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BILINEAR GENERATING FUNCTIONS FOR LAGUERRE AND LAURICELLA POLYNOMIALS IN SEVERAL VARIABLES

L. CARLITZ *)

1. Erdélyi [1] has defined the Laguerre polynomial in k variables by means of

$$(1.1) \quad (1-u_1-\dots-u_k)^{-\alpha-1} \exp \left\{ -\frac{x_1 u_1 + \dots + x_k u_k}{1-u_1-\dots-u_k} \right\} = \\ = \sum_{n_1, \dots, n_k=0}^{\infty} L_{n_1, \dots, n_k}^{(\alpha)}(x_1, \dots, x_k) u_1^{n_1} \dots u_k^{n_k}.$$

This is equivalent to

$$(1.2) \quad L_{n_1, \dots, n_k}^{(\alpha)}(x_1, \dots, x_k) = \\ = \frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1! \dots n_k!} \sum_{r_j=0}^{n_j} (-1)^{r_1+\dots+r_k} \binom{n_1}{r_1} \dots \binom{n_k}{r_k} \frac{x_1^{r_1} \dots x_k^{r_k}}{(\alpha+1)_{r_1+\dots+r_k}}.$$

He obtained the following bilinear generating function.

$$(1.3) \quad \sum_{n_1, \dots, n_k=0}^{\infty} \frac{n_1! \dots n_k!}{\Gamma(\alpha+n_1+\dots+n_k+1)} u_1^{n_1} \dots u_k^{n_k} \cdot \\ \cdot L_{n_1, \dots, n_k}^{(\alpha)}(x_1, \dots, x_k) L_{n_1, \dots, n_k}^{(\alpha)}(y_1, \dots, y_k) = \\ = \frac{(u; x, y)^{-\frac{1}{2}\alpha}}{(1-U)^{\alpha+1}} \exp \left(-\frac{X+Y}{1-U} \right) I_{\alpha}(2\sqrt{(u; x, y)}),$$

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where

$$(1.4) \quad U = \sum_{j=1}^k u_j, \quad X = \sum_{j=1}^k u_j x_j, \quad Y = \sum_{j=1}^k u_j y_j, \quad W = \sum_{j=1}^k u_j x_j y_j$$

and

$$(1.5) \quad (u; x, y) = (1-U)^{-2}(W-UW+XY).$$

(There is a slight error in the definition of $(u; x, y)$ as given in [1]). For $k=1$, (1.3) reduces to the Hardy-Hille formula [2, p. 101]:

$$(1.6) \quad \sum_{n=0}^{\infty} \frac{n! u^n}{\Gamma(\alpha + n + 1)} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) = \\ \frac{(xyu)^{-\frac{1}{2}\alpha}}{1-u} \exp \left(-\frac{(x+y)u}{1-u} \right) I_{\alpha} \left(\frac{2\sqrt{xyu}}{1-u} \right).$$

The object of the present paper is to give an elementary proof of (1.3) and indeed of the following more general bilinear formula.

$$(1.7) \quad \sum_{n_1, \dots, n_k=0}^{\infty} \frac{(\alpha)_{n_1+\dots+n_k}}{n_1! \dots n_k!} u_1^{n_1} \dots u_k^{n_k} \cdot \\ \cdot F_A(-n_1, \dots, -n_k; \gamma; \alpha; y_1, \dots, y_k) = \\ \cdot F_A(-n_1, \dots, -n_k; \beta; \alpha; x_1, \dots, x_k) \cdot \\ = (1-U)^{-\alpha+\beta+\gamma} (1-U+X)^{-\beta} (1-U+Y)^{-\gamma} \cdot \\ \cdot F \left[\beta, \gamma; \alpha; \frac{(1-U)W+XY}{(1-U+X)(1-U+Y)} \right].$$

where

$$(1.8) \quad F_A(-n_1, \dots, -n_k; \beta; \alpha; x_1, \dots, x_k) = \\ = \sum_{r_j=0}^{n_j} (-1)^{r_1+\dots+r_k} \binom{n_1}{r_1} \dots \binom{n_k}{r_k} \frac{(\beta)_{r_1+\dots+r_k}}{(\alpha)_{r_1+\dots+r_k}} x_1^{r_1} \dots x_k^{r_k}.$$

For $k=1$, (1.7) reduces to

$$(1.9) \quad \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} u^n F(-n, \beta; \alpha; x) F(-n, \gamma; \alpha; y) = \\ = (1-x)^{-\alpha+\beta+\gamma} (1-u+xu)^{-\beta} (1-u+yu)^{-\gamma} \cdot \\ \cdot F \left[\beta, \gamma; \alpha; \frac{xyu}{(1-u+xu)(1-u+yu)} \right],$$

a formula due to Weisner [3].

2. It will be convenient to define

$$(2.1) \quad \Phi(n; r, s) = \sum \binom{n_1}{r_1} \dots \binom{n_k}{r_k} \binom{n_1}{s_1} \dots \binom{n_k}{s_k} \cdot \\ \cdot \frac{u_1^{n_1}}{n_1!} \dots \frac{u_k^{n_k}}{n_k!} x_1^{r_1} \dots x_k^{r_k} y_1^{s_1} \dots y_k^{s_k},$$

where the summation is over all nonnegative n_j, r_j, s_j such that

$$(2.2) \quad n_1 + \dots + n_k = n, r_1 + \dots + r_k = r, s_1 + \dots + s_k = s.$$

Thus $\Phi(n; r, s)$ is homogeneous of weight n in $u_1, \dots, u_k$, of weight r in $x_1, \dots, x_k$ and of weight s in $y_1, \dots, y_k$. Alternatively we may define $\Phi(n; r, s)$ by means of

$$(2.3) \quad \sum_{n, r, s=0}^{\infty} \Phi(n; r, s) = \exp(U + X + Y + W),$$

where U, X, Y, W are defined by (1.4).

We rewrite (1.3) in the form

$$(2.4) \quad \sum_{n_1, \dots, n_k=0}^{\infty} \frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1! \dots n_k!} L_{n_1, \dots, n_k}^{(\alpha)}(x_1, \dots, x_k) \cdot \\ \cdot L_{n_1, \dots, n_k}^{(\alpha)}(y_1, \dots, y_k) = \\ = (1-U)^{-\alpha-1} \exp \left(-\frac{X+Y}{1-U} \right) \sum_{n=0}^{\infty} \frac{(u; x, y)^n}{n! (\alpha+1)_n}.$$

Making use of (1.2) and (2.2), it is easily seen that the left hand side of (2.4) is equal to

$$(2.5) \quad \sum_{n, r, s=0}^{\infty} (-1)^{r+s} \Phi(n; r, s) \frac{(\alpha+1)_n}{(\alpha+1)_r(\alpha+1)_s}.$$

Hence that part of the left hand side of (2.4) that is of weight r in $x_1, \dots, x_k$ and of weight s in $y_1, \dots, y_k$ is equal to

$$(2.6) \quad \sum_{n=0}^{\infty} \Phi(n; r, s) \frac{(\alpha+1)_n}{(\alpha+1)_r(\alpha+1)_s}.$$

We set the right hand side of (2.4) equal to

$$(2.7) \quad \sum_{n, r, s=0}^{\infty} \psi(n; r, s),$$

where $\psi(n; r, s)$ is homogeneous of degree n in $u_1, \dots, u_k$, of weight r in $x_1, \dots, x_k$ and of weight s in $y_1, \dots, y_k$. Then clearly (2.4) is equivalent to

$$(2.8) \quad \sum_{n=0}^{\infty} \Phi(n; r, s) \frac{(\alpha+1)_r(\alpha+1)_s}{(\alpha+1)_n} = \sum_{n=0}^{\infty} \psi(n; r, s) \quad (r, s=0, 1, 2, \dots).$$

The right hand side of (2.4) is equal to

$$(2.9) \quad (1-U)^{-\alpha-1} \sum_{r, s=0}^{\infty} (-1)^{r+s} \frac{X^r Y^s}{r! s!} (1-U)^{-r-s} \\ \cdot \sum_{j=0}^{\infty} \frac{[(1-U)W + XY]^j}{j!(\alpha+1)_j} (1-U)^{-2j}.$$

Since $(1-U)W + XY$ is of weight one in $x_1, \dots, x_k$ and of weight one in $y_1, \dots, y_k$, it follows from (2.9) that

$$(2.10) \quad \sum_{n=0}^{\infty} \psi(n; r, s) = \\ = (-1)^{r+s} (1-U)^{-\alpha-r-s-1} \sum_{j=0}^{\min(r, s)} \frac{X^{r-j} Y^{s-j} [(1-U)W + XY]^j}{j!(r-j)!(s-j)!(\alpha+1)_j}.$$

In the next place, (2.8) is equivalent to

$$(2.11) \quad \sum_{n, r, s=0}^{\infty} \Phi(n; r, s)(\alpha+1)_n = \sum_{n, r, s=0}^{\infty} \psi(n; r, s)(\alpha+1)_r(\alpha+1)_s.$$

But, by (2.1) and (2.2),

$$\begin{aligned} & \sum_{n, r, s=0}^{\infty} (-1)^{r+s} \Phi(n; r, s)(\alpha+1)_n \\ &= \sum_{n_j=0}^{\infty} \frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1! \dots n_k!} u_1^{n_1} \dots u_k^{n_k} \\ & \cdot \sum_{r_j, s_j=0}^{n_j} (-1)^{r_1+\dots+r_k+s_1+\dots+s_k} \binom{n_1}{r_1} \dots \binom{n_k}{r_k} \binom{n_1}{s_1} \dots \binom{n_k}{s_k} \\ & x_1^{r_1} \dots x_k^{r_k} y_1^{s_1} \dots y_k^{s_k} = \sum_{n_j=0}^{\infty} \frac{(\alpha+1)_{n_1+\dots+n_k}}{n_1! \dots n_k!} u_1^{n_1} \dots u_k^{n_k} (1-x_1)^{n_1} \dots (1-x_k)^{n_k} \\ & (1-y_1)^{n_1} \dots (1-y_k)^{n_k} = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{n!} \left\{ \sum_{j=1}^k u_j (1-x_j)(1-y_j) \right\}^n. \\ (2.12) \quad & = \{1-U+X+Y-W\}^{-\alpha-1}. \end{aligned}$$

On the other hand, by (2.10),

$$\begin{aligned} & \sum_{n, r, s=0}^{\infty} \Psi(n; r, s)(\alpha+1)_r(\alpha+1)_s = \\ &= (1-U)^{-\alpha-1} \sum_{r, s=0}^{\infty} (-1)^{r+s} (\alpha+1)_r (\alpha+1)_s (1-U)^{-r-s} \\ & \cdot \sum_{j=0}^{\min(r, s)} \frac{X^{r-j} Y^{s-j} [(1-U)W + XY]^j}{j!(r-j)!(s-j)!(\alpha+1)_j} = \\ &= (1-U)^{-\alpha-1} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{j!} [(1-U)W + XY]^j (1-U)^{-2j} \\ & \cdot \sum_{r, s=0}^{\infty} (-1)^{r+s} \frac{(\alpha+j+1)_r (\alpha+j+1)_s}{r! s!} X^r Y^s (1-U)^{-r-s} = \\ &= (1-U)^{-\alpha-1} \sum_{j=0}^{\infty} \frac{(\alpha+1)_j}{j!} [(1-U)W + XY]^j (1-U)^{-2j}. \end{aligned}$$

$$\begin{aligned}
& \cdot \left(1 + \frac{X}{1-U}\right)^{-\alpha-j-1} \left(1 + \frac{Y}{1-U}\right)^{-\alpha-j-1} - \\
& = (1-U)^{\alpha+1} (1-U+X)^{-\alpha-1} (1-U+Y)^{-\alpha-1} \sum_{j=0}^{\infty} \\
& \quad \frac{(\alpha+1)_j}{j!} \frac{[(1-U)W + XY]^j}{(1-U+X)^j (1-U+Y)^j} \\
& = (1-U)^{\alpha+1} \{(1-U+X)(1-U+Y) - (1-U)W + XY\}^{-\alpha-1} = \\
& = (1-U)^{\alpha+1} \{(1-U)^2 + (1-U)(X+Y) - (1-U)W\}^{-\alpha-1} = \\
& = (1-U+X+Y-W)^{-\alpha-1},
\end{aligned}$$

so that

$$(2.13) \quad \sum_{n, r, s=0}^{\infty} \psi(n; r, s) (\alpha+1)_r (\alpha+1)_s = (1-U+X+Y-W)^{-\alpha-1}.$$

In view of (2.12) and (2.13), (2.11) is satisfied. This completes the proof of (2.4) and therefore of (1.3).

3. We shall now prove (1.7). By (1.8) the left hand side of (1.7) is equal to

$$\begin{aligned}
& \sum_{n_j=0}^{\infty} \frac{(\alpha)_{n_1+\dots+n_k}}{n_1! \dots n_k!} u_1^{n_1} \dots u_k^{n_k} \cdot \\
& \sum_{r_j, s_j}^{\infty} (-1)^{r_1+\dots+r_k+s_1+\dots+s_k} \binom{n_1}{r_1} \dots \binom{n_k}{r_k} \binom{n_1}{s_1} \dots \binom{n_k}{s_k} \cdot \\
& = \frac{(\beta)_{r_1+\dots+r_k} (\gamma)_{s_1+\dots+s_k}}{(\alpha)_{r_1+\dots+r_k} (\alpha)_{s_1+\dots+s_k}} x_1^{r_1} \dots x_k^{r_k} y_1^{s_1} \dots y_k^{s_k} = \\
& = \sum_{n, r, s=0}^{\infty} (-1)^{r+s} (\alpha)_n \Phi(n; r, s) \frac{(\beta)_r (\gamma)_s}{(\alpha)_r (\alpha)_s},
\end{aligned}$$

where $\Phi(n; r, s)$ is defined by (2.1).

Now by (2.8) we have

$$(3.1) \quad \sum_{n=0}^{\infty} \Phi(n; r, s) \frac{(\alpha)_n}{(\alpha)_r (\alpha)_s} = \sum_{n=0}^{\infty} \Psi'(n; r, s) \quad (r, s=0, 1, 2, \dots),$$

where $\psi'(n; r, s)$ is the result of replacing $\alpha+1$ by α in $\Psi(n; r, s)$. Consequently the left member of (1.7) is equal to

$$(3.2) \quad \sum_{n, r, s=0}^{\infty} \Psi'(n; r, s)(\beta)_r(\gamma)_s.$$

Then, by (2.10), this sum is equal to

$$\begin{aligned} & (1-U)^{-\alpha} \sum_{r, s=0}^{\infty} (-1)^{r+s} (\gamma)_s (1-U)^{-r-s} \\ & \cdot \sum_{j=0}^{\min(r, s)} \frac{X^{r-j} Y^{s-j} [(1-U)W + XY]^j}{j!(r-j)!(s-j)!(\alpha)_j} = \\ & = (1-U)^{-\alpha} \sum_{j=0}^{\infty} \frac{(\beta)_j (\gamma)_j}{j!(\alpha)_j} [(1-U)W + XY]^j (1-U)^{-2j} \\ & \cdot \sum_{r, s=0}^{\infty} (-1)^{r+s} \frac{(\beta+j)_r (\gamma+j)_s}{r!s!} X^r Y^s (1-U)^{-r-s} = \\ & = (1-U) \sum_{j=0}^{\infty} \frac{(\beta)_j (\gamma)_j}{j!(\alpha)_j} [(1-U)W + XY]^j (1-U)^{-2j} \\ & \cdot \left(1 + \frac{X}{1-U} \right)^{-\beta-j} \left(1 + \frac{Y}{1-U} \right)^{-\gamma-j} = \\ & = (1-U)^{-\alpha+\beta+\gamma} (1-U+X)^{-\beta} (1-U+Y)^{-\gamma} \sum_{j=0}^{\infty} \frac{(\beta)_j (\gamma)_j}{j!(\alpha)_j} \\ & \left\{ \frac{(1-U)W + XY}{(1-U+X)(1-U+Y)} \right\}^j = (1-U)^{-\alpha+\beta+\gamma} (1-U+X)^{-\beta} (1-U+Y)^{-\gamma} \\ & F \left[\beta, \gamma; \alpha; \frac{(1-U)W + XY}{(1-U+X)(1-U+Y)} \right]. \end{aligned}$$

This completes the proof of (1.7).

4. Exactly as in proving (1.7), we can establish the following more general result.

$$(4.1) \quad \sum_{n_1, \dots, n_k=0}^{\infty} \frac{(\alpha)_{n_1+\dots+n_k}}{n_1! \dots n_k!} u_1^{n_1} \dots u_k^{n_k}.$$

$$\begin{aligned}
 & \cdot F_A(-n_1, \dots, -n_k; \beta; \beta'; x_1, \dots, x_k) F_A(-n_1, \dots, -n_k; \gamma; \gamma'; y_1, \dots, y_k) = \\
 & = (1-U)^{-\alpha} \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j (\gamma)_j}{j! (\beta)_j (\gamma')_j} [(1-U)W + XY]^j (1-U)^{-2j} \cdot \\
 & \cdot F \left[\alpha+j, \beta+j; \beta'+j; -\frac{X}{1-U} \right] F \left[\alpha+j, \gamma+j; \gamma'+j; -\frac{Y}{1-U} \right].
 \end{aligned}$$

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