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ON THE SOLUTION OF SCHRÖDINGER-LIKE WAVE EQUATIONS

by J. DE GRAAF and L. A. PELETIER *)

§ 1. Introduction.

In this paper we shall prove the unique solvability of the initial value problem for a system of inhomogeneous linear second order partial differential equations which can be denoted in matrix notation as

$$(1) \quad Su \equiv u_t + iDu_{xx} + Au_x + Bu = f,$$

where $u(x, t) = \text{col}(u_1(x, t), \dots, u_n(x, t))$. The real variable x runs through the interval $(-\infty, \infty)$. The real variable t satisfies $0 \leq t \leq T < \infty$. D , A and B are constant matrices. D and A have the following restrictive properties:

- (2) (i) $D = \text{diag}(d_{11}, \dots, d_{nn})$. D is real.
(ii) $A = A^\dagger$,

where A^\dagger is the hermitian transpose of A .

S is a differential operator defined by (1).

$f(x, t)$ is a vector valued function, $f = \text{col}(f_1(x, t), \dots, f_n(x, t))$.

For the case $f=0$ some special properties of the solution will be investigated.

One method of dealing with this problem is based on the theory of semi-groups. See [2] and [3].

However, an elementary and very elegant proof for a related problem has been given by Ladyženskaia [1]. She deals with the initial

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value problem for the operator equation

$$(3) \quad u_t - iS_1(t)u = f,$$

where for almost all t , $u(t)$ and $f(t)$ are elements of a separable Hilbert-space H . Here the linear operator $S_1(t)$ has the following four properties:

- (i) The domain $D(S_1)$ of S_1 is dense everywhere in H .
- (ii) S_1 is self-adjoint.
- (4) (iii) S_1 establishes a one-to-one mapping of $D(S_1)$ on to H .
- (iv) $S_1 > 0$.

In the present problem the operator corresponding to S_1 in neither positive nor self-adjoint; on the other hand it does not depend on t . We shall prove existence and, in passing, uniqueness, by suitably modifying Ladyzenskaja's method, using an auxiliary operator which does have the properties (i) - (iv).

§ 2. Some notations.

R : the interval $(-\infty, \infty)$ of the real numbers.

Q : a strip in the x - t -plane containing all points satisfying the inequalities $-\infty < x < \infty$ and $0 \leq t \leq T < \infty$.

We consider the vector valued functions of n complex components $u = \text{col}(u_1, \dots, u_n)$, $u_k = u_k(x, t)$ defined on R and Q respectively; in the former case t is fixed.

$L_2(R)$, $L_2(Q)$: Hilbert-spaces containing all square integrable n -component vector valued functions on R and Q respectively. The inner products and norms belonging to them are defined by

$$(u, v) = \int_{-\infty}^{\infty} u^\dagger v dx \quad \|u\| = (u, u)^{1/2} \quad u, v \in L_2(R)$$

$$(u, v)_Q = \int_0^T \int_{-\infty}^{\infty} u^\dagger v dx dt \quad \|u\|_Q = (u, u)_Q^{1/2} \quad u, v \in L_2(Q)$$

u^\dagger being the hermitian transpose of u .

$W_2^m(R)$: Sobolev space. This space contains all vector valued $L_2(R)$ — functions $u(x)$ whose generalised derivatives $D^k u$, ($k=1, 2, \dots, m$) are also elements of $L_2(R)$. The innerproduct and norm are, respectively,

$$(u, v)_m = (D^m u, D^m v) + \dots + (u, v)$$

$$\|u\|_m = (u, u)_m^{1/2}.$$

$W = L_2(Q) \times L_2(R)$ i.e. the Cartesian product, being the set of all pairs $[v; \varphi]$ with $v \in L_2(Q)$ and $\varphi \in L_2(R)$. The innerproduct and norm are defined by:

$$([v; \varphi], [w; \psi]) = (v, w)_Q + (\varphi, \psi)$$

$$\| [v; \varphi] \|_W = (\|v\|_Q^2 + \|\varphi\|^2)^{1/2} = ([v; \varphi], [v; \varphi])^{1/2}.$$

REMARK. All integrations applied in the definitions of norms and inner products, are in the sense of Lebesgue. All differentiations in the present paper are meant in generalized sense, although the classical notation will be retained.

§ 3. Existence and uniqueness of the solution.

We consider equation (1) with the properties (2), $f \in L_2(Q)$ and $u(x, 0) \in L_2(R)$. Let $D(\mathcal{O})$ be a linear manifold in $L_2(Q)$ containing the functions $u(x, t)$ which have generalized derivatives u_t, u_x, u_{xx} in $L_2(Q)$ and for which at each $t \in [0, T]$, $u \in W_2^2(R)$ and $u_t \in W_2^1(R)$. Further we define $D_0(\mathcal{O})$ and $D^0(\mathcal{O})$, being linear sub-sets of $D(\mathcal{O})$, containing $D(\mathcal{O})$ -functions vanishing at $t=0$ and $t=T$ respectively.

$D_0(\mathcal{O}) \cap D^0(\mathcal{O}) = D_0^0(\mathcal{O})$ is dense everywhere in $L_2(Q)$.

Define on $D(\mathcal{O})$ the operator \mathcal{O} as follows

$$\mathcal{O}u = [Su; u(t=0)].$$

The range $R(\mathcal{O})$ of operator \mathcal{O} establishes a linear manifold in the Hilbert-space W .

LEMMA. 1 The linear manifold $R(\mathcal{O})$ is dense everywhere in W .

PROOF. The lemma holds if and only if there is no element in W , except $[0; 0]$, normal to the linear manifold $R(\mathcal{O})$. Let $[\nu; \psi] \in W$ be normal to $R(\mathcal{O})$. Then for every $u \in D(\mathcal{O})$

$$(5) \quad \int_0^T dt \int_{-\infty}^{\infty} \nu^\dagger S u dx + \left[\int_{-\infty}^{\infty} \psi^\dagger u dx \right]_{t=0} = 0.$$

From $\nu(x, t)$ construct the function $\varphi(x, t)$ as solution of

$$\tilde{\Delta}\varphi = \varphi - \varphi_{xx} = \int_t^T \nu(x, \tau) d\tau.$$

As the auxiliary operator $\tilde{\Delta}$ establishes a one-to-one mapping of $W_2^2(R)$ onto $L_2(R)$, φ exists, is uniquely determined, and belongs to $W_2^2(R)$ for every $t \in [0, T]$. It is easily seen that

$$(6) \quad \nu = \tilde{\Delta}\varphi_t.$$

Now we choose for any $a \in [0, T]$ the function $u \in D(\mathcal{O})$ as follows:

$$(7) \quad u(x, t) = \begin{cases} 0 & 0 \leq t \leq a \\ \int_a^t \varphi(x, \tau) d\tau & a \leq t \leq T \end{cases}$$

By using the definition of $D(\mathcal{O})$ one can verify that, for every $a \in [0, T]$, u is an element of $D(\mathcal{O})$.

Substitution of (6) and (7) in (5) yields

$$\begin{aligned} & \int_a^T dt \int_{-\infty}^{\infty} dx (\tilde{\Delta}\varphi_t)^\dagger S \left(\int_a^t \varphi d\tau \right) = \\ & = \int_{-\infty}^{\infty} dx \int_a^T dt (\tilde{\Delta}\varphi_t)^\dagger \left\{ \varphi + \int_a^t (iD\varphi_{xx} + A\varphi_x + B\varphi) d\tau \right\} = 0. \end{aligned}$$

After partial integration and using the properties of φ we obtain

$$\int_{-\infty}^{\infty} dx \int_a^T dt \{ (\Delta\varphi_t)^\dagger \varphi - (\tilde{\Delta}\varphi)^\dagger (iD\varphi_{xx} + A\varphi_x + B\varphi) \} = 0.$$

Add to this expression its complex conjugate, then after some partial integrations:

$$(8) \quad \int_a^T \frac{\partial}{\partial t} (\tilde{\Delta}\varphi, \varphi) dt - \int_a^T ((B+B^\dagger)\tilde{\Delta}\varphi, \varphi) dt = 0.$$

Further, there exists a unitary matrix U , such that

$$B+B^\dagger = U^\dagger \Lambda U$$

where Λ is a real diagonal matrix. Let K , in absolute sense be the greatest eigenvalue of $(B+B^\dagger)$, then

$$(9) \quad |((B+B^\dagger)\tilde{\Delta}\varphi, \varphi)| = |(U^\dagger \Lambda U \tilde{\Delta}\varphi, \varphi)| = |(\Lambda \tilde{\Delta}U\varphi, U\varphi)| \leq \\ \leq K(\tilde{\Delta}U\varphi, U\varphi) = K(\tilde{\Delta}\varphi, \varphi).$$

With this result we obtain, as $\varphi(T)=0$, from (8) the inequality

$$(\tilde{\Delta}\varphi, \varphi)_{t=a} \leq K \int_a^T (\tilde{\Delta}\varphi, \varphi) dt,$$

which holds for all $a \in [0, T]$. This is essentially Gronwall's inequality [6]. Therefore $(\tilde{\Delta}\varphi, \varphi) = 0$ for every $t \in [0, T]$.

Since $\tilde{\Delta} > 0$, it immediately follows that $\varphi = 0$, and hence $\nu = 0$ (cf. (6)). As $W_2^2(R)$ is dense everywhere in $L_2(R)$, ψ must also be zero. This completes the proof of our lemma.

REMARK. The auxiliary operator $\tilde{\Delta}$ has the properties (4).

Ladyženskaja constructs the functions φ by means of the operator S_1 (cf. (3)). This is not possible in our case and, obviously, not necessary.

LEMMA 2. For every $u \in D(\mathcal{O})$ the following inequalities hold.

$$(10) \quad \begin{aligned} (i) \quad & (u, u)_{t=t_1} \leq c_1 \left\{ (u, u)_{t=0} + \int_0^{t_1} \|Su\|^2 dt \right\} \\ (ii) \quad & \int_0^{t_1} (u, u) dt \leq c_2 \left\{ (u, u)_{t=0} + \int_0^{t_1} \|Su\|^2 dt \right\} \end{aligned}$$

c_1 and c_2 are constants.

PROOF. Premultiply the expression

$$u_t + iDu_{xx} + Au_x + Bu = Su$$

by u^\dagger and integrate it over the entire x -axis.

Then, adding to the result its complex conjugate and keeping in mind that the operators $iD\frac{\partial^2}{\partial x^2}$ and $A\frac{\partial}{\partial x}$ are skew-hermitian, we obtain

$$\frac{d}{dt}(u, u) = -((B+B^\dagger)u, u) + (Su, u) + (u, Su).$$

Using (9) and the expansion of $\|u - Su\|^2$ we obtain

$$\frac{d}{dt}(u, u) - \beta(u, u) \leq \|Su\|^2,$$

with $\beta = K + 1$.

We multiply the left-hand side of the last inequality by $e^{-\beta t}$. The result may be written

$$\frac{d}{dt} e^{-\beta t} (u, u) \leq \|Su\|^2.$$

Integration of this inequality from 0 to t_1 leads to the desired inequalities (10), where

$$c_1 = e^{\beta T} \quad \text{and} \quad c_2 = T e^{\beta T}.$$

THEOREM 1. The operator \mathcal{O} has a closure $\bar{\mathcal{O}}$, where $R(\bar{\mathcal{O}}) = \overline{R(\mathcal{O})} = W$. The operator equation $\mathcal{O}u = [f; \varphi_0]$ is uniquely solvable for every $f \in L_2(Q)$ and $\varphi_0 \in L_2(R)$, and for all t the solution u is an element of $L_2(R)$, which depends continuously on t . Finally $Su = f$ and $u(x, t) \rightarrow \varphi_0(x)$ as $t \rightarrow 0$ in the sense of the L_2 - norm.

PROOF.

(i) $R(\mathcal{O})$ is dense everywhere in W (lemma 1), so every element in W can be approximated by sequence of elements belonging to $R(\mathcal{O})$. From lemma 2 it is clear that the originals of the elements of a Cauchy sequence in $R(\mathcal{O})$ establish a converging sequence in $L_2(Q)$. The union of $D(\mathcal{O})$ and the limit points of these sequences will be denoted by $D(\bar{\mathcal{O}})$.

(ii) Let $\{u_n\} \subset D(\mathcal{O})$ be a sequences converging to $u \in L_2(Q)$ such that $\{\mathcal{O}u_n\} = \{[f_n; \varphi_n]\}$ converges to $[f; \varphi_0] \in W$.

When we succeed in proving that $u=0$ implies $[f; \varphi_0] = [0, 0]$ the operator has a closure $\bar{\mathcal{O}}$.

Multiply the expression $(Su_n)^\dagger = f_n^\dagger$ from the right with $\zeta \in D^0(\mathcal{O})$ and integrate over Q . By means of partial integration the differentiations may be carried over to ζ . This leads to

$$\int_{-\infty}^{\infty} \varphi_n^\dagger \zeta(x, 0) dx - \int_0^T dt \int_{-\infty}^{\infty} u_n^\dagger \tilde{S} \zeta dx = - \int_0^T dt \int_{-\infty}^{\infty} f_n^\dagger \zeta dx$$

where $\tilde{S}\zeta = \zeta_t + iD\zeta_{xx} + A\zeta_x - B^\dagger\zeta$.

For $n \rightarrow \infty$ our assumptions lead to

$$\int_{-\infty}^{\infty} \varphi_0^\dagger \zeta(x, 0) dx = - \int_0^T dt \int_{-\infty}^{\infty} f^\dagger \zeta dx$$

holding for every $\zeta \in D^0(\mathcal{O})$.

Restricting ourselves apriori to test functions $\zeta \in D_0^0(\mathcal{O})$ we immediately obtain $f=0$ as $D_0^0(\mathcal{O})$ is a dense subset of $L_2(Q)$. Then also $\varphi_0=0$ as $W_2^2(R)$ is a dense subset of $L_2(R)$.

(iii) Let $\{u_n\} \subset D(\mathcal{O})$ be a sequence converging to $u \in D(\bar{\mathcal{O}})$. It follows from (10 i) that the $u_n(t)$, as elements of $L_2(R)$, converge uniformly with respect to $t \in [0, T]$ to $u(t)$. Furthermore, because $\partial u_n / \partial t \in L_2(Q)$, they are — as elements of $L_2(R)$ — strongly continuous with respect to t . The limit function $u(t)$ must therefore be strongly continuous as well.

§ 4. Some properties of the solution if $f=0$.

Concerning the $(n \times n)$ -matrices D , A and B we restrict ourselves to those cases where the matrix

$$(11) \quad (D - Az + iBz^2)$$

has n independent eigenvectors for all z within a sufficiently small circle around the origin of the complex z -plane. This property, together with condition (2), ensures that the matrix operator

$$(12) \quad P(k, t) = e^{(iDk^2 - iAk - B)t}$$

is uniformly bounded for $k \in R$ and $t \in [0, T]$, (see appendix).

If we take the initial value $u(x, 0) = g(x) \in W_2^3(R)$, the solution may be represented by

$$(13) \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(k, t) \bar{g}(k) e^{ikx} dk,$$

where

$$(14) \quad \bar{g}(k) = \int_{-\infty}^{\infty} g(x) e^{-ikx} dx.$$

This is easily verified by direct substitution: The differentiation may be carried out under the integral sign, as the required derivatives of the integrals converge uniformly since

$$\left| \int_N^{\infty} (ik)^2 P \bar{g} e^{ikx} dx \right| \leq \|P\| \int_N^{\infty} |k^3 \bar{g}| \frac{dk}{k} \leq \|P\| \|k^3 \bar{g}\| \left(\int_N^{\infty} \frac{1}{k^2} dx \right)^{1/2}.$$

THEOREM 2. For an arbitrary initial condition $u(x, 0) = g(x) \in L_2(R)$ the solution may be represented by (13).

PROOF. As $W_2^3(R)$ is dense everywhere in $L_2(R)$, it is possible to find a sequence $\{g_n\} \subset W_2^3(R)$ which converges to $g(x)$ in the sense of the $L_2(R)$ -norm. Then the solutions $u_n(x, t)$, corresponding to initial values $g_n(x)$ also converge in the L_2 -norm i.e.

$$\lim_{n \rightarrow \infty} u_n(x, t) = u(x, t)$$

$u(x, t)$ being a solutions and $u(x, 0) = g(x)$.

Finally, as the Fourier-Plancherel operator (14) is bounded, the order of integration and passing to the limit may be interchanged.

This proves the theorem.

THEOREM 3.

$u(x, 0) \in W_2^p(R)$; $p \geq 1$, implies: $u(x, t) \in W_2^p(R)$ for all t .

PROOF. Let $u(x, 0) = g(x) \in W_2^p(R)$.

Using representation (13), we find

$$\begin{aligned} \frac{\partial^q u}{\partial x^q} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^q P(k, t) \bar{g}(k) e^{ikx} dk; \quad (q=0, 1, \dots, p-1). \\ (15) \quad \frac{\partial^p u}{\partial x^p} &= \frac{\partial}{\partial x} \left\{ \frac{\partial^{p-1} u}{\partial x^{p-1}} - \frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^{p-1} P(k, t) \bar{g}(k) dk \right\} \end{aligned}$$

because the second term between pointed brackets is independent of x .

Note:

$$\left| \int_{-\infty}^{\infty} (ik)^{p-1} P \bar{g} dk \right| \leq \| (1+k^2)^{1/2} (ik)^{p-1} P \bar{g} \| \| (1+k^2)^{-1/2} \|.$$

From (15) it follows that

$$\frac{\partial^p u}{\partial x^p} = - \frac{\partial}{\partial x} \frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^{p-1} P(k, t) \bar{g}(k) (e^{ikx} - 1) dk.$$

Finally, according to Titchmarsh p. 69 [5], we have for almost all x

$$\frac{\partial^p u}{\partial x^p} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^p P(k, t) \bar{g}(k) e^{ikx} dk.$$

Consequently $\partial^p u / \partial x^p \in L_2(\mathbb{R})$, and for all t $u(x, t) \in W_2^p(\mathbb{R})$.

Appendix.

A. Consider the $(n \times n)$ -matrix $(D - Az + iBz^2)$, defined on the complex z -plane. The matrix has the properties

(i) $D = \text{diag}(d_{11}, \dots, d_{nn})$; D is real.

(ii) $A = A^\dagger$.

(iii) There exists a positive number ρ so that $(D - Az + iBz^2)$ has n independent eigenvectors for $|z| \leq \rho$.

REMARK. Sufficient (although not necessary) for (iii) to hold is that all diagonal elements of D are all different from one another.

Let

$$(a) \quad (D - Az + iBz^2)v_j(z) = \lambda_j(z)v_j(z) \quad j=1, 2, \dots, n.$$

For $|z| \leq \rho$ all $\lambda_j(z)$ are analytic in the origin, and the $v_j(z)$ may be chosen so. (cf. [4]). Substitute in (a) the power series

$$\lambda_j(z) = \lambda_j^{(0)} + \lambda_j^{(1)}z + \lambda_j^{(2)}z^2 + \dots$$

$$v_j(z) = v_j^{(0)} + v_j^{(1)}z + v_j^{(2)}z^2 + \dots$$

After identification of equal powers we obtain

$$(b) \quad Dv_j^{(0)} = \lambda_j^{(0)}v_j^{(0)}$$

$$(c) \quad Dv_j^{(1)} - Av_j^{(0)} = \lambda_j^{(0)}v_j^{(1)} + \lambda_j^{(1)}v_j^{(0)}.$$

From (b) it follows that $\lambda_j^{(0)} = d_{jj}$. Take $v_j^{(0)\dagger}v_j^{(0)} = 1$.

Premultiply (c) by $v_j^{(0)\dagger}$. After some reshuffling this yields

$$\lambda_j^{(1)} = -v_j^{(0)\dagger} A v_j^{(0)}.$$

Obviously, both $\lambda_j^{(0)}$ and $\lambda_j^{(1)}$ are real when A is hermitian.

B. THEOREM. There exists a positive number M , so that for all $t \in [0, T]$ and $k \in R$

$$\| e^{(iDk^3 - ikA - B)t} \| \leq M.$$

PROOF. In view of condition (iii) there exists a number $K > 0$ such that the exponent divided by ik^2t is diagonalizable for every k , $|k| > K$.

(i) There exists a positive number M_1 so that for all k , such that $|k| \leq K$ and all $t \in [0, T]$, $\| e^{(iDk^3 - ikA - B)t} \| \leq M_1$ as the variables are bounded.

(ii) For k , $|k| > K$, construct from the normalized eigenvectors a matrix $S(k)$ such that

$$ik^2t \left(D - \frac{1}{k} A + \frac{1}{k^2} iB \right) = S^{-1}(k) [D_1(k, t) + D_2(k, t)] S(k).$$

$$D_1(k, t) = \text{diag} [ik^2td_{jj} + ikt\lambda_j^{(1)}]$$

$$D_2(k, t) = \text{diag} \left[it \left(\lambda_j^{(2)} + \frac{1}{k} \lambda_j^{(3)} + \dots \right) \right].$$

Then $\| e^{D_1(k, t)} \| = 1$, as $\lambda_j^{(1)}$ is real. $e^{D_2(k, t)}$, $S(k)$ and $S^{-1}(k)$ are uniformly bounded for k , $|k| > K$, so there exists a number M_2 satisfying the requirement that for all k , $|k| > K$.

$$\| e^{(iDk^3 - ikA - B)t} \| \leq M_2.$$

(iii) Finally, take $M = \max (M_1, M_2)$.

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