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A NOTE ON FOURIER COEFFICIENTS

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1. Riesz in his famous paper [4] posed a problem whether exists a continuous function f of bounded variation for which the sequences (na_n) and (nb_n) , a_n , b_n being the Fourier coefficients [7] of f , converge and at least one of the two limits be different from zero. Moreover Steinhaus [5] proved that if f is a continuous functions of bounded variation and $na_n \rightarrow a$ and $nb_n \rightarrow b$, then $a=b=0$, which was later improved by Alexits [1] who proved that if f is a function which has only removable discontinuities i.e. $f(x+0)=f(x-0)$ and if $na_n \rightarrow a$, $nb_n \rightarrow b$, then $a=b=0$.

The object of this note is to further improve the results of Steinhaus an Alexits and prove the following theorem.

THEOREM. If a_n , b_n are the Fourier coefficients of a 2π -periodic and L -integrable function f . If the sequences (na_n) and (nb_n) are summable to a and b respectively by a regular Nörlund method (N, p) [6] satisfying the conditions that p_n 's are non-negative, non-increasing, $p_0=1$ and

$$(1.1) \quad \frac{p_n+1}{p_n} \geq \frac{p_n}{p_n-1} (n > 0)$$

and if f has a removable discontinuity at $x=0$, then $a=b=0$.

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2. To prove the theorem we require the following lemma.

LEMMA. If (N, p) is a regular Nörlund method where (p_n) is a non-negative and non-increasing sequence, then so is (N, q) where $q_k = (n-k)\Delta p_k (\Delta p_k = p_k - p_{k+1})$. Moreover

$$Q_n = \sum_{k=0}^n (n-k)\Delta p_k = \sum_{k=0}^n p^k = P_n.$$

PROOF.

$$\begin{aligned} Q_n &= \sum_{k=0}^n (n-k)\Delta p_k = \sum_{k=0}^n k\Delta p_{n-k} \\ &= b \sum_{k=0}^n (p_{n-k} - p_{n-k-1}) = \sum_{k=0}^n k p_{n-k} - \sum_{k=1}^{n+1} (k-1)p_{n-k} \\ &= -(n+1)p_{-1} + \sum_{k=1}^{n+1} p_{n-k} \\ &= P_n (\text{regarding } p_{-1} = 0). \end{aligned}$$

Then

$$\frac{q_n}{Q_n} = \frac{q_n}{P_n} = \frac{(n-n)\Delta p_n}{P_n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Also

$$q_k = (n-k)\Delta p_k = (n-k)(p_k - p_{k+1}) \geq 0,$$

which shows that (N, q) is a regular method.

3. Proof of the theorem: Since f has a removable discontinuity at $x=0$, by Fejer's theorem [7], the Fourier series of f is $(C, 1)$ -summable to $\frac{1}{2} \{f(0+0) - f(0-0)\} = 1$ at $x=0$ i. e. the series

$$(3.1) \quad \sum a_k$$

is $(C, 1)$ summable to L . Let σ_n and s_n denote the n -th Cesaro mean

and partial sum of (3.1) respectively. Then by Abel's lemma we have

$$\begin{aligned} \frac{1}{P_n} \sum_{k=0}^n p_{n-k} k a_k &= \frac{1}{P_n} \sum_{k=0}^{n-1} \Delta p_{n-k} \sum_{v=0}^k v a_v + p_0 \sum_{v=0}^n v a_v \\ &= \frac{1}{P_n} \sum_{k=0}^n \Delta p_{n+k} \sum_{v=0}^k v a_v (p_{-1}=0) \\ &= \frac{1}{P_n} \sum_{k=0}^n \Delta p_{n-k} \left\{ k s_k - \sum_{v=0}^{k-1} s_v \right\} \\ &= \frac{1}{P_n} \sum_{k=0}^n k \Delta p_{n-k} s_k - \frac{1}{P_n} \sum_{k=0}^n \Delta p_{n-k} \sum_{v=0}^{k-1} s_v \\ &= \frac{1}{P_n} \sum_{k=0}^n \Delta p_{n-k} s_k - \frac{1}{P_n} \sum_{k=0}^n q_{n-k} \sigma k \\ &= \frac{1}{Q_n} \sum_{k=0}^n q_{n-k} - s_k \frac{1}{Q_n} \sum_{k=0}^n k \Delta p_{n-k} \sigma k \end{aligned}$$

where $q_k = (n-k)\Delta p_k$ and $Q_n = \sum_{k=0}^n q_k = p_n$. By lemma (N, q) is a regular Nörlund method hence making $n \rightarrow \infty$ we get

$$a = \lim_{n \rightarrow \infty} \frac{1}{Q_n} \sum_{k=0}^n q_{n-k} s_k - L.$$

Since any two Nörlund methods are consistent, the method (N, q) is consistent with $(C, 1)$ and hence

$$a = L - L = 0.$$

Now by [6, pp. 69, Th. 23] and condition (1.1) it follows that (nb_n) is $(C, 1)$ -summable to b i.e.

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n k b_k = b.$$

Lukacs [2] has proved that at the point where $g(x \pm 0)$ exists

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx) = \frac{f(x+0) - f(x-0)}{\pi}$$

and in this case at $x=0$,

$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^{\infty} b_k = \frac{f(0+0) - f(0-0)}{x} = 0.$$

It follows from (3.2), (3.3) and the inequalities

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^{\infty} k u_k \leq \lim_{\infty \leftarrow u} \frac{1}{\log n} \sum_{k=1}^{\infty} u_k$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{n=1}^{\infty} k u_k \geq \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^{\infty} u_k$$

that

$$b = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n k b_k = \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=0}^n b_k = 0,$$

which completes the proof of the theorem.

4. Special Cases: Lastly we note that making the choice of p_n in (N, p) to be

$$p_k = 1 \quad \text{then} \quad P_n = n + 1$$

or

$$p_k = \frac{1}{k+1} \quad \text{then} \quad P_k \sim \log n \quad \text{as} \quad n \rightarrow \infty$$

we see that the result of this note holds in particular if the sequences (na_n) and (nb_n) are $(C, 1)$ -summable or harmonic summable to a and b respectively.

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