$, let H_p be a group of order $p \cdot \tau(p)$ with trivial centre

$$\text{(e.g. } H_p = gp\{x, y \mid x^p = y^{\tau(p)} = 1,\ x^{-1}yx = y^{\lambda(p)}, \text{ where } 1 \neq \lambda(p) \equiv 1 \bmod \tau(p)\}).$$

Now, the direct product $G = D \prod_{p \in P} H_p$ has trivial centre and is FL.

By the way, FL-groups with centre 1 are necessarily thin (all p-Sylow subgroups finite), because otherwise there would be a group of type $C(p^\infty)$ in the centre (cf. Černikov [48], theorem 1).

Next we look at the class $\mathrm{zn}F$ from a different point of view.

REMARK. Although it is true that $\mathrm{n}F$ and L^*F are the same class, it is not true in general, that $\mathrm{n}X = \mathrm{L}^*X$. But if X is N_0-closed, then clearly $\mathrm{n}X = \mathrm{L}^*X$. Therefore we have to distinguish between the operators N and L^*.

2.3. If $G/Z(G)$ is locally normal, G is locally normal a $\mathrm{z}F$-group, but not conversely:

$$\mathrm{zn}F < \mathrm{L}^*\mathrm{z}F.$$

PROOF. For any finite subset $S \subset G$, there exists, by assumption, N with $SZ \leq N/Z \trianglelefteq G/Z$ such that $|N:Z| < \infty$. Obviously $S \subseteq N$, $Z \leq Z(N)$, $N \in \mathrm{z}F$, whence $G \in \mathrm{L}^*\mathrm{z}F$.

A counterexample is given by the following group:

Take

$$X = C \prod_{i=1}^{\infty} X_i, \quad \text{the cartesian product of groups} \quad X_i = gp\{x_i \mid x_i^3 = 1\}$$

and

$$Y = D \prod_{i=1}^{\infty} Y_i, \quad \text{the direct product of groups} \quad Y_i = gp\{y_i \mid y_i^2 = 1\}$$

and let G be the split extension of X by Y defined by the relations $x_i^{y_i} = x_i^2$ and $x_j^{y_i} = x_j$ for $j \neq i$. Then $Z(G) = 1$ and G is not locally normal, for the element $(x_1, x_2, \ldots) \in X$ has infinitely many conjugates in G. But G is locally normal-$\mathrm{z}F$, because any finite subset lies in the join of X with finitely many Y_i's, $XY_1 \ldots Y_n$, say. Groups of this type are centre-by-finite, because $Z(XY_1 \ldots Y_n) = C \prod_{i=n+1}^{\infty} X_i$.

REMARK. The class $\mathrm{z}F$ is not N_0-closed. More specific information is given by the following result (due to W. Specht):

The product of a normal Z_nF-subgroup and a normal Z_mF-subgroup is a $\mathrm{Z}_{n+m}F$-subgroup, where Z_nF denotes the class of groups having its n^{th} therm of the upper central series of finite index.

A direct consequence of Schur's theorem 1.5. is:

2.4. If G is locally normal-$\mathrm{Z}F$, it is also locally normal-$\mathrm{K}F$, but not conversely: $\mathrm{L}^*\mathrm{Z}F < \mathrm{L}^*\mathrm{K}F$.

The following counterexample, given by M. J. Tomkinson, represents a generalization of Neumann's example 1.5. using cartesian products with amalgamated centre instead of direct products with amalgamated centre.

Let

$Q = C\prod_{i=1}^{\infty} Q_i$, where $Q_i = gp\{a_i, b_i \mid a_i^2 = b_i^2 = [a_{i'}, b_i], [a_i, b_i]^2 = 1\}$ (quaternion group).

$Z(Q) = C\prod_{i=1}^{\infty} C_i$, where $C_i = gp\{c_i\}$, $c_i = [a_i, b_i]$.

$Z(Q)$ has exponent 2, and thus may be written $Z(Q) = D\prod_{\gamma\in\Gamma} C_\gamma$, where $C_\gamma = gp\{c_\gamma \mid c_\gamma^2 = 1\}$; note that each C_i appears as a C_γ.

Let Q_1 be the group generated by all elements $c_\gamma c_\delta^{-1}$. Then $|Z(Q) : Q_1| = 2$ and $Z(Q/Q_1) = Z(Q)/Q_1$.

Let H be the split extension of Q by $X := D\prod_{i=1}^{\infty} X_i$, where $X_i = \{x_i \mid x_i^2 = 1\}$, such that the following relations hold:

$$a_i^{x_i} = b_i\,; \qquad b_i^{x_i} = a_i$$

$$a_j^{x_i} = a_j\,; \qquad b_j^{x_i} = bj \qquad\qquad \text{for } j \neq i.$$

Finally, let $G = H/Q_1$.

$QX_1 \ldots X_n/Q_1 \in \mathrm{K}F$, and therefore $G \in \mathrm{L}^*\mathrm{K}F$.

But $gp\{(a_1, a_2, \ldots, a_n, \ldots)^H\} = Q$ and $Q/Q_1 \notin \mathrm{Z}F$.

This proves $G \notin \mathrm{L}^*\mathrm{Z}F$.

2.5. If G is locally normal-$\mathrm{K}F$, G' is locally normal, but not conversely:

$$\mathrm{L}^*\mathrm{K}F < \mathrm{K}\mathrm{L}^*F.$$

PROOF. By 1.14.,

$$\text{L}^*\text{K}F \leq \text{L}FC < \text{KL}F < \text{K}P,$$

thus it remains to show: G' is FC.

We even prove slightly more: Denote by $F(G)$ the fully invariant subgroup consisting of all elements of G that have only a finite number of conjugates in G. Then $G' \leq F(G)$.

For, if $x \in G'$, let $S_x \subset G$ denote the finite subset of all elements constituting the commutator word x. By assumption, there is $N \trianglelefteq G$, such that $S_x \subseteq N$ and N' finite. Therefore we have $x \in \langle S_x \rangle' \leq N' \trianglelefteq$ $\trianglelefteq | N \trianglelefteq G$, i.e. x lies in a finite normal subgroup N' of G and consequently has only a finite number of conjugates in G.

For a counterexample take the one given in 1.11.

REMARK. In fact, we have shown that locally normal-KF groups are FC-nilpotent of class 2 (cf. F. Haimo [53] Corr. 3).

Finally we have

2.6. G' locally normal implies G' locally finite, but not conversely:

$$\text{KL}^*F < \text{KL}F.$$

Again, as in 1.9., the group S_∞, whose comumtator A_∞ is locally finite but not FC, may serve as a counterexample.

We are now able to give a simple proof for a theorem (due to Černikov [57], theorem 1) characterizing FC-groups:

2.7. The class of FC-groups is exactly the class of central extensions of torsion-free abelian groups by locally normal groups.

PROOF. For G an FC-group, take A a maximal torsion-free subgroup of $Z(G)$ such that $Z(G)/A$ is periodic (Zorn).

By 1.8., $G/Z(G)$ is periodic, hence G/A is periodic and therefore locally normal.

Conversely, if $1 \to A \to G \to G/A \to 1$ is an exact sequence with A torsion-free contained in $Z(G)$ and G/A locally normal, then

$$G/Z(G) \cong \frac{G/A}{Z(G)/A}$$

is locally normal.

Now, by 1.14., $\mathrm{ZL}^*F < \mathrm{K}P$, i.e. G' is periodic. Since

$$G = G/1 = G/(A \cap G') \precsim G/A \times G/G',$$

G is isomorphic to a subgroup of the direct product of an *FC*-group and an abelian group, which again is *FC*.

Inserting the results of this section in 1.14., we get:

$$\begin{array}{ccccccccccccc} & & \nearrow \mathrm{Z}FL \searrow & & & & \nearrow \mathrm{ZL}F & \to & \mathrm{L}FC \searrow & & & & \\ G \cap FC \to & \mathrm{Z}F & & FC \to & \mathrm{ZN}F & & & \nearrow & & \mathrm{KL}F & \to \mathrm{K}P & \to & PT \\ & & \searrow \mathrm{K}F \nearrow & & & \searrow \mathrm{L}^*\mathrm{Z}F & \to & \mathrm{L}^*\mathrm{K}F & \to & \mathrm{KL}^*F \nearrow & & & \end{array}$$

REFERENCES

[07] SCHUR, I.: *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. reine angew. Math., 132 (1907), 85-137.

[48] BAER, R.: *Finiteness properties of groups*, Duke Math. J., 15 (1948), 1021-1032.

[48] CERNIKOV, S. N.: *Infinite groups with finite layers*, Translation in Transl. AMS, series 1, vol. 1 (Algebra), 1962, 339-391.

[53] HAIMO, F.: *The FC-chain of a group*, Canad. J., 5 (1953), 498-511.

[54] NEUMANN, B. H.: *Group covered by permutable subsets*, J. London Math. Soc., 29 (1954), 236-248.

[57] CERNIKOV, S. N.: *Über die Struktur von Gruppen mit endlichen Klassen konjugierter Elemente*, Dokl. Akad. Nauk SSSR, 115 (1957), 60-63 (Russian).

[59] EREMIN, I. I.: *Gruppen mit endlichen Klassen konjugierter abelscher Untergruppen*, Math. Sbornik, 47 (1959), 45-54 (Russian).

[63] CERNIKOV, S. N.: *Endlichkeitsbedingungen in der Gruppen-theorie*, Berlin: VEB Deutscher Verlag der Wiss. Mathem. Forschungsberichte XX (1963).

[68] HERSTEIN, I. N.: *Noncommutative Rings*, New York: Wiley 1968.

[70] GRUENBERG, K. W.: *Cohomological Topics in Group Theory*, Berlin-Heidelberg-New York: Springer 1970.

Manoscritto pervenuto in redazione il 30 settembre 1971.