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RINGS OF CONTINUOUS FUNCTIONS WITH VALUES IN A NON-ARCHIMEDEAN ORDERED FIELD

G. De Marco *) and M. Richter **)}

1. Introduction.

In a recent paper, [DMW], investigations have been made over the ring $C(X, F)$ of continuous functions from a topological space $X$ into a proper subfield $F$ of the field of real numbers. Here we show that the techniques used in [DMW] lead, with suitable modifications, to analogous results if $F$ is a non-archimedean ordered field.

Our main concern here are the residue class fields of $C(X, F)$ and $C^*(X, F)$ (the ring of bounded functions). It does not seem easy to give a full description of these fields by known concepts. The analogue of [DMW, 2.1] reads: If a residue class field of $C^*(X, F)$ contains one new element of a completion of $F$, it actually contains the whole completion (Theorem 3.3). But here the analogy ends: $F$ is never dense in a residue class field $K$ of $C^*(X, F)$ unless $K = F$ or measurable cardinals are considered (Proposition 3.2, Corollary 3.4).

In II we describe those properties of ordered fields which will be used later. In III and IV residue class fields of $C^*(X, F)$ and $C(X, F)$ are studied and in V we take the special case that $F$ is real closed.

No inquiry is made about the structure spaces of $C(X, F)$ and $C^*(X, F)$ since it is clear that such an analysis can be carried out as in [DMW, 1] without essential modifications.

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II. General remarks on order fields.

The term « ordered » always means « totally ordered ». Throughout the whole paragraph, the letter $K$ denotes an ordered field. With the order topology, $K$ is a topological field. On its subsets, $K$ induces an order and a topology, which is not, in general, the topology of the induced order.

**Proposition 2.1.** Let $F$ be a subfield of $K$. Then $F$ (with the topology of the induced order) is a topological subfield of $K$ if and only if $F$ is cofinal in $K$. Otherwise, $F$ is a discrete subspace of $K$.

**Proof.** $F$ is cofinal in $K$ if and only if $F^+ = \{ x \in F : x > 0 \}$ is coinitial in $K^+ = \{ y \in K : y > 0 \}$. And this is clearly equivalent to the fact that the neighborhoods of 0 in $F$ are the sets $V \cap F$, $V$ neighborhood of 0 in $K$. The last assertion is then also clear.

If $S \subseteq K$, $cl_K(S)$ denotes the closure of $S$ in the order topology of $K$.

**Proposition 2.2.** The subset $S$ of $K$ is topologically dense in $K$ if and only if it is order dense in $K$. If $F$ is a subfield of $K$, then $cl_K(F)$ is a subfield of $K$, which is cofinal in $K$ if and only if $F$ is. If $F$ is not cofinal in $K$, then $cl_K(F) = F$.

**Proof.** The last statement follows from 2.1. The remaining are obvious.

Given a non-empty $S \subseteq K$, we denote by $\omega(S)$ its cofinality type (which is an initial ordinal). A cut $\gamma$ of $K$ is an ordered pair of sets $(A, B)$ such that $K = A \cup B$ and $A < B$ (i.e., $a \in A, b \in B$ imply $a < b$). The cut $\gamma$ is said to be trivial if either one of the following is true: (i) $A = \emptyset$; (ii) $B = \emptyset$; (iii) $A$ has a maximum; (iv) $B$ has a minimum. A cut $\gamma = (A, B)$ is called a Cauchy cut if $A$ and $B$ are non empty and $B - A = \{ b - a : a \in A, b \in B \}$ is coinitial in $K^+$. If $F$ is a subfield of $K$ and $\gamma = (A, B)$ is a cut of $F$, an element $a \in K$ such that $A \leq a \leq B$ is said to fill $\gamma$. Given $a \in K$, put $A_a = \{ x \in F : x < a \}$, $B_a = \{ y \in F : y > a \}$. The cofinality type of $A_a$ is denoted by $\omega_{F^+}(a)$, the cointiality type of $B_a$ by $\omega_{F^-}(a)$. If $a \notin F$, $(A_a, B_a)$ is a non-trivial cut of $F$ which we denote by $\gamma_F(a)$. 

PROPOSITION 2.3. Let $F$ be a cofinal subfield of $X$, and let $a$ belong to $K \setminus F$. Then the following are equivalent

(i) $a \in \text{cl}_K(F)$

(ii) $\gamma_F(a)$ is a Cauchy cut of $F$.

Consequently, $\text{cl}_K(F) = \{ a \in K : a \text{ fills a Cauchy cut of } F \}$.

**Proof.** (i) $\Rightarrow$ (ii). Since $F$ is cofinal in $K$, $A_a$ and $B_a$ are nonempty. Assume that $B_a - A_a$ is not cointial in $F^+$, that is $B_a - A_a > \varepsilon > 0$ for some $\varepsilon \in F$. Put $V_\varepsilon = \{ x \in K : |x| < \varepsilon \}$. By (i), $a \in F + V_\varepsilon$, that is $a = c + u$ for some $c \in F$ and $u \in V_\varepsilon$. If $u > 0$, then $c \in A_a$; and $c + \varepsilon > c + u = a$, hence $c + \varepsilon \in B_a$. But this implies $\varepsilon \in B_a - A_a$, a contradiction. Analogously $u < 0$, $c \in B_a$ and $c - \varepsilon \in A_a$ imply $\varepsilon \in B_a - A_a$.

(ii) $\Rightarrow$ (i). If $a \notin \text{cl}_K(F)$, there is $\varepsilon \in K^+$ such that $(a - \varepsilon, a + \varepsilon) \cap F = \emptyset$. Hence $A_a < a - \varepsilon < a + \varepsilon < B_a$ imply $B_a - A_a > 2 \varepsilon$ that is, $B_a - A_a$ is not cointial in $F^+$; hence $B_a - A_a$ cannot be cointial in $F^+$.

Concerning the topology of $K$, we prove:

**PROPOSITION 2.4.** $K$ is connected if and only if $K$ is a copy of $\mathbb{R}$ (the real numbers field). Otherwise, $K$ admits a base $\mathcal{V}(K)$ of open-and-closed neighborhoods of 0, $\mathcal{V}(K) = \{ V_\alpha : \alpha \in \omega(K) \}$ such that $\alpha < \beta$ implies $V_\beta \subseteq V_\alpha$. If $K$ is non archimedean, the $V_\alpha$ may be chosen as subrings of $K$.

**Proof.** Assume $K \neq \mathbb{R}$. If $K$ is archimedean, $K$ is a copy of a proper subfield of $\mathbb{R}$, and $\omega(F) = \omega$, the first infinite ordinal. Take $a \in \mathbb{R} \setminus K$ and put $V_n = \{ x \in K : |x| < |a|/n \}$ for each $n \in \omega_0$.

If $K$ is non archimedean, let $\{ a_\alpha : \alpha \in \omega(K) \}$ be a subset of $K^+$, cointital in $K^+$, such that $a_\alpha < 1$ for all $\alpha \in \omega(K)$. Put $U_\alpha = \{ x \in K : |a| < a_\alpha/n \}$ for all positive integers $n$. Since $K$ is non archimedean, the $U_\alpha$ are neighborhoods of 0, and they clearly are a base. If $x, y \in U_\alpha$, then

$$|x + y| \leq |x| + |y| \leq a_{\alpha/2n} + a_{\alpha/2n} = a_{\alpha/n}$$

and

$$|x \cdot y| = |x| |y| \leq a_{\alpha/n} \cdot 1$$

for all positive integers $n$. Thus the $U_\alpha$ are subrings of $K$; hence they are open-and-closed, and the family of all intersection $W_\alpha = \bigcap_{\beta \leq \alpha} U_\beta$ clearly
contains a subfamily \( V_\alpha \) satisfying the required conditions.

From now on, \( \mathcal{U}(K) \) will always denote such a neighborhood base.

As every topological field, \( K \) is said to be complete if it is complete with respect to the uniform structure \( \mathcal{U} \) generated by the entourages \( W_\alpha = \{(x, y) \in K \times K : x - y \in V_\alpha, V_\alpha \in \mathcal{U}(K)\} \). The completion \( \overline{K} \) of the uniform space \( (K, \mathcal{U}) \) may be given a structure of an ordered field, whose topology is that of the completion. The field \( K \) is obviously topologically dense in \( \overline{K} \) (and hence order dense), and \( \overline{K} \) is unique up to an order-preserving field isomorphism which is the identity on \( K \). By 2.3 only the Cauchy cuts of \( K \) are filled in \( \overline{K} \). Clearly, \( \overline{K} \) fills all the Cauchy cuts of \( K \). For each \( a \in \overline{K} \), \( \omega_f^- (a) = \omega_f^+(a) = \omega(K) = \omega(K) \).

**Proposition 2.5.** Let \( F \) be a cofinal subfield of \( K \), and let \( F \) be a completion of \( F \). Then \( \text{cl}_K(F) \) is isomorphic to a subfield of \( F \) containing \( F \).

**Proof.** Extension theorem for uniformly continuous mappings.

We refer the reader to [GJ, chapter 13] for terminology and basic facts on real closed fields. By [S], if \( F \) is real closed, then \( F \) is real closed.

**Proposition 2.6.** Let \( K \) be a real closed extension of the real closed field \( F \). If every Cauchy cut of \( F \) is filled in \( K \), then there exists a copy \( E \) of \( F \) in \( K \), such that \( F \subseteq E \subseteq K \).

**Proof.** Let \( \Phi \) be the set of all isomorphisms \( \varphi : L \to K \), where \( L \) is a real-closed extension of \( F \) in \( \overline{F} \), which are the identity on \( F \). If \( \varphi, \psi \in \Phi \), we write \( \varphi \preceq \psi \) if \( \psi \) is an extension of \( \varphi \). Let \( \varphi \in \Phi \), \( \varphi : L \to K \), and \( a \in F \setminus L \); assume that \( b \) fills in \( K \) the Cauchy cut \( \gamma_f(a) \). Then \( b \notin \varphi[L] \), since \( \varphi \) is order-preserving; hence \( b \) is transcendental over the real closed field \( \varphi[L] \). Thus \( \varphi \) has an extension \( \varphi' : L(a) \to \varphi[L](b) \), such that \( \varphi'(x) = \varphi(x) \) for \( x \in L \), and \( \varphi'(a) = b \). By [GJ, 13.12], \( \varphi' \) is order preserving. Hence \( \varphi' \) has an extension \( \psi \) from the real closed field \( L' \) (which is the algebraic closure of \( L(a) \) in \( \overline{F} \)) to the algebraic closure of \( \varphi[L](b) \) in \( K \).

By Zorn's lemma, \( \Phi \) has a maximal element \( \varphi \) and we have shown that the domain of \( \varphi \) is all of \( \overline{F} \).
III. Residue class fields.

Let $X$ be a topological space, $F$ a non-archimedean ordered field. We denote by $C = C(X, F)$ the lattice ordered ring of all continuous functions from $X$ to $F$; $C^* = C^*(X, F)$ will be the subring of $C$ consisting of all $f \in C$ such that $f[X]$ has a lower and an upper bound in $F$.

If $P$ (resp. $P^*$) is a prime ideal of $C$ (resp. $C^*$) the ring $C/P$ (resp. $C^*/P^*$) is totally ordered under the quotient ordering. (The proofs are analogous to those given in [GJ, chapter 5].) The canonical mapping of $C$ onto $C/P$ (resp. of $C^*$ onto $C^*/P^*$) maps the constants onto a copy of $F$, which, again, will be denoted by $F$.

A set $E$ of idempotents of $C$ such that $\sum_{e \in E} e(x) = 1$ for all $x \in X$ is called a partition of unity into idempotents, or simply a partition ($|E|$-partition if we want to emphasize its cardinality). A partition $E$ is said to be strongly contained in the ideal $I$ of $C$ (resp. $C^*$) if $\sum_{e \in S} e \in I$ for all $S \subseteq E$ such that $|S| < |E|$.

In what follows, $M^*$ is a maximal ideal of $C^*$ and $K$ is the ordered field $C^*/M^*$. Observe that $F$ is cofinal in $K$. The next lemma, analogous to [DMW, Lemma 2.1] is the key step in the study of the residue class fields of $C^*$ and $C$.

**Lemma 3.1.** Let $f \in C^*$. If $a = M^*(f)$ (the residue class of $f$ modulo $M^*$) does not belong to $f[X]$, then $M^*$ strongly contains either an $\omega_F^-(a)$-partition or an $\omega_F^+(a)$-partition.

**Proof.** Assume first $f[X] < a$. We show that, in this case, $M^*$ strongly contains an $\omega_F^-(a)$-partition. Put $A_a = \{x \in F : x < a\}$, and choose a well-ordered cofinal subset of $A_a$, say $\{a_\alpha : \alpha \in \omega_F^-(a)\}$ (where $a_\alpha < a_\beta$ if $\alpha < \beta$, and for each limit ordinal $\gamma < \omega(a)$, $\sup\{a_\alpha : \alpha < \gamma\}$ does not exist in $F$) and for each $\alpha$ choose $V_\alpha \in \mathfrak{V}(F)$ (see 2.4) in such a way that $a_{\alpha + 1} > a_\alpha + V_\alpha$. Put $L_\alpha = \{x \in F : x < a_\alpha + v$ for some $v \in V_\alpha\}$. Clearly the $L_\alpha$ are open-and-closed, $L_{\alpha + 1} \supseteq L_\alpha$, and $\bigcup_{\alpha < \beta} L_\alpha = A_\alpha$. Thus, $Z_a = a \in \omega_F(a)$ $= f^*[L_\alpha \setminus \bigcup_{\alpha < \beta} L_\beta]$ is open and closed in $X$, the $Z_\alpha$ are pairwise disjoint, and $\bigcup_{\alpha \in \omega_F(a)} Z_\alpha = X$. Furthermore, given $\alpha_0 \in \omega_F(a)$, there exists $\alpha \geq \alpha_0$ such that $Z_\alpha \neq \emptyset$, since, otherwise, we would have $f[X] < a_{\alpha_0 + 1} < a$, which is
impossible. This shows that \( \{ e_\alpha : \alpha \in \omega_\infty^{-}(a) \} \), where \( e_\alpha \) is the characteristic function of \( Z_\alpha \), is an \( \omega_\infty^{-}(a) \)-partition. For each \( \alpha \in \omega_\infty^{-}(a) \), put \( g_\alpha = f \lor a_{\alpha + 2} - f \). Since \( M^*(g_\alpha) = M^*(f \lor a_{\alpha + 2}) - M^*(f) = a \lor a_{\alpha + 2} - a = 0 \), \( g_\alpha \in M^* \). Define \( h_\alpha \) on \( X \) to be \( 1/g_\alpha \) on \( \bigcup_{\beta \leq \alpha} Z_\beta = f^*\{L_\alpha\} \), to be zero otherwise. Observe that \( |h_\alpha| < (a_{\alpha + 2} - a_{\alpha + 1})^{-1} \). Hence \( h_\alpha \in C^* \), and \( \sum_{\beta \leq \alpha} e_\beta = h_\alpha g_\alpha \in M^* \).

This proves that \( \{ e_\alpha : \alpha \in \omega_\infty^{-}(a) \} \) is strongly contained in \( M^* \).

In an analogous way it can be shown that if \( f[X] > a \), then \( M^* \) strongly contains an \( \omega_\infty^+(a) \)-partition. It only remains to prove that we may always assume either \( f \subseteq X \subseteq a \) or \( f \subseteq X \supset a \).

If \( \alpha \in F \), simply use \( f \land a \) (or \( f \lor a \)) instead of \( f \).

If \( \alpha \notin F \), put \( B_\alpha = \{ y \in F : y > a \} \); \( A_\alpha \), \( B_\alpha \) are open-and-closed in \( F \), since \( F \) is cofinal in \( K \). Let \( e \) be the characteristic function of \( f^*\{A_\alpha\} \). Then \( f = fe + f(1 - e) \), and since \( M^* \) is prime, exactly one of the two idempotents \( e, 1 - e \) belongs to \( M^* \). The conclusion is now obvious.

**Proposition 3.2.**

(i) If \( M^* \) contains no partitions, then \( C^*/M^* = F \).

(ii) If \( M^* \) contains a countable partition, then some non-Cauchy cut of \( F \) is filled in \( K \).

(iii) If \( M^* \) contains an \( \alpha \)-partition, with \( \alpha \) non-measurable, then \( M^* \) contains a countable partition.

**Proof.**

(i) By Lemma 3.1, \( M^*(f) \in F \) for all \( f \in C^* \).

(ii) Let \( \{ e_n : n \in \mathbb{N} \} \) (\( \mathbb{N} \) is the set of natural numbers) be a partition contained in \( M^* \). Put \( f = \sum_{n \in \mathbb{N}} ne_n \). Since \( F \) is non-archimedean, \( f \in C^* \) and \( M^*(f) \) fills the non-Cauchy-cut \( \gamma = (A, B) \), where \( B = \{ y \in F : y > \mathbb{N} \} \) and \( A = F \setminus B \).

(iii) Repeat the proof given in [DMW], theorem 2.2.

**Theorem 3.3.** The following statements (i), (ii), (iii) are equivalent and imply (iv). If \( F \) is not complete, then all four are equivalent.

(i) For some \( f \in C^* \), \( M^*(f) \in \text{cl}_K(F) \setminus f[X] \).

(ii) \( M^* \) strongly contains an \( \omega(F) \)-partition.

(iii) \( M^* \) contains a unit of \( C = C(X, F) \).

(iv) \( \text{cl}_K(F) \) is a completion of \( F \).
**PROOF.** (i) implies (ii). Put \( a = M^*(f) \). By 2.5, \( \omega_F^-(a) = \omega_F^+(a) = \omega(F) \). By 3.1, \( M^* \) strongly contains an \( \omega(F) \)-partition.

(ii) implies (iii). Let \( \{ e_\alpha : \alpha \in \omega(F) \} \) be an \( \omega(F) \)-partition strongly contained in \( M^* \) and let \( \{ a_\alpha : \alpha \in \omega(F) \} \) be a cofinal subset of \( F \), such that \( \alpha < \beta \) implies \( a_\beta < a_\alpha \). Put \( u = \sum_{\alpha \in \omega(F)} a_\alpha e_\alpha \). Then \( u \) is a unit of \( C \), and since the partition is strongly contained in \( M^* \), we have \( 0 \leq M^*(u) = M^*(\sum_{\beta \leq \alpha} a_\beta e_\beta) \leq a_\alpha \) for all \( \alpha \in \omega(F) \). Since \( F \) is cofinal in \( K \), \( M^*(u) = 0 \).

(iii) implies (i). Apply Lemma 3.1.

(i) implies (iv). By 2.3 and 2.5, we have to show that every Cauchy cut of \( F \) is filled in \( K \). Let \( \gamma = (A, B) \) be a non-trivial Cauchy cut of \( F \), and let \( \{ a_\alpha : \alpha \in \omega(F) \} \) be a cofinal subset of \( A \), such that \( a_\alpha < a_\beta \) whenever \( \alpha < \beta \). Take an \( \omega(F) \)-partition strongly contained in \( M^* \). It can be verified, arguing as above, that \( M^*(\sum_{\alpha \in \omega(F)} a_\alpha e_\alpha) \) fills \( \gamma \).

That (iv) implies (i) if \( F \) is not complete is obvious by 2.3.

**COROLLARY 3.4.** If \( cl_K(F) \setminus F \neq \emptyset \), then \( cl_K(F) \) is a completion of \( F \). If \( \omega(F) \) is non-measurable, \( K \) is a completion of \( F \) if and only if \( F = K \) and \( F \) is complete.

**PROOF.** The first part is proved in the same way as (iv) was derived from (i) in 3.3. For the second statement, apply 3.3 and 3.2.

**IV.** \( K = C(X, F)/M \).

In this section, \( M \) will denote a maximal ideal of \( C = C(X, F) \) and \( K \) the field \( C/M \).

**THEOREM 4.1.** The following are equivalent:

(i) \( M \) does not contain strongly an \( \omega(F) \) partition.

(ii) \( F \) is cofinal in \( K \).

(iii) \( M \cap C^* \) is maximal in \( C^* \).

Furthermore, if (iii) holds then \( C/M \) is isomorphic to \( C^*/M \cap C^* \).

**PROOF.** Assume (i) holds. If \( F \) is not cofinal in \( K \), there exists \( f \in C \) such that \( M(f) > F \). Arguing as in Lemma 3.1, we see that \( M \) contains an \( \omega(F) \)-partition and (ii) is proved.
Now assume (ii) and put $P' = M \cap C^*$. The natural mapping $j$ of $C^*/P^*$ into $C/M$ is one-to-one, and $C/M$ is the field of fractions of $j[C^*/P^*]$. Let $M^*$ be the maximal ideal of $C^*$ containing the prime ideal $P^*$. By an argument analogous to [DMW, lemma 2], it can be shown that if $u \geq 0, u \in M^\star \setminus P^\star$, then $0 < u < F^\star$. But, then, $F$ is not cofinal in $K$. Hence $M \cap C^*$ is maximal in $K$.

If $M \cap C^*$ is a maximal ideal of $C^*$ which contains no unit of $C$, then by Theorem 3.1, $M \cap C^*$ does not strongly contain any $\omega(F)$-partition, whence $M$ cannot contain strongly an $\omega(F)$-partition and (i) holds.

V. Real closed fields.

This section is devoted to a brief investigation of the residue class fields of $C(X, F)$ and $C'(X, F)$ with $F$ real-closed. The first natural question is: are these residue class fields also real closed? The answer to this question is affirmative and a proof may be given following [G], 13.4], once we have proved that in a real-closed field $K$ the roots of a polynomial depend continuously on the coefficients. This fact is well-known in the case $K = \mathbb{R}$, but its proof makes use of Rouché's theorem. Hence we give an elementary proof of this fact. Let $K$ be a real-closed field, and let $L$ be its algebraic closure. The topology and the absolute value on $L$ are defined in the usual way.

Let $n$ be a positive integer. For each $a = (a_0, \ldots, a_{n-1}) \in L^n$ we denote by $\rho_1 a, \ldots, \rho_n a$ the «real parts» of the roots (in $L$) of the polynomial $p_a(t) = \sum_{v=0}^{n-1} a_v t^v + t^n$, (listing each according to its multiplicity) indexed so that $\rho_1 a \leq \ldots \leq \rho_n a$ (see [G], 13.3]). Put also

$$||a|| = \Max \{ |a_0|, \ldots, |a_{n-1}| \}.$$

**Theorem 5.1.** The functions $\rho_1, \ldots, \rho_n$ are continuous functions from $L^n$ to $K$.

**Proof.** The theorem is an immediate consequence of the following:

**Proposition 5.2.** Let $a \in L^n$ be given, and let $\varepsilon > 0$ be such that $|x_a - y_a| > \varepsilon$ whenever $x_a, y_a$ are distinct roots of $p_a(t)$. Let $x_a$ be a
root of $p_a(t)$ of multiplicity $r_a$. Then, for each $\varepsilon$, $0 < \varepsilon \leq \varepsilon_a$, there exists $\delta > 0$ such that, if $\|a - b\| < \delta$, then $p_b(t)$ has (counting multiplicities) exactly $r_a$ roots $x_b$ such that $|x_b - x_a| < \varepsilon$.

**Proof.** It is enough to show that under the above assumptions, $p_b(t)$ has at least $r_a$ roots in $x_a + U_\varepsilon$. We first show that $p_b(t)$ has at least one root in $x_a + U_\varepsilon$. In fact, for every $b \in L^n$

$$|p_b(x_a)| = |p_b(x_a) - p_a(x_a)| \leq \sum_{v=0}^{n-1} |a_v - b_v| \cdot |x_a|^v \leq \|a - b\| \left( \sum_{v=0}^{n-1} |x_a|^v \right)$$

and

$$|p_b(x_a)| = \prod_{j=1}^{n} |x_a - x_j|$$

where the $x_j$ are the roots of $p_b(t)$. Thus, for

$$\delta = \varepsilon^n \cdot \left( \sum_{v=0}^{n-1} |x_a|^v \right)^{-1},$$

we have $|x_a - x_j| < \varepsilon$

for at least one $j$.

Dividing the polynomial $p_a(t)$ by $t - \xi (\xi \in L)$ we obtain

$$p_a(t) = (t - \xi)p_{\phi(a, \xi)}(t) + p_d(\xi)$$

where $\phi : L^n \times L \rightarrow L^{n-1}$ is a continuous function (its components are polynomials in $\xi$, having $a_0, \ldots, a_{n-1}, 1$ as coefficients).

We have already seen that the proposition is true for $r_a = 1$. Also, the proposition is trivial for $n = 1$. Assume that it is true for $n - 1$ ($n > 1$), and that $r_a > 1$. Then $x_a$ is a root of $p_{\phi(a, x_a)}(t)$, of multiplicity $r_a - 1$. Given $\varepsilon > 0$, we can find $\eta > 0$ such that for every $c \in L^{n-1}$ satisfying $\|c - \phi(a, x_a)\| < \eta$, $p_c(t)$ has exactly $r_a - 1$ roots in $x_a + U_\varepsilon$, by the induction hypothesis. Since $\phi$ is continuous, there exists $\delta_1 > 0$ such that whenever $\|a - b\|, |\xi - x_a| < \delta_1$, then

$$\|\phi(b, \xi) - \phi(a, x_a)\| < \eta.$$ 

By what we have above shown, we can find $\delta < \delta_1$ such that $\|a - b\| < \delta$ implies $|x_b - x_a| < \min \{\delta_1, \varepsilon\}$ for at least one root $x_b$ of $p_b(t)$. Thus, if
\[ \| a - b \| < \delta, \ p_q(b, x_b)(t) \] has \( r_a - 1 \) roots in \( x_a + U_\varepsilon \). Hence \( p_b(t) \) has at least \( r_a \) roots in \( x_a + U_\varepsilon \).

Following now the proof of \([G J, 13.4]\) (and observing, in the bounded case, that the roots of \( p_a(t) \) are bounded by \( n(1 + \| a \|) \)), we have

**Theorem 5.3.** Let \( F \) be a real closed field, \( M \) (resp. \( M^* \)) a maximal ideal of \( \mathbb{C} = C(X, F) \) (resp. \( \mathbb{C}^* = C^*(X, F) \)). Then \( C/M \) (resp. \( C^*/M^* \)) is real-closed.

Also we have

**Theorem 5.4.** Let \( F \) be a real closed field, \( M \) a maximal ideal of \( \mathbb{C} = C(X, F) \), \( K = C/M \). If \( F \) is not cofinal in \( K \), then there is a copy \( E \) of \( F \) such that

\[ F \subseteq E \subseteq K. \]

**Proof.** Since \( M \) contains an \( \omega(F) \)-partition, every Cauchy cut \( F \) is filled in \( K \), as is easy to see. Thus Theorem 2.6. applies.

**Remark.** This copy of \( F \) in \( K \) is not unique, unless \( F \) is already complete. It could be shown that there are at least \( \text{trdeg}_F(F) \times I \) such copies, where \( I = \{ x \in K : |x| < F^+ \} \).

**References**


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