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SIGNAL AND NOISE IN NONLINEAR DEVICES

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ABSTRACT - Independent signal and noise are presented to the input of a nonlinear device, and we ask how the output time function is to be decomposed into an output signal and output noise. On the basis of two requirements on this decomposition we determine that the signal output is just the conditional expectation of the output with respect to the original signal. This makes the output signal and noise uncorrelated. The decomposition is invariant to linear filtering. The signal output of a zero-memory device is given by another zero-memory device acting on the input signal. The output signal of a bandpass nonlinearity is written down in terms of an integral. For the bandpass hard limiter in Gaussian noise this gives the output signal amplitude very quickly in terms of Bessel functions. The antocorrelation of the output noise of a zero-memory device in Gaussian noise is derived. Another possible definition of signal output is investigated and rejected.

1. Introduction.

When independent signal and noise are components of the input to a nonlinear device such as a detector or bandpass limiter, these two components are inextricably mixed in the output. Shutterly [11] (and see also Campbell [6]) expresses the output of a zero-memory device as a sum of products of signal and noise functions. However, the concept of « output signal-to-noise ratio » is applied to nonlinear

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devices, which suggests that the output might be decomposed into two uncorrelated components, to be called output signal and output noise. Davenport [7] made such a decomposition for the hard limiter, but in the autocorrelation or spectral domain. The purpose of this paper is to make the same decomposition in the time domain on the basis of general and simple requirements on such a decomposition. This would be of help in tracing the progress of signal and noise through a receiver containing nonlinear devices. We will define the output signal and noise for a very broad class of black box devices, and then specialize the result to zero memory devices and bandpass zero-memory devices, including as a special case the bandpass limiter.

2. Definition of Output Signal and Noise.

Suppose the « nonlinear device » has two time-varying real inputs $s(t)$ and $n(t)(-\infty<t<\infty)$, and one output $y(t)$, where the signal s and noise n are sample functions of independent real stationary processes.

The sample functions s and n are points of independent sample spaces Ω_s and Ω_n which have probability measures P_s and P_n. Assume that these processes have finite variance, $E(s^2(t))<\infty$, $E(n^2(t))<\infty$ for all t. For convenience we will take $E(n(t))=0$.

The output $y(t)(-\infty<t<\infty)$ of the device is determined by the input functions s and n. Therefore we can consider y to be a sample function of a random process $Y(t, s, n)$ on the product probability space $\Omega_s\times\Omega_n$ with the product probability $P_s\cdot P_n$. We require that the device be time invariant; in other words, if the inputs are $s(t-\delta)$ and $n(t-\delta)$, then the output is $y(t-\delta)$. This guarantees that Y is a stationary process. We also ask that

$$E(y^2(t))=\int_{\Omega_s}\int_{\Omega_n} Y^2(t, s, n)P_s(ds)P_n(dn)<\infty.$$

Our aim is to determine what is meant by the « signal » and « noise » portions of $y(t)$. In Davenport's treatment of the bandpass limiter [7] [8] the autocorrelation or spectrum of the output was calculated. These broke up naturally into a part due to signal only, and a part due to noise and intermodulation between signal and noise. What we want is a decomposition of the output in the time domain,

$$y(t)=s_y(t)+n_y(t) \tag{2.1}$$

into output signal and noise portions.

We will place the following conditions on the decomposition (2.1):

i) For each $t(-\infty<t<\infty)$, $s_y(t)$ is a random variable of finite variance on the original signal sample space Ω_s. The $s_y(t)$ form a stationary process.

ii) For each t, the random variable $n_y(t)=y(t)-s_y(t)$ is uncorrelated with all random variables in the space $S=L^2(\Omega_s, P_s)$ of signal random variables of finite variance, i. e., with all random variables $f(s)$ on Ω_s such that $E(f^2)<\infty$. Thus

$$E(n_y(t)f)=E(n_y(t))E(f) \tag{2.2}$$

for all f in S.

It is enough to require (2.2) for all f of the form $f(s)=F(s(t_1), \ldots, s(t_k))$, where $t_1, \ldots, t_k$ are distinct times and F is a function of k variables such that $E[f^2(s)]<\infty$.

It is too much to require that n_y be *independent* of the original signal process. Thus n_y will in general depend both on input signal and input noise.

Conditions (i) and (ii) imply that

$$E[y(t_1)y(t_2)]=E[s_y(t_1)s_y(t_2)]+E[n_y(t_1)n_y(t_2)]+2E(n_y)E(s_y) \tag{2.3}$$

since for fixed t', $s_y(t')$ is a signal random variable which can replace f in (2.2). Hence the power spectrum of y is, except possibly for a dc component, the sum of the power spectra of s_y and n_y.

To see how far conditions (i) and (ii) determine s_y and n_y, we write (2.2) as

$$E[(n_y-E(n_y))f]=0 \tag{2.4}$$

for all f in S. Then write $y(t)$ as

$$y(t)=[s_y(t)+E(n_y)]+[n_y(t)-E(n_y)].$$

Since $E(n_y)$ is just a constant, it belongs to S. Therefore $s_y(t)+E(n_y)$ is in S by (i), and $n_y(t)-E(n_y)$ is orthogonal to S by (2.4). Thus for fixed t, $s_y(t)+E(n_y)$ is the projection $p(t)=P(t, s)$ of the random variable $y(t)$ onto S, the « signal space ».

Furthermore, the random variables

$$\begin{aligned} s_y(t)&=p(t)+c, \\ n_y(t)&=y(t)-s_y(t), \end{aligned} \tag{2.5}$$

where c is any constant, satisy (i) and (ii). (It may be shown that the random variables $p(t)$ form a stationary process.). Thus these conditions determine the output signal and noise within a constant.

The projection $p(t)$ has another significance. We know that $p(t)$ is an integrable random variable on Ω_s satisfying $E(p(t)f)=E(y(t)f)$ for all bounded measurable f on Ω_s. This implies that $p(t)$ is just the *conditional expectation* of the random variable $y(t)$ with respect to all the random variables $s(t')(-\infty<t'<\infty)$: $p(t)=E(y(t)\mid s(t'),\text{ all } t')$. Since (Ω_s, P_s) and (Ω_n, P_n) are independent probability spaces, it may easily be verified that this projection or conditional expectation may be written

$$P(t, s)=\int_{\Omega_n} Y(t, s, n)P_n(dn). \tag{2.6}$$

We now set $c=0$ in (2.5) and adopt as our definitions of output signal and noise

$$\begin{aligned} s_y(t)&=E(y(t)\mid s(t'),\text{ all } t'), \\ n_y(t)&=y(t)-s_y(t). \end{aligned} \tag{2.7}$$

Then $E(s_y(t))=E(y(t))$, $E(n_y(t))=0$. Equation (2.6) shows that the signal portion of the output at time t is obtained by fixing the input signal and averaging the output at time t over all possible noise inputs belonging to the noise sample space Ω_n. (For causal devices the output is determined by the past $s(t')$, $n(t')(t' \leq t)$, but this does not falsify our statements.)

The definition (2.7) may also be of use for non-stationary input signals. The conditions (i) and (ii), with stationarity removed, yield that $s_y(t) = p(t) + c(t)$, where $c(t)$ is an arbitrary deterministic function of time. Some further condition (like c = constant) is needed to define s_y well enough.

3. Linear Filters and Zero-Memory Devices.

The conditional expectation in (2.7) is of course in no convenient form for calculation, being an average over a whole function space. However, in the special case of a zero-memory device followed by a linear filter (acting on the sum of the input signal and noise), the conditional expectation reduces to ordinary integrals over real variables.

a) *Linear Filter.* Let

$$y(t) = (Hx)(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \qquad (-\infty < t < \infty), \tag{3.1}$$

where $x = s + n$ and the impulse response h satisfies

$$\int_{-\infty}^{\infty} (1+t^2)h^2(t)dt < \infty. \tag{3.2}$$

The condition (3.2) on h ensures that with probability one the integral (3.1) exists for all t and that $E(y^2) < \infty$, given that $E(x^2) < \infty$. We will show that

$$s_y(t) = (Hs)(t), \quad n_y(t) = (Hn)(t), \tag{3.3}$$

where s_y and n_y are defined by (2.7). The condition (3.2) and $E(s^2) < \infty$ ensures that $(Hs)(t)$ is a random variable on Ω_s with finite variance, so all we have to do is verify the projection property

$$E[(Hs)(t)f] = E[y(t)f] \tag{3.4}$$

for all random variables f on Ω_s of finite variance. Thus

$$
\begin{aligned}
(3.5)\qquad E\left[f\cdot\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau\right] &= \int_{-\infty}^{\infty}h(t-\tau)E[(s(\tau)+n(\tau))f]d\tau \\
&= \int_{-\infty}^{\infty}h(t-\tau)E[s(\tau)f]d\tau = E\left[f\cdot\int_{-\infty}^{\infty}h(t-\tau)s(\tau)d\tau\right],
\end{aligned}
$$

which is (3.4). With the definition (2.7), then, a linear filter does not mix the signal and noise.

This property extends further. Suppose we follow the general non-linear device of section 2 by a linear filter H. Thus the output of the composite device is $z(t)=(Hy)(t)=(Hs_y)(t)+(Hn_y)(t)$. If we replace s and n in (3.5) by s_y and n_y, and note that $E(fn_y(t))=0$ by (ii) and $En_y=0$, we see that the analog of (3.4),

$$E[(Hs_y)(t)f]=E[(Hy)(t)f],$$

holds, and hence

$$(3.6)\qquad s_{Hy}(t)=(Hs_y)(t),\ n_{Hy}(t)=(Hn_y)(t).$$

We emphasize that $s_{Hy}(t)$ is the projection of $(Hy)(t)$ onto the *original* signal space. The result (3.6) says that decomposition (2.7) is not affected by passage through a linear filter. The signal and noise outputs of a device followed by a linear filter are obtained by letting the filter act on the signal and noise outputs of the device. In particular, this will allow us to treat bandpass non-linearities.

b) Zero-Memory Device.

Here we let

$$y(t)=F(s(t),\ n(t))$$

where F, the characteristic of the device, is a real-valued function of two variables such that

$$E[F^2(s(t),\ n(t))]<\infty.$$

For fixed t we are dealing with only the two random variables $s(t)$ and $n(t)$. Hence (2.6) and (2.7) become

$$s_y(t)=\int_{-\infty}^{\infty} F(s(t),\ n)p(n)dn=G(s(t)), \tag{3.7}$$

where p is the probability density of $n(t)$. As far as the signal is concerned, the device acts like another device of characteristic G, which of course depends on the noise density $p(n)$. We will call G the *signal characteristic* of the device. In the next section this will be illustrated by the hard limiter. Blachman [1] [2] has considered this idea that the signal output is a zero-memory function of the signal input, obtained by averaring the output over the noise.

Now suppose the input signal and noise to this device F are narrowband about a center frequency ω_0. Let the signal have the representation $s(t)=V(t)\sin(\omega_0 t+\theta(t))$, $V(t)\geq 0$, where $V\sin\theta$ and $V\cos\theta$ are narrowband about zero frequency with bandwidths small compared with ω_0. Also, assume that the random variables $V(t)$ and $\theta(t)$, t fixed, are independent, that $\theta(t)$ is uniformly distributed in $[0,\ 2\pi]$, and that the distribution of $V(t)$ is independent of t. Then

$$\begin{aligned} E[G^2(s(t))] &= E[G^2(V(t)\sin(\omega_0 t+\theta(t)))] \\ &= E\left[\frac{1}{2\pi}\int_0^{2\pi} G^2(V(t)\sin\Phi)d\Phi\right] < \infty . \end{aligned} \tag{3.8}$$

Hence with probability one, we can expand $G(V(t)\sin\Phi)$ in a Fourier series on $0\leq\Phi\leq 2\pi$:

$$G(V(t)\sin\Phi)=\sum_{k=-\infty}^{\infty} c_k(V(t))e^{ik\Phi},$$

convergent in $L^2(0,\ 2\pi)$, where the Fourier coefficients c_k are given by

$$c_k(V)=\frac{1}{2\pi}\int_0^{2\pi} G(V\sin\Phi)e^{-ik\Phi}d\Phi. \tag{3.9}$$

The same change of variables as was used in (3.8) will give that

$$G(s(t)) = \sum_{k=-\infty}^{\infty} c_k(V(t)) e^{ik(\omega_0 t + \theta(t))}. \tag{3.10}$$

For fixed t this converges in the mean. The terms in (3.10) are the signal components in the narrow frequency zones about each $\pm k\omega_0 (k=0, 1, 2, \ldots)$. We can find a (nonrealizable) filter H satisfying (3.2) whose complex transfer function is 1 in the k^{th} zone and 0 in all other zones (by making the transfer function smooth enough). If such a filter passes the k^{th} harmonic unchanged and annihilates all the others, then by (3.6) the signal output of the device consisting of the zero-memory device followed by H is

$$2 \operatorname{Re} [c_k(V(t)) e^{ik(\omega_0 t + \theta(t))}].$$

In the first zone, $k=1$, there are in-phase and quadrature components sharing the original phase modulation $\theta(t)$. The amplitude modulation is distorted by zero-memory characteristics $\operatorname{Im} c_k$ and $\operatorname{Re} c_k$.

For zero-memory devices of form $F(s(t)+n(t))$, Blachman [1] [3] and Doyle [9] obtain the output signal amplitude in the first zone by averaging over the noise the component of the total bandpass output in phase with the input signal. The resulting integral may be transformed into (3.9) ($k=1$) under the condition that $n(t) = A(t) \sin \psi(t)$, where $A(t)$ and $\psi(t)$ are independent random variables and $\psi(t)$ is uniformly distributed.

4. Bandpass Limiter.

An example of a zero-memory device is the ideal limiter, where

$$y(t) = F(s(t), n(t)) = \operatorname{sgn}(s(t)+n(t)),$$

$$\begin{aligned} \operatorname{sgn} x &= 1 \qquad (x \geq 0) \\ &= -1 \qquad (x < 0). \end{aligned}$$

Henceforth the input noise $n(t)$ will be a stationary Gaussian process, with $E(n)=0$, $E(n^2)=\sigma^2$. We easily calculate from (3.7) that the signal at the output of the hard limiter is

$$s_y(t)=G(s(t))=g\left(\frac{s(t)}{\sigma}\right),$$

(4.1)
$$g(x)=\sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{1}{2}z^2}dz.$$

The signal characteristic G is a smooth limiter. For more general noise densities, G has the shape of the distribution function of $n(t)$. For large signal-to-noise ratios the signal characteristic G is itself like a hard limiter. For small signal-to-noise rations, G is almost linear, i.e.

$$G(s)\approx\sqrt{\frac{2}{\pi}}\frac{s}{\sigma}$$

Consider now the case of narrow-band signal and noise inputs. The harmonic expansion (3.10) of $s_y(t)$ can be written

(4.2)
$$s_y(t)=\sum_{k=1}^{\infty} b_k(\nu(t))\sin(k\omega_0 t+k\theta(t)),\ \nu(t)=\frac{V(t)}{\sigma},$$

(4.3)
$$b_k(\nu)=\int_{-\pi}^{\pi} g(\nu\sin\Phi)\sin k\Phi d\Phi \qquad (k=1,\ 2,\ \ldots);$$

if k is even then $b_k=0$.

Tausworthe [12] and Blachman [2] obtained the expression (4.3) for the signal amplitude in the k^{th} zone. We can avoid the usual hypergeometric functions and express (4.3) directly in terms of Bessel functions. Integrate (4.3) by parts to obtain

(4.4)
$$b_k(\nu)=\sqrt{\frac{2}{\pi}}\frac{\nu}{k\pi}\int_{-\pi}^{\pi} e^{-\frac{1}{2}\nu^2\sin^2\Phi}\cos k\Phi\cos\Phi d\Phi$$

$$=\sqrt{\frac{2}{\pi}}\frac{\nu}{k\pi}\int_{-\pi}^{\pi} e^{-\frac{1}{4}\nu^2(1-\cos 2\Phi)}\,\tfrac{1}{2}[\cos(k-1)\Phi+\cos(k+1)\Phi]d\Phi$$

$$= \sqrt{\frac{2}{\pi}}\frac{\nu}{k}e^{-\frac{1}{4}\nu^2}\left[\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e^{\frac{1}{4}\nu^2\cos\theta}\cos\left(\frac{k-1}{2}\theta\right)d\theta\right.$$

$$\left.+\frac{1}{4\pi}\int_{-2\pi}^{2\pi} e^{\frac{1}{4}\nu^2\cos\theta}\cos\left(\frac{k+1}{2}\theta\right)d\theta\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{\nu}{k}e^{-\frac{1}{4}\nu^2}\left[I_{\frac{1}{2}(k-1)}\left(\frac{1}{4}\nu^2\right)+I_{\frac{1}{2}(k+1)}\left(\frac{1}{4}\nu^2\right)\right] (k \text{ odd}),$$

where the I_m are modified Bessel functions of the first kind.

The signal power in the k^{th} zone is

$$\tfrac{1}{2}E\left[b_k^2\left(\frac{V(t)}{\sigma}\right)\right].$$

From this we can compute signal-to-noise ratio in this zone, since the total power there is $8/(\pi k)^2$, and the signal and noise portions are uncorrelated by our assumption (ii) on the signal-noise decomposition.

5. Noise and International Output.

In the case of a zero-memory device in Gaussian noise with mean 0, variance σ we will expand the noise portion $n_y(t)$ of the output in a Hermite series. This is a convenient form for computing the output noise autocorrelation and spectrum [5] [13]. We write

$$y(t)=F(s(t),\ n(t))=\sum_{r=0}^{\infty}\frac{1}{r!}a_r(s(t))H_r\left(\frac{n(t)}{\sigma}\right), \tag{5.1}$$

where

$$H_r(x)=(-1)^r e^{\frac{1}{2}x^2}\frac{d^r}{dx^r}e^{-\frac{1}{2}x^2},$$

$$a_r(s)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}F(s,\ n)H_r\left(\frac{n}{\sigma}\right)e^{-\frac{1}{2}\frac{n^2}{\sigma^2}}\frac{dn}{\sigma},$$

and the expansion in (5.1) converges in the mean. (The situation is analogous to (3.9)). But the $r=0$ term in (5.1) is just the signal portion $s_y(t)$ since $H_0=1$. Thus

$$n_y(t)=\sum_{r=1}^{\infty}\frac{1}{r!}a_r(s(t))H_r\left(\frac{n(t)}{\sigma}\right). \tag{5.2}$$

Because

$$(1-\rho^2)^{-1/2}\exp\left[-\frac{1}{2(1-\rho^2)}(x_1^2-2\rho x_1x_2+x_2^2)\right]$$
$$=\sum_{r=0}^{\infty}\frac{1}{r!}\rho^r\exp\left[-\tfrac{1}{2}(x_1^2+x_2^2)\right]H_r(x_1)H_r(x_2)$$

(Mehler's formula), the autocorrelation of n_y is

$$E[n_y(t)n_y(t+\tau)]=\sum_{r=1}^{\infty}E[a_r(s(t))a_r(s(t+\tau))]\frac{1}{r!}\rho^r(\tau), \tag{5.3}$$

where $\rho(\tau)=\sigma^{-2}E[n(t)n(t+\tau)]$.

Let $s(t)$ have the stationary narrow-band form $V(t)\sin\Phi(t)$, $\Phi(t)=\omega_0t+\theta(t)+\alpha$, where α is a uniformly distributed constant phase. Write

$$a_r(V\sin\Phi)=\sum_{k=-\infty}^{\infty}a_{rk}(V)e^{ik\Phi}, \tag{5.4}$$

$$a_{rk}(V)=\frac{1}{2\pi}\int_0^{2\pi}a_r(V\sin\Phi)e^{-ik\Phi}d\Phi.$$

Let $V_1=V(t)$, $V_2=V(t+\tau)$, $\Phi_1=\Phi(t)$, $\Phi_2=\Phi(t+\tau)$, etc. It can be shown that the terms in (5.4) are uncorrelated in the sense that

$$E[a_{rk}(V_1)e^{ik\Phi_1}\overline{a_{rl}(V_2)e^{ik\Phi_2}}]=0 \qquad (k\neq l).$$

Hence

$$E[n_y(t)n_y(t+\tau)] \tag{5.5}$$
$$=\sum_{r=1}^{\infty}\sum_{k=-\infty}^{\infty}E[a_{rk}(V_1)\overline{a_{rk}(V_2)}e^{ik(\theta_1-\theta_2)}]e^{-ik\omega_0\tau}\frac{1}{r!}\rho^r(\tau).$$

This displays the autocorrelation of n_y as a series of intermodulation terms in the usual way.

Let us calculate the coefficients a_{rk} more explicitly for the hard limiter, $F(s, n) = \operatorname{sgn}(s+n)$. For convenience let $E(n^2)=1$. Then

$$a_r(s) = \frac{(-1)^r}{(2\pi)^{1/2}} \left[\int_{-s}^{\infty} \frac{d^r}{dn^r} e^{-\frac{1}{2}n^2} dn - \int_{-\infty}^{-s} \frac{d^r}{dn^r} e^{-\frac{1}{2}n^2} dn \right]$$
$$= (-1)^{r-1} \left(\frac{2}{\pi}\right)^{1/2} H_{r-1}(s) e^{-\frac{1}{2}s^2},$$

where s is to be replaced by s/σ. Then

$$a_{rk}(V) = \frac{(-1)^{r-1}}{(2\pi^3)^{1/2}} \int_0^{2\pi} H_{r-1}\left(\frac{V}{\sigma} \sin \Phi\right) \exp\left(-\frac{V^2 \sin^2 \Phi}{2\sigma^2} - ik\Phi\right) d\Phi.$$

This procedure was used by Tikhonov and Amiantov [13], and of course other expressions for the a_{rk} exist [4] [7] [8] [10] [11] [13]. We display the expansions (5.3) and (5.5) to show the connection between the existing theory and our decomposition $y = s_y + n_y$.

6. Investigation of Another Definition of Signal Output.

An alternative definition of signal output might have been the corresponding « wide sense » conditional expectation $w(t)$, the projection of $y(t)$ onto the subspace generated by linear combinations of the random variables $s(t')$, $-\infty < t' < \infty$ (or $t' \leq t$). Then $y(t_1) - w(t_1)$ is orthogonal to $w(t_2)$, just as $y(t_1) - p(t_1)$ is orthogonal to $p(t_2)$, so this gives a decomposition in the spectral domain as before. In general, for a zero-memory device, $w(t)$ is not given by a zero-memory transformation of the signal, even in the absence of noise. A sufficient condition that $w(t)$ be a zero-memory function of $s(t)$, and thus $w(t) = \text{const} \cdot s(t)$, is that

$$E(s(t') \mid s(t)) = f(t', t) s(t) \tag{6.1}$$

for all t, t', where $f(t', t)$ is not a random variable. We will proceed to prove this: We can evaluate f by taking the expectation of (6.1) multiplied by $s(t)$:

$$E[s(t)E(s(t') \mid s(t))] = E(s(t)s(t')) = f(t', t)E(s^2),$$

$$f(t', t) = \frac{E(s(t')s(t))}{E(s^2)} \quad (=0 \text{ if } E(s^2)=0), \tag{6.2}$$

the normalized autocorrelation. If G is a function such that $E[G^2(s(t))] < \infty$ then

$$E[G(s(t))s(t')] = E[G(s(t))E(s(t') \mid s(t))] =$$
$$= \frac{E(G(s)s)}{E(s^2)} E(s(t)s(t')) = AE(s(t)s(t')).$$

Is should be noted that (6.1) is weaker than Barret and Lampard's condition [14], being merely a geometric condition on the joint distribution of $s(t)$, $s(t')$.

Given a zero-memory device $y(t) = F(s(t), n(t))$ and the noise process, let G be the signal characteristic, defined by (3.7). For all t, t',

$$E[y(t)s(t')] = E[[G(s(t)) + n_y(t)]s(t')] =$$
$$= E[G(s(t))s(t')] = E[As(t)s(t')].$$

This demonstrates that $As(t$ı is the projection $w(t)$.

We see that under the condition (6.1), which is satisfied by Gaussian and sine-wave signals, the « linear » definition $w(t)$ of the signal output does not include distortion higher harmonics. These parts are classified as « noise » even in the case of no input noise. For this reason we would not want to consider $w(t)$ the entire signal output.

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