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An existence theorem for solutions of second order non-linear ordinary differential equations in the complex domain

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AN EXISTENCE THEOREM FOR SOLUTIONS OF SECOND ORDER NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN *)

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1. Introduction.

In this paper we treat second order differential polynomials $\Omega(y) = \sum_{m,j,k\geq 0} f_{mjk}(x)\, y^m (y')^j (y'')^k$, where the coefficients $f_{mjk}(x)$ are complex functions, defined and analytic in a sectorial region which is approximately of the form,

$$a < \arg(x - \lambda e^{i(a+b)/2}) < b \tag{1}$$

(for some $\lambda \geq 0$) and where as $x \to \infty$ in this region, each non-zero $f_{mjk}(x)$ has an asymptotic expansion in terms of *logarithmic monomials* (i. e. functions of the form

$$M(x) = Kx^{\alpha_0} (\log x)^{\alpha_1} (\log\log x)^{\alpha_2} \dots (\log_q x)^{\alpha_q} \text{ for complex } K \neq 0$$

and real α_j). Thus the class of differential polynomials we treat contains, in particular, those having rational functions for coefficients). In [1 ; § 43] and [5 ; § 122], existence theorems were proved for solu-

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tions of $\Omega(y)=0$ which are asymptotically equivalent to logarithmic monomials as $x \to \infty$ over a filter base consisting essentially of the sectors (1) as $\lambda \to \infty$. In this paper, we prove an existence theorem (§ 3 below) for solutions of $\Omega(y)=0$ which are of larger rate of growth than all logarithmic monomials as $x \to \infty$ over such a filter base. (We are using here the concepts of asymptotic equivalence ($\sim$) and larger rate of growth ($>$) as $x \to \infty$, introduced in [4 ; § 13]. For the reader's convenience, these concepts are reviewed in § 2 below).

For a given Ω, the corresponding first order differential polynomial, $G(z) = \sum_{m+j+k=p} f_{mjk}(x) z^j (z' + z^2)^k$, where $p = \max\{m+j+k : f_{mjk} \neq 0\}$, plays an important role in our existence theorem. More specifically, we are interested in the *critical* monomials of $G(z)$ (i. e. those logarithmic monomials N for which there is a function $h \sim N$ such that $G(h)$ is not $\sim G(N)$). In [1 ; §§ 21, 26], an algorithm was introduced for finding the set of *all* critical monomials of a given differential polynomial. To apply our existence theorem here, we look for critical monomials N of G such that $N > x^{-1}$. (Thus $N = cx^{-1+\gamma_0}$ $(\log x)^{\gamma_1} \dots (\log_t x)^{\gamma_t}$ where $(\gamma_0, \gamma_1, \dots, \gamma_t)$ is lexicographically greater than $(0, 0, \dots, 0)$). Then if (a, b) is an interval on which the function $I(\varphi) = \cos(\gamma_0 \varphi + \arg c)$ is positive, the theorem in § 3 below asserts the existence, in sectorial subregions of the region (1), of at least a one-parameter family of solutions of $\Omega(y)=0$, each having the form $\exp \int W$ for some $W \sim N$, provided certain subsidiary conditions are fulfilled. (The solutions given by § 3 are shown to be automatically of larger rate of growth than all logarithmic monomials). The subsidiary conditions are of two main types. One type requires that N *not* be a critical monomial of certain other first order differential polynomials. This type of condition is fulfilled in general, since by [1 ; § 29 (b), 21, 17 (Remark (2))], we see that for any nonzero first order differential polynomial, there are *finitely* many critical monomials such that any other critical monomial is a constant multiple of one of these. The other type of condition is also seen to be fulfilled in general since it requires that one or two logarithmic monomials which arise, do not have certain special forms. (In this connection, see § 2 (d) and Remark (c) after § 3). It should be noted that it is easy to test the conditions in any given example, since those of the first type can be tested using the algorithm in [1; §§ 21, 26], while those of the second type can be tested by inspection. Also it should be remarked that these types of conditions are similar to those imposed in the existence theorems in [1; § 42] and [4; § 127] for solutions of first order equations which are $\sim$ to logarithmic monomials. In fact, some of our conditions make it possible to apply, at the outset, a result in [4] to assert the existence of a solution $\sim N$ of the first order differential equation $G(z) = 0$. The remaining conditions then enable us to use this solution of $G(z) = 0$ to transform the second order differential equation $\Omega(y) = 0$ into a quasi-linear form. Our conditions play an essential role in effecting this transformation, since they permit us, at a crucial stage, to assert the existence of a particular type of solution of a certain second order non-homogeneous linear differential equation (see § 4).

Of course one can obtain information on the existence of solutions of $\Omega(y) = 0$ which are of *smaller* rate of growth than all logarithmic monomials, by making the change of variable $y = w^{-1}$, multiplying by a suitable power of w and then applying the theorem in § 3 to the resulting equation.

In § 7, we apply our results to an example.

2. Concepts from [4, 6].

(a) [4; § 94]. Let $-\pi \leq a < b \leq \pi$. For each non-negative real-valued function g on $(0,(b-a)/2)$, let $E(g)$ be the union (over $\delta \in (0,(b-a)/2)$) of all sectors, $a + \delta < \arg(x - h(\delta)) < b - \delta$ where $h(\delta) = g(\delta) \exp(i(a+b)/2)$. The set of all $E(g)$ (for all choices of g) is denoted $F(a, b)$ and is a filter base which converges to ∞. Each $E(g)$ is simply-connected by [4; § 93]. If W is analytic in $E(g)$ then the symbol $\int W$ will stand for a primitive of W in $E(g)$. If x and x_0 are in $E(g)$, then the contour of integration for $\int_{x_0}^{x} W$ will be any rectifiable path in $E(g)$ from x_0 to x. A statement is said to hold *except in finitely many directions* (*briefly*, *e. f. d.*) in $F(a, b)$, if there are finitely many points $r_1 < \dots < r_q$ in (a, b) such that the statement holds in each of $F(a, r_1), F(r_1, r_2), \dots, F(r_q, b)$ separately.

(b) [4 ; § 13]. If f is analytic in some $E(g)$, then $f \to 0$ in $F(a, b)$ means that for any $\varepsilon > 0$, there is a g_1 such that $|f(x)| < \varepsilon$ for all $x \in E(g_1) \cdot f < 1$ in $F(a, b)$ means that in addition to $f \to 0$, all functions $\theta_j^k f \to 0$ where $\theta_j f = (x \log x \dots \log_{j-1} x) f'$. Then $f_1 < f_2, f_1 \sim f_2, f_1 \approx f_2$ and $f \lessapprox f_2$ mean respectively, $f_1 / f_2 < 1$, $f_1 - f_2 < f_2, f_1 \sim c f_2$ for some constant $c \neq 0$, and finally either $f_1 < f_2$ or $f_1 \approx f_2$. If $f < 1$, then by [4 ; § 28], $(x \log x \dots \log_q x) f' < 1$ for all $q \geq 0$. If $M = K x^{\alpha_0} (\log x)^{\alpha_1} \dots (\log_t x)^{\alpha_t}$, then by simple calculation, $M'/M \lessapprox x^{-1}$. If M is not constant, then it follows from [4 ; § 28], that $\psi < M$ implies $\psi' < M'$. If for every real $\alpha, f < x^{\alpha}$, we say f is *trivial* in $F(a, b)$.

(c) [6 ; p. 247]. A logarithmic differential field (briefly an *LDF*) over $F(a, b)$, is a differential field D of functions (each analytic in some $E(g)$), for which there is an integer $q \geq 0$ such that D contains all logarithmic monomials of rank $\leq q$ (i. e. those of the form $K x^{\alpha_0} (\log x)^{\alpha_1} \dots (\log_q x)^{\alpha_q}$), and such that every non-zero element of D is $\sim$ to a logarithmic monomial of rank $\leq q$. (For a fixed q, the set of rational combinations of logarithmic monomials of rank $\leq q$, is the simplest example of an *LDF*).

(d) [4 ; § 43, 100]. If in $F(a, b)$, W is $\sim$ to a monomial of the form,

$$K x^{-1} (\log x)^{-1} \dots (\log_{k-1} x)^{-1} (\log_k x)^{-1+t} (\log_{k+1} x)^{c_1} \dots (\log_{k+s} x)^{c_s}$$

where $k \geq 0$ and $t > 0$, then we say W is in the *divergence class* in $F(a, b)$. The *indicial function* of W is the function on (a, b) defined by $IF(W)(\varphi) = \cos(\delta_{0k} t\varphi + \arg K)$ where δ_{0k} is the Kronecker delta. Clearly $IF(W)$ has at most finitely many zeros unless $k > 0$ and K is purely imaginary.

3. The Main Theorem :

$$\text{Let } \Omega(y) = \sum_{m, j, k \geq 0} f_{mjk}(x)\, y^m (y')^j (y'')^k$$

be a second order differential polynomial with coefficients in an LDF over $F(a, b)$. Let $A = \{m + j + k : f_{mik} \not\equiv 0\}$. Let

$$G(z) = \sum_{m+j+k=p} f_{mjk}(x)\, z^j (z' + z^2)^k \text{ where } p = \max A.$$

Let N be a critical monomial of G such that (i) $N > x^{-1}$, (ii) $IF(N) > 0$ on (a, b), (iii) N is not a critical monomial of $z(\partial G/\partial z) + z'(\partial G/\partial z')$, (iv) $\partial G/\partial z' \not\equiv 0$ and N is not a critical monomial of $\partial G/\partial z'$, and (v) if $A - \{p\}$ is non-empty and $r = \max A - \{p\}$, then N is not a critical monomial of $H(z) = \sum_{m+j+k=r} f_{mjk}(x)\, z^j (z' + z^2)^k$. Let $T = (N'/N) + (\partial G(N)/\partial z)(\partial G(N)/\partial z')^{-1}$. (Then under the assumptions (iii) and (iv), T is automatically in the divergence class by [1 ; § 40 (b)]). We assume T satisfies the following three conditions: (vi) $IF(T) \not\equiv 0$, (vii) if it is the case that $T \approx N$, say $T \sim \sigma N$ where σ is a non-zero constant, then we require that $\sigma \notin \{p - r, 2(p - r)\}$ and $IF((\sigma + (r - p))N) \not\equiv 0$, (viii) if it is the case that $T \approx N'/N$, then we require that T is not $\sim N'/N$. (*see Remark* (c) *after proof*).

Then under these conditions, e. f. d. in $F(a, b)$ there exists a function $u_0 \sim N$ such that $G(u_0) = 0$ and such that if x_0 is in the domain of u_0, then the equation $\Omega(y) = 0$ possesses solutions $y_c^* \sim c \exp \int_{x_0}^{x} u_0$ for every constant $c \neq 0$.

The solutions y_c^* have the following properties:

(A) For every real α, $y_c^* > x^\alpha$.

(B) For each $c \neq 0$, there is a function $W_c \sim N$ such that y_c^* is of the form $\exp \int W_c$.

PROOF: In this proof, we will make use of the lemmas which are stated and proved in §§ 4, 5 and 6.

By conditions (iii) and (iv) (and § 6) it follows from [1 ; § 40 (b)] that there exists a logarithmic monomial $Q(x)$, such that

$Q(x)\, G(N + Nw)$ is normal, (i. e. has the form

(1) $$Q(x)\, G(N + Nw) = \Sigma\, d_{jk}(x)\, w^j (w')^k$$

where in $F(a, b)$, $d_{00} < 1$, $d_{10} \sim 1$, $d_{j0} \lesssim 1$ for $j > 1$, $1/d_{01}$ is in the divergence class, and there exists $q \geq 1$ such that for $k \geq 1$ and $j + k \geq 2$, we have $d_{jk} \lesssim d_{01}(L_q)^{k-1}$ where $L_q(x) = x \log x \ldots \log_{q-1} x$). By differentiating (1) with respect to w and evaluating at $w = 0$, we obtain, $Q(N(\partial G(N)/\partial z) + N'(\partial G(N)/\partial z')) \sim 1$ since $d_{10} \sim 1$. Similarly, differentiating (1) with respect to w' and evaluating at $w = 0$, we obtain $QN(\partial G(N)/\partial z') \sim d_{01}$. Thus clearly,

(2) $1/d_{01} \sim T$ (where T is as defined in the hypothesis).

By assumption, $IF(T) \not\equiv 0$. Letting I be any open subinterval of (a, b) on which $IF(T)$ is nowhere zero, it follows from [4 ; § 111] (if $IF(T) > 0$) or [4 ; § 117] (if $IF(T) < 0$) that there exists a function $w_0 < 1$ in $F(I)$ such that $Q(x)\, G(N + Nw_0) = 0$. Letting $u_0 = N + Nw_0$, we have,

(3) $u_0 \sim N$ in $F(I)$ and $G(u_0) = 0$.

Let x_0 be a point in the domain of u_0 and let $y_c = c \exp \int_{x_0}^{x} u_0$ for each constant $c \neq 0$. We now assert that

(4) For each real α, $y_c > x^\alpha$ in $F(I)$.

To prove (4), let α be given and let $z_c = x^{-\alpha} y_c$. Then by simple calculation it is easily seen that z_c is a solution of $z - (u_0 - \alpha x^{-1})^{-1} z' = 0$. But every non-zero solution of this equation is > 1 by [3 ; p. 271, Lemma δ], since $u_0 - \alpha x^{-1} \sim N$ and $IF(N) > 0$ on I. Thus $z_c > 1$ so $y_c > x^\alpha$.

Note that if $A = \{p\}$, the theorem is proved, for in this case $\Omega(y_c) = y_c^{\,p}\, G(u_0) = 0$ and letting δ be a value of $\log c$, we have

$y_c = \exp\left(\delta + \int_{x_0}^{x} u_0\right)$ which is of the form $y_c = \exp \int u_0$. Thus we may assume $A \neq \{p\}$.

We now write the equation $\Omega(y) = 0$ in the form,

$$(5) \qquad Q \sum_{m+j+k=p} f_{mjk}\, y^m (y')^j (y'')^k = -Q \sum_{m+j+k<p} f_{mjk}\, y^m (y')^j (y'')^k .$$

In what follows let c be a non-zero constant. Since $y_c' = y_c u_0$ and $y_c'' = y_c (u_0' + u_0^2)$, it follows that under the change of dependent variable $y = y_c(1+v)$ and multiplication by y_c^{-p}, (5) is transformed into an equation of the form,

$$(6) \qquad \sum g_{ijk}\, v^i (v')^j (v'')^k = \sum h_{ijk}\, v^i (v')^j (v'')^k$$

where it is easily verified that (7) - (13) below hold:

(7) Each h_{ijk} is trivial (by (4)),

(8) Each g_{ijk} is $<$ some power of x (since it is true of u_0, u_0' and each f_{ijk}),

$$(9) \qquad g_{000} = Q \sum_{m+j+k=p} f_{mjk}\, u_0^j (u_0' + u_0^2)^k ,$$

$$(10) \qquad h_{000} = -Q \sum_{m+j+k<p} f_{mjk}\, u_0^j (u_0' + u_0^2)^k (y_c)^{m+j+k-p}$$

$$(11) \qquad g_{100} = pQ \sum_{m+j+k=p} f_{mjk}\, u_0^j (u_0' + u_0^2)^k ,$$

$$(12) \qquad g_{010} = Q \sum_{m+j+k=p} f_{mjk} \left(j u_0^{j-1} (u_0' + u_0^2)^k + 2k\, u_0^{j+1} (u_0' + u_0^2)^{k-1} \right),$$

$$(13) \qquad g_{001} = Q \sum_{m+j+k=p} f_{mjk}\, k u_0^j (u_0' + u_0^2)^{k-1} .$$

By (9) and (11), $g_{000} = QG(u_0)$ and $g_{100} = pQG(u_0)$, so by (3),

$$(14) \qquad g_{000} = 0 \text{ and } g_{100} = 0.$$

Now by definition of r as being max $(A - \{p\})$, we have by (10),

$$h_{000} = - Q \sum_{t=0}^{r} \Gamma_t(u_0)\, y_c^{\,t-p} \tag{15}$$

where $\Gamma_t(z) = \sum_{m+j+k=t} f_{mjk}\, z^j (z' + z^2)^k$. By hypothesis, N is not a critical monomial of $H(z) = \Gamma_r(z)$. Thus $\Gamma_r(N) \not\equiv 0$ by [1 ; § 5], and since $u_0 \sim N$, we have $\Gamma_r(u_0) \sim \Gamma_r(N)$. Since $\Gamma_r(N)$ clearly belongs to an LDF (namely the field generated by the original and set of logarithmic monomials of rank $\leq$ rank N), and since $\Gamma_r(N)$ is not zero, there is a logarithmic monomial B such that $\Gamma_r(N) \sim B$. Hence,

$$\Gamma_r(u_0) \sim B \text{ in } F(I). \tag{16}$$

Now for each $t \leq r$, clearly $\Gamma_t(u_0)$ is $<$ some power of x (since it is true of u_0, u_0' and each f_{mjk}). Thus by (16), $(\Gamma_r(u_0))^{-1}\Gamma_t(u_0)$ is $<$ some power of x for each $t < r$. But if $t < r$, then by (4), y_c^{r-t} is $>$ all powers of x. Hence $(\Gamma_r(u_0))^{-1}\Gamma_t(u_0) < y_c^{r-t}$ and so $\Gamma_t(u_0)\, y_c^{t-p} < \Gamma_r(u_0)\, y_c^{r-p}$ for each $t < r$. Hence by (15) and (16),

$$h_{000} \sim - Q B y_c^{r-p}. \tag{17}$$

From (13), we have,

$$g_{001} = Q \partial G(u_0)/\partial z'. \tag{18}$$

But from (1),

$$Q G(z) = \sum d_{jk} ((z - N)/N)^j ((Nz' - zN')/N^2)^k. \tag{19}$$

Differentiating this relation with respect to z', and then evaluating at $z = u_0$, we obtain (since $u_0 = N + N w_0$),

$$Q \partial G(u_0)/\partial z' = N^{-1} \sum d_{jk}\, k w_0^j (w_0')^{k-1}. \tag{20}$$

Now if $j \geq 1$ and $k \geq 1$, then $d_{jk} \lessapprox (L_q)^{k-1} d_{01}$. Since $w_0 < 1$, we have $w_0' < L_q^{-1}$ (see § 2 (b)). Hence $d_{jk}\, w_0^j (w_0')^{k-1} < d_{01}$ if $j \geq 1$

and $k \geq 1$. Thus clearly

(21) $$\Sigma\, d_{jk}\, kw_0^j\,(w_0')^{k-1} \sim d_{01}$$

so by (18), (19) and (20),

(22) $$g_{001} \sim N^{-1}\, d_{01}\,.$$

Finally, we determine g_{010}. From (12), clearly

(23) $$g_{010} = Q\partial G\,(u_0)/\partial z.$$

Differentiating (19) with respect to z and evaluating at $z = u_0$, we obtain (since $u_0 = N + Nw_0$),

(24) $Q\partial G\,(u_0)/\partial z = \psi_1 + \psi_2$, where $\psi_1 = (-\,N'/N^2)\,\Sigma\, d_{jk}\, kw_0^j\,(w_0')^{k-1}$

and $\psi_2 = N^{-1}\,\Sigma\, d_{jk}\, jw_0^{j-1}\,(w_0')^k$. By (21), we have

(25) $$\psi_1 \sim (-\,N'/N^2)\, d_{01}\,.$$

We now compute ψ_2. We have $d_{10} \sim 1$ and $d_{j0} \lesssim 1$ for $j > 1$. Thus since $w_0 < 1$, we have $d_{j0}\, w_0^{j-1} < 1$ for $j > 1$. Now if $j \geq 1$ and $k \geq 1$, then $d_{jk} \lesssim d_{01}\,(L_q)^{k-1}$. Since $1/d_{01}$ is in the divergence class, there exists $t > 0$ such that $1/d_{01} > L_t^{-1}$. Thus $d_{01} < L_t$. Letting $m = \max\,\{t, q\}$, we have $w_0' < L_m^{-1}$ since $w_0 < 1$ (see § 2 (b)). Thus for $j \geq 1$, $k \geq 1$, we have $d_{jk}\, w_0^{j-1}\,(w_0')^k < 1$, so clearly,

(26) $$\psi_2 \sim N^{-1}\,.$$

Thus by (23) and (24), we have

(27) $g_{010} = \psi_1 + \psi_2$ (where ψ_1 satisfies (25) and ψ_2 satisfies (26)).

Now let $S_{ijk} = g_{ijk} - h_{ijk}$, so (6) may be written,

(28) $$\Sigma\, S_{ijk}\, v^i\,(v')^j\,(v'')^k = 0,$$

where by (7), (8), (14), (17), (22), we have in $F(I)$,

(29) $$S_{000} \sim QBy_c^{r-p}, S_{100} \text{ is trivial and } S_{001} \sim N^{-1} d_{01}.$$

Now since d_{01} is $\sim$ to a logarithmic monomial, we have in $F(a, b)$ that one of the following must hold: $d_{01} N'/N$ is $> 1, \approx 1$ or < 1. We distinguish these three cases:

Case I: $d_{01} N'/N > 1$. Hence $\psi_2 < \psi_1$ so we have,

(30) $$S_{010} \sim g_{010} \sim (-N'/N^2) d_{01}.$$

We consider the equation,

(31) $$S_{001} v'' + S_{010} v' + S_{000} = 0$$

which may be written $(S_{010}/S_{001})^{-1} v'' + v' = -S_{000}/S_{010}$. By

(29) and (30), $S_{010}/S_{001} \sim -N'/N$ and $-S_{000}/S_{010} \sim QBN^2(N' d_{01})^{-1} y_c^{r-p}$. Now y_c^{r-p} is of the form $\exp \int (r-p) u_0$, and since N is a logarithmic monomial, $N'/N \lesssim x^{-1}$ (see § 2 (b)), so $N'/N < u_0$. Since $IF((r-p) u_0) < 0$ on I, it follows by § 4, that e. f. d. in $F(I)$, equation (31) has a solution $v_0 = R_1 y_c^{r-p}$ where,

(32) $$R_1 \sim -QB/(r-p)^2 N d_{01}.$$

Thus by (4) (since $r < p$), v_0 is trivial, and hence,

(33) $$v_0^m \text{ is trivial for } m \geq 1.$$

Let J be any open subinterval of I, such that v_0 exists on $F(J)$. A simple calculation shows that $v_0'/v_0 = M$ where $M = (r - p) u_0 + (R_1'/R_1)$ and hence $M \sim (r-p) u_0$ (see § 2 (b)). Thus since v_0 solves (31), clearly,

(34) $$S_{001}(M^2 + M') v_0 + S_{010} M v_0 + S_{000} = 0.$$

We now subject equation (28) to the change of variable $v = v_0 + v_0 u$, and divide the result by v_0. Since $v' = v_0 \Phi_1(u)$ where $\Phi_1(u) = M + Mu + u'$ and $v'' = v_0 \Phi_2(u)$ where $\Phi_2(u) = M' + M^2 + (M' + M^2) u + 2Mu' + u''$, we obtain

$$\Sigma S_{ijk} v_0^{i+j+k-1} (1 + u)^i (\Phi_1(u))^j (\Phi_2(u))^k = 0. \tag{35}$$

We denote this equation by

$$\Sigma t_{ijk} u^i (u')^j (u'')^k = 0. \tag{36}$$

Now clearly,

$$t_{000} = \Sigma S_{ijk} v_0^{i+j+k-1} M^j (M' + M^2)^k. \tag{37}$$

Iu view of (33), (34) and the fact that S_{100} is trivial we see that t_{000} is trivial in $F(J)$. Now,

$$t_{100} = \Sigma S_{ijk} v_0^{i+j+k-1} (i + j + k) M^j (M' + M^2)^k. \tag{38}$$

By (33) and (34), we may write $t_{100} = (-S_{000}/v_0) + q_1$ where q_1 is trivial in $F(J)$. By (29) and the fact that $v_0 = R_1 y_c^{r-p}$ where R_1 satisfies (32), we obtain $t_{100} \sim (r-p)^2 N d_{01}$. Now,

$$t_{010} = \Sigma S_{ijk} v_0^{i+j+k-1} (2kM^{j+1} (M' + M^2)^{k-1} + jM^{j-1} (M' + M^2)^k). \tag{39}$$

By (33) therefore $t_{010} = S_{010} + 2MS_{001} + q_2$, where q_2 is trivial in $F(J)$. But by (29) (and the fact that $M \sim (r-p) N$), we see that $2M S_{001} \sim 2(r-p) d_{01}$. But $N'/N \lesssim x^{-1}$ (see § 2 (b)) and so $N' < N^2$. Thus by (30), clearly $t_{010} \sim 2(r-p) d_{01}$. Now,

$$t_{001} = \Sigma S_{ijk} v_0^{i+j+k-1} kM^j (M' + M^2)^{k-1} \tag{40}$$

and so by (33) and (29), we have $t_{001} \sim N^{-1} d_{01}$. Finally in view of (33), (and the fact that each S_{ijk} is $<$ some power of x, by (7) and (8)), we clearly see that if $i + j + k \geq 2$ then t_{ijk} is trivial in

$F(J)$. Hence if we divide equation (36) by $(r-p)^2 N d_{01}$, we obtain,

(41) $$\Sigma\, a_{ijk}\, u^i (u')^j (u'')^k = 0, \text{ where in } F(J),$$

(42) $$a_{100} \sim 1,\ a_{010} \sim 2/(r-p)N,\ a_{001} \sim 1/(r-p)^2 N^2, \text{ and } a_{ijk}$$

is trivial if $i+j+k \neq 1$.

Now let $W_1 = W_2 = (p-r)N$. Define $E_2 = a_{001} W_1 W_2 - 1$. From (42), $a_{001} W_1 W_2 \sim 1$, so $E_2 < 1$ in $F(J)$. Now define $E_1 = -W_1 (a_{010} + (1+E_2)(W_1^{-1} + W_2^{-1} + W_1' (W_2 W_1^2)^{-1})$. Since $W_1' = (p-r)N'$, we see that $W_1'(W_2 W_1^2)^{-1} = N'/(p-r)^2 N^3$ which is $< 1/N$ since $N'/N \lessapprox x^{-1}$ (see § 2 (b)). But $W_1^{-1} + W_2^{-1} = 2/(p-r)N$ and $a_{010} \sim -2/(p-r)N$ by (42) so clearly $E_1 < -W_1/N$. Thus $E_1 < 1$ in $F(J)$. Finally if we define $E_0 = a_{100} - (1 + E_2 + E_1)$, then $E_0 < 1$ since $a_{100} \sim 1$. Since $IF(W_j) > 0$ on J for $j = 1, 2$, it follows from § 5, that equation (41) possesses a solution $u^* < 1$ in $F(J)$. Thus the function $v^* = v_0 + v_0 u^*$ is a solution of (28), and so the function $y_c^* = y_c(1+v^*)$ is a solution of $\Omega(y) = 0$. In view of what I and J represent, it is clear that such a y_c^* exists e. f. d. in $F(a, b)$. Since v_0 is trivial by 33), v^* is trivial in $F(J)$. Thus clearly $y_c^* \sim y_c$ and Part (A) follows from (4). Since $y_c^* = y_c(1+v^*)$, clearly $(y_c^*)' = y_c^*(u_0 + (v^*)'/(1+v^*))$. Since v^* is trivial in $F(J)$, so is $(v^*)'$ (see § 2 (b)). Thus we may write $(y_c^*)' = y_c^* W_c$, where $W_c \sim u_0$. Thus if x_1 is a point in the domain of W_c, then for some constont K, $y_c^* = K \exp \int_{x_1}^{x} W_c$. Since $y_c^* \not\equiv 0$, $K \neq 0$. Thus for for any value of $\log K$, $y_c^* = \exp\left(\log K + \int_{x_1}^{x} W_c\right)$ which is of the form $\exp \int W_c$. This proves Part (B) in Case I.

Case II : $d_{01} N'/N \approx 1$. In this case, $d_{01} N'/N \sim \delta$ where δ is a non-zero constant. By (2) and hypothesis (viii), $\delta \neq 1$. Thus in (27), we have $g_{010} \sim (1-\delta)/N$ so by (7),

(43) $$S_{010} \sim (1-\delta)/N.$$

We again consider the equation (31). In this case, by (29) and (43), $S_{010}/S_{001} \sim (1-\delta)/d_{01}$ and $-S_{000}/S_{010} \sim -QBN(1-\delta)^{-1} y_c^{r-p}$. As in Case I, y_c^{r-p} is of the form $\exp \int (r-p)\, u_0$, and in this case, $1/d_{01} < N$ since $N'/N \lesssim x^{-1}$ (see § 2 (b)). Hence by § 4, e. f. d. in $F(I)$, equation (31) possesses a solution $v_0 = R_2\, y_c^{r-p}$, where,

$$R_2 \sim -QB/(r-p)^2 N d_{01}\,. \tag{44}$$

As in Case I, (33) holds. Let J be any open subinterval of I such that v_0 exists on $F(J)$. A simple calculation shows that $v_0'/v_0 = M$ where $M = (r-p)\, u_0 + R_2'/R_2$. Thus, as in Case I, $M \sim (r-p)\, u_0$ and since v_0 solves (31), we have that (34) holds. We now subject equation (28) to the change of variable $v = v_0 + v_0 u$, and then divide by v_0. If we denote the result by

$$\Sigma\, t_{ijk}\, u^i (u')^j (u'')^k = 0, \tag{45}$$

then as in Case I, the coefficients t_{000}, t_{100}, t_{010} and t_{001} are given by (37), (38), (39) and (40) respectively, and (in view of (33)), we also have

$$t_{ijk} \text{ is trivial in } F(J) \text{ for } i+j+k \geq 2. \tag{46}$$

In view of (29), (33) and (34), it follows from the representation in (37) that t_{000} is trivial in $F(J)$, and it follows from the representation in (38), that $t_{100} = (-S_{000}/v_0) + q_3$ where q_3 is trivial in $F(J)$. Since $v_0 = R_2\, y_c^{r-p}$ where R_2 is as in (44), we have using (29) that $t_{100} \sim (r-p)^2 N d_{01}$. In view of (29) and (33), it follows from the representation in (39) that $t_{010} = S_{010} + 2MS_{001} + q_4$ where q_4 is trivial in $F(J)$. Now $2MS_{001} \sim 2(r-p)\, d_{01}$ by (29). But in this case $d_{01} > 1/N$ since $1/d_{01} \approx N'/N \lesssim x^{-1}$ (see § 2 (b)). Thus clearly from (43), $t_{010} \sim 2(r-p)\, d_{01}$. Finally from (29), (33) and the representation in (40), we see that $t_{001} \sim N^{-1} d_{01}$. Hence if we divide equation (45) by $(r-p)^2 Nd_{01}$, we obtain an equation,

(47) $$\Sigma\, a_{ijk}\, u^i (u')^j (u'')^k = 0, \text{ where in } F(J),$$

(48) $$a_{100} \sim 1,\ a_{010} \sim 2/(r-p)\,N,\ a_{001} \sim 1/(r-p)^2 N^2 \text{ and } a_{ijk}$$

is trivial if $i+j+k \neq 1$. But these are *precisely* the same asymptotic properties we found for the coefficients of (41) in Case I (see (42)). Thus letting $W_1 = W_2 = (p-r)\,N$, we see that if we define E_2, E_1, E_0 as in Case I, we will find that $E_j < 1$ in $F(J)$ $(j = 0, 1, 2)$, and so by § 5, it follows that equation (47) possesses a solution $u^* < 1$ in $F(J)$. Thus as in Case I, the equation $\Omega(y) = 0$ possesses the solution $y_c^* = y_c(1 + v_0 + v_0 u^*)$ e. f. d. in $F(a, b)$. The proof the solution y_c^* has the desired properties in this case follows *exactly* as in Case I.

Case III : $d_{01} N'/N < 1$. In this case, in (27) we have $g_{010} \sim N^{-1}$, so we have

(49) $$S_{010} \sim N^{-1}.$$

We again consider the equation (31). In this case by (29) and (49), $S_{010}/S_{001} \sim 1/d_{01}$ and $-S_{000}/S_{010} \sim -QBN\, y_c^{r-p}$. As in Case I, y_c^{r-p} is of the form $\exp \int (r-p)\, u_0$. Let $U = (1/d_{01}) + (r-p)\,N$. Since $1/d_{01}$ and N are both $\sim$ to logarithmic monomials, one of the following three cases must hold:

(a) $1/d_{01} < N$, (b) $1/d_{01} > N$ and (c) $1/d_{01} \approx N$. In Case (a), $U \sim (r-p)\,N$ so $IF(U) < 0$ on I. In Case (b), $U \sim 1/d_{01}$ so by (2), $IF(U)$ is nowhere zero on I. In Case (c), $1/d_{01} \sim \sigma N$ for some constant $\sigma \neq 0$. By (2) and hypothesis (vii), $\sigma \neq p - r$ so $U \sim (\sigma + (r-p))\,N$ and $IF(U) \not\equiv 0$ on I. Thus in all three cases $IF(U) \not\equiv 0$. Hence by § 4, e. f. d. in $F(I)$, equation (31) possesses a solution $v_0 = R_3\, y_c^{r-p}$, where

(50) $$R_3 \sim -QB/(r-p)\, d_{01}\, U.$$

As in Case I (33) holds. Let J be any open subinterval of I such that v_0 exist on $F(J)$. A simple calculation shows that $v_0'/v_0 = M$

where $M = (r - p) u_0 + R_3'/R_3$. Thus $M \sim (r - p) u_0$, and since v_0 solves (31), we have that (34) holds. We now subject equation (28) to the change variable $v = v_0 + v_0 u$ and then divide by v_0. If we denote the result by

$$(51) \qquad \Sigma\, t_{ijk}\, u^i (u')^j (u'')^k = 0,$$

then as in Case I, the coefficients $t_{000}, t_{100}, t_{010}$ and t_{001} are given by (37), (38), (39), and (40) respectively, and (in view of (33)), we also have

$$(52) \qquad t_{ijk} \text{ is trivial in } F(J) \text{ if } i + j + k \geq 2.$$

In view of (29), (33) and (34), it follows from the representation in (37) that

$$(53) \qquad t_{000} \text{ is trivial in } F(J),$$

and it follows from (38) that $t_{100} = (-S_{000}/v_0) + q_5$ where q_5 is trivial in $F(J)$. Since $v_0 = R_3\, y_c^{r-p}$ where R_3 is as in (50), it follows from (29) that

$$(54) \qquad t_{100} \sim (r - p)\, d_{01}\, U \text{ in } F(J).$$

In view of (29) and (33) it follows from (39) that $t_{010} = S_{010} + 2MS_{001} + q_6$, where q_6 is trivial in $F(J)$. By (29), $2MS_{001} \sim 2(r-p)\, d_{01}$, and by (49), $S_{010} \sim N^{-1}$. Let $U^{\#} = (1/N) + 2(r - p)\, d_{01}$. By hypothesis (vii), $1/d_{01}$ is not $\sim 2(p - r) N$, so $U^{\#}$ is $\approx$ to either N^{-1} or d_{01} and so is nontrivial. Hence clearly,

$$(55) \qquad t_{010} \sim U^{\#} \text{ in } F(J).$$

Finally from (29) and (33), it follows from (40), that

$$(56) \qquad t_{001} \sim d_{01}/N \text{ in } F(J).$$

Hence if we divide equation (51) by $(r - p)\, d_{01}\, U$, we obtain an equation

$$(57) \qquad \Sigma\, a_{ijk}\, u^i (u')^j (u'')^k = 0,$$

where by (52)-(56), we have in $F(J)$,

(58) $\qquad a_{100} \sim 1,\ a_{010} \sim U^{\#}/(r-p)\,d_{01}\,U,\ a_{001} \sim 1/(r-p)\,NU,$

(59) $\qquad a_{ijk}$ is trivial if $i+j+k \neq 1$.

To proceed any further, we must consider each of the possibilities for the asymptotic behavior of U. Since $1/d_{01}$ and N are $\sim$ to logarithmic monomials, one of the following cases must hold in $F(a,b)$: $1/d_{01} < N$, $1/d_{01} > N$, $1/d_{01} \approx N$. We distinguish these three subcases.

Subcase A: $1/d_{01} < N$. Thus in this subcase, $U \sim (r-p)\,N$ and $U^{\#} \sim 2\,(r-p)\,d_{01}$. Hence by (58) and (59), $a_{100} \sim 1$, $a_{010} \sim 2/(r-p)\,N$, $a_{001} \sim 1/(r-p)^2 N^2$ and a_{ijk} is trivial if $i+j+k \neq 1$. But these are *precisely* the same asymptotic properties we found for the coefficients of (41) in Case I (see (42)). Thus exactly as in Case I (i. e. by letting $W_1 = W_2 = (p-r)\,N$, and defining E_1, E_2, E_0 as in Case I), we can apply § 5 to show that equation (57) possesses a solution $u^* < 1$ in $F(J)$. Thus $\Omega(y) = 0$ possesses the solution $y_c^* = y_c(1 + v_0 + v_0 u^*)$ in $F(J)$ and hence e. f. d. in $F(a,b)$. The proof that y_c^* has the desired properties in this case follows exactly as in Case I.

Subcase B: $1/d_{01} > \mathrm{N}$. Thus in this subcase, $U \sim 1/d_{01}$ and $U^{\#} \sim 1/N$. Thus by (58) and (59), we have $a_{100} \sim 1$, $a_{010} \sim 1/(r-p)\,N$, $a_{001} \sim d_{01}/(r-p)\,N$. Let $W_1 = (p-r)\,N$ and $W_2 = -1/d_{01}$. Define $E_2 = a_{001}\,W_1\,W_2 - 1$. Then $E_2 < 1$, in $F(J)$ since clearly $a_{001} W_1 W_2 \sim 1$. Now define $E_1 = -W_1\,(a_{010} + (1 + E_2)\,(W_1^{-1} + W_2^{-1} + W_1'(W_2 W_1^2)^{-1}))$. By this subcase, $W_2^{-1} < W_1^{-1}$. Also, $W_1'\,(W_2 W_1^2)^{-1} = d_{01}\,N'/(r-p)\,N^2$ which is also $< W_1^{-1}$ since by this case $d_{01}N'/N < 1$. Hence since $a_{010} \sim -W_1^{-1}$, we see that $E_1 < -W_1\,W_1^{-1}$ so $E_1 < 1$ in $F(J)$. Finally if we define $E_0 = a_{100} - (1 + E_2 + E_1)$ then $E_0 < 1$ in $F(J)$ since $a_{100} \sim 1$. Thus since $IF(W_1) > 0$ on J while $IF(W_2) \not\equiv 0$ on J (since $IF(W_2) \equiv -IF(T)$ by (2) and $IF(T) \not\equiv 0$ by assumption (vi)), it follows from § 5 that e. f. d. in $F(J)$, equation (57) possesses a solution $u^* < 1$. In view of what J

and I represent, clearly such a u^* exists e. f. d. in $F(a, b)$. Thus (28) possesses the solution $v^* = v_0 + v_0 u^*$, and hence $\Omega(y) = 0$ possesses, e. f. d. in $F(a, b)$, the solution $y_c^* = y_c(1 + v_0 + v_0 u^*)$. The proof that the solution y_c^* has the desired properties follows exactly as in Case I.

Subcase C; $1/d_{01} \approx N$. Thus there is a non-zero constant σ such that $1/d_{01} \sim \sigma N$. Since $T \sim 1/d_{01}$ we have by assumption (vii) that,

$$\sigma \notin \{p - r, 2(p - r)\} \text{ and } IF(\lambda N) \not\equiv 0 \text{ where } \lambda = \sigma + (r - p). \tag{60}$$

In this subcase, clearly $U \sim \lambda N$ and $U^{\#} \sim (\sigma + 2(r - p))/\sigma N$, so by (58) and (59) we have $a_{100} \sim 1$, $a_{010} \sim (\sigma + 2(r - p))/(r - p)\lambda N$, $a_{001} \sim \sim 1/\lambda(r - p)N^2$ and a_{ijk} is trivial for $i + j + k \neq 1$. Let $W_1 = = (p - r)N$ and $W_2 = -\lambda N$. Define $E_2 = a_{001} W_1 W_2 - 1$. Then $E_2 < 1$ in $F(J)$ since $a_{001} W_1 W_2 \sim 1$. Now define $E_1 = - W_1(a_{010} + + (1 + E_2)(W_1^{-1} + W_2^{-1} + W_1'(W_2 W_1^2)^{-1}))$. Clearly, since $\lambda = = \sigma + (r - p)$, we have $W_1^{-1} + W_2^{-1} = (\sigma + 2(r - p))/\lambda(p - r)N$. Now $W_1'(W_2 W_1^2)^{-1} = N'/\lambda(r - p)N^3$ which is $< W_1^{-1} + W_2^{-1}$ since by this subcase $N'/N^3 \approx d_{01} N'/N^2$ which is $< 1/N$ by this case. Since $a_{010} \sim -(W_1^{-1} + W_2^{-1})$ clearly, we have that $E_1 < < - W_1(W_1^{-1} + W_2^{-1})$ so $E_1 < 1$ in $F(J)$. Finally if we define $E_0 = a_{100} - (1 + E_2 + E_1)$, then $E_0 < 1$ in $F(J)$ since $a_{100} \sim 1$. Thus since $IF(W_1) > 0$ on J, while $IF(W_2) \not\equiv 0$ on J (in view of (60) and the fact that $IF(W_2) = - IF(\lambda N)$), it follows from § 5 that e. f. d. in $F(J)$, equation (57) possesses a solution $u^* < 1$. In view of what J and I represent, clearly such a u^* exists e. f. d. in $F(a, b)$. Thus (28) possesses the solution $v^* = v_0 + v_0 u^*$, and hence $\Omega(y) = 0$ possesses, e. f. d. in $F(a, b)$, the solution $y_c^* = = y_c(1 + v_0 + v_0 u^*)$. The proof that the solution y_c^* has the desired properties follows exactly as in Case I. This concludes the proof of the theorem.

Remarks: (a) If I is any open subinterval of (a, b) on which $IF(T) > 0$, then in view of (2), it follows from [4; § 111] that the equation $Q(x)\, G(N + Nw) = 0$ actually possesses a whole one-parameter family of solutions < 1 in $F(I)$. By using this family for w_0 in the proof, it is clear that e. f. d. in $F(I)$, we can actually expect a *two-parameter* family of solutions of $\Omega(y) = 0$, each having the form $\exp \int W$ for some $W \sim N$.

(b) In each case in the proof we showed that for each $c \neq 0$, the equation $\Omega(y) = 0$ possesses an exact solution $y_c^* = y_c(1 + v_c + v_c u_c^*)$ where $y_c = c \exp \int_{x_0}^{x} u_0$, $u_c^* < 1$ and $v_c = R_c y_c^{r-p}$ for some $R_c \sim$ to a logarithmic monomial. This representation clearly gives more information about y_c^* than either of the statements $y_c^* \sim y_c$ or $y_c^* = \exp \int W_c$ for some $W_c \sim N$, and furtermore, is useful in estimating how good an approximation y_c is to the exact solution y_c^*.

(c) In general one can make no specific statement about the relation between N and its corresponding T. In fact for *any* given logarithmic monomials N and T^*, where T^* is in the divergence class, one can always produce a second order $\Omega(y)$, whose corresponding $G(z)$ has N as a critical monomial (which automatically satisfies (iii), (iv) and (v) of § 3), and where T^* is the corresponding T for N. To see this, consider $\Omega(y) = (1/T^* N) yy'' - (1/T^* N)(y')^2 + ((1/N) - (N'/T^* N^2)) yy' - y^2$. Here $G(z) = (1/T^* N) z' + ((1/N) - (N'/T^* N^2)) z - 1$, and hence $G(Nw) = (w'/T^*) + w - 1$. Since T^* is in the divergence class, an easy application of [1; § 26] to $G(Nw)$, followed by [1; § 30 (b)] shows that N is a critical monomial of G satisfying (iii) by § 6. Clearly (iv) and (v) are also satisfied. A simple calculation shows that $T^* = (N'/N) + (\partial G(N)/\partial z)(\partial G(N)/\partial z')^{-1}$. In view of this example, one can view the situations where $T \approx N$ or $T \approx N'/N$ (i. e. those where one would have to verify (vii) or (viii) of § 3), as being somewhat special situations.

4. Lemma: Let W, V and H be functions $\sim$ to logarithmic monomials in some $F(I)$. Let $V > x^{-1}$ and $W \not\sim - V$. (Then clearly $W + V \gtrapprox V$, so $W + V$ is also in the divergence class). Let $IF(V) \not\equiv 0$ and $IF(W + V) \not\equiv 0$. Let w_0 be a function of the form $\exp \int V$. Then e. f. d. in $F(I)$, the equation $(v''/W) + v' = H w_0$ possesses a solution $v_0 = R w_0$ where $R \sim WH/V(W + V)$.

PROOF: Let $z_0 = Hw_0$. Under the change of variable $w = v'$, the equation

(a) $(v''/W) + v' = z_0$ becomes

(b) $(w'/W) + w = z_0$.

Under the change of dependent variable $w = xWz_0 u$, followed by division by xz_0, equation (b) becomes,

(c) $u' + Uu = x^{-1}$,

where $U = W + V + (H'/H) + (W'/W) + x^{-1}$. Now since $W \not\sim -V$, clearly $W + V$ is $\sim$ to a logarithmic monomial which is $\gtrapprox V$ and hence $> x^{-1}$. Since H and W are $\sim$ to logarithmic monomials, H'/H and W'/W are $\lessapprox x^{-1}$. (see § 2 (b)). Thus,

(d) $U \sim W + V > x^{-1}$.

Hence in some element of $F(I)$, U is nowhere zero, so (c) may be written $u + u'/U = 1/xU$. By (d), $1/xU < 1$ and $IF(U) \not\equiv 0$. Letting J be any open subinterval of I on which $IF(U)$ is nowhere zero, it follows from [3; p. 271, Lemma δ] that there is a function $u_1 < 1$ in $F(J)$ such that $u_1 + u_1'/U = 1/xU$. But $xu_1' < 1$ (see § 2 (b)), so $u_1 \sim 1/xU$. Now clearly the function $w_1 = xWu_1 z_0$ is a solution of (b). Let J_1 be any open subinterval of J on which $IF(V)$ is nowhere zero. Since z_0 is of the form $H \exp \int V$, and $xWu_1 \sim W/W + V$ (by (d)), it follows by [2; § 10 (b)] that the equation $v' = w_1$ possesses a solution $v_0 = Rw_0$ where $R \sim WH/V(W + V)$ in $F(J_1)$. (Note: [2; § 10 (b)] was proved under the hypothesis that $V \approx x^{-1+t}$ for some $t > 0$, but it is easily seen that the *exact* same proof holds if the condition $V \approx x^{-1+t}$ is replaced by the condition that V be $\approx$ to a logarithmic monomial which is $> x^{-1}$). Then v_0 is clearly a solution of (a) proving the lemma.

5. LEMMA: (a) Let $\Phi(y) = A_2 y'' + A_1 y' + A_0 y$ where A_0, A_1 and A_2 are analytic in some $F(I)$. Let W_1, W_2 be $\sim$ to logarithmic

monomials is $F(I)$. Define E_2, E_1 and E_0 as follows: $E_2 = A_2 W_1 W_2 - 1$, $E_1 = - W_1 (A_1 + (1 + E_2)(W_1^{-1} + W_2^{-1} + W_1' (W_2 W_1^2)^{-1}))$, and $E_0 = A_0 - (1 + E_2 + E_1)$. Then,

$$(1) \qquad \Phi = \dot{W}_2 \dot{W}_1 + E_2 \dot{W}_2 \dot{W}_1 + E_1 \dot{W}_1 + E_0$$

where $\dot{W}_j$ is the operator $\dot{W}_j y = y - y' / W_j$.

(b). Let $\Psi(y) = a_{000} + \Phi(y) + \sum_{i+j+k \geq 2} a_{ijk} y^i (y')^j (y'')^k$,

where a_{000} and a_{ijk} (for $i + j + k \geq 2$) are trivial in $F(I)$, and where $\Phi(y)$ is a second order linear differential polynomial for which there exist functions W_1 and W_2 in the divergence class in $F(I)$ (see § 2 (d)) and functions E_0, E_1, E_2 each < 1 in $F(I)$, such that (1) above holds. Then for any open subinterval J of I on which both $IF(W_1)$ and $IF(W_2)$ are nowhere zero, the equation $\Psi(y) = 0$ possesses at least one solution $y^* < 1$ in $F(J)$.

PROOF: Part (a) is proved by directly computing the right side of (1) and showing that (1) holds if E_2, E_1 and E_0 are defined as in the statement.

To prove (b), let $A_0 y = y$, $A_1 y = \dot{W}_1 y$ and $A_2 y = \dot{W}_2 \dot{W}_1 y$. A simple calculation shows that y' and y'' are linear polynomials in $A_0 y$, $A_1 y$, and $A_2 y$ (with coefficients which are each $<$ some power of x). Thus when $\Psi(y)$ is written as a polynomial in the $A_j y$, we obtain (using (1)),

$$\Psi(y) = a_{000} + (1 + E_2) A_2 y + E_1 A_1 y + E_0 A_0 y + \\ + \sum_{m+n+q \geq 2} b_{mnq} (A_0 y)^m (A_1 y)^n (A_2 y)^q,$$

where each b_{mnq} is trivial (since each a_{ijk} is trivial). Since E_0, E_1, E_2 are each < 1 in $F(I)$, it follows directly from the remark after [5; § 99] (which states that formulas (99.6) and (99.7) are sufficient for a strong factorization sequence), that (W_1, W_2) is a *strong factorization sequence* (see [5; § 88 (b)]) for $\Psi(y)$. Hence if J is any open subinterval of I on which both $IF(W_1)$ and $IF(W_2)$ are nowhere zero, then by [5; § 99, Part 5], the equation $\Psi(y) = 0$ possesses at least one solution $y^* < 1$ in $F(J)$, proving (b).

6. LEMMA: Let $G(z) = \Sigma\, g_{ij} z^i (z')^j$ have coeffiicients in an LDF over $F(a, b)$. Let N be a critical monomial of $G(z)$. Then N is a simple non-parametric critical monomial of G (as defined in [1; §§ 28 (d), 14]) if and only if N is not a critical monomial of $\Gamma(z) = = z(\partial G/\partial z) + z'(\partial G/\partial z')$.

PROOF: (For notation and definitions used in this proof, see [1; §§ 7, 9, 28]). Let N have exponent β and multiplicity m in G. Let $H(z) = G(Nz)$, say $H(z) = \Sigma\, h_{ij}(x) z^i (z')^j$. By [1; § 30 (b)], 1 is a critical monomial of H having exponent β and multiplicity m. Now for each $s \geq 0$, there is a logarithmic monomial $Q(x)$ such that $[1, s, H](u, v) = \Sigma\, r_{ij}(u) v^i (v')^j$, where

$$\text{(a)} \qquad r_{ij}(u) = h_{ij}(e_s(u))\,(e_s(u) \dots e_1(u))^{-j}\, Q(e_s(u)),$$

(where $e_j(u)$ is the jth iteration of the exponential function). Thus by [1; § 27], for s sufficiently large,

(b) $r_{i\beta} \lesssim 1$, r_{ij} is trivial if $j \neq \beta$ and 1 is a root of multiplicity m of $C(v) = \Sigma\, r_{i\beta}(\infty) v^i$, (where $r_{i\beta}(\infty)$ is the constant defined by: $r_{i\beta}(\infty) = 0$ if $r_{i\beta} < 1$, while if $r_{i\beta} \approx 1$, $r_{i\beta} \sim r_{i\beta}(\infty)$).

Now let $\Lambda(z) = \Gamma(Nz)$. Then clearly, we have $\Lambda(z) = \Sigma(i+j) h_{ij} z^i (z')^j$. Thus for all $s \geq 0$, there is a logarithmic monomial $R(x)$ such that $[1, s, \Lambda](u, v) = \Sigma\, t_{ij} v^i (v')^j$, where

$$\text{(c)} \qquad t_{ij}(u) = (i+j)\, h_{ij}(e_s(u))\,(e_s(u)) \dots e_1(u))^{-j}\, R(e_s(u)).$$

Letting $B(u) = R(e_s(u))/Q(e_s(u))$, we have by (a) and (c) that

$$\text{(d)} \qquad [1, s, \Lambda](u, v) = B\, \Sigma\, (i+j)\, r_{ij}(u)\, v^i (v')^j.$$

Now assume N is not critical of $\Gamma(z)$. Then by [1; § 30 (b)], 1 is not critical of Λ. Now if β were > 0, then in view of (b) and (d), clearly $[1, s, \Lambda](u, 1)$ would be trivial, so by [1; §§ 3, 11 (b)], 1 would be critical of Λ which is a contradiction. Thus $\beta = 0$ proving hat N is non-parametric by [1; § 28]. In view of (b) and the fact that $\beta = 0$, we may write (d) as,

$$\text{(e)} \qquad [1, s, \Lambda](u, v) = B\,(dC/dv) + \Psi(u, v)$$

where all coefficients of Ψ are trivial. If $m > 1$, then 1 is a root of dC/dv, so by (e) and [1; §§ 3, 11 (b)], we would again obtain the contradiction that 1 is critical of Λ. Thus $m = 1$, so N is simple also.

Conversely, suppose N is simple and non-parametric in G. Then $\beta = 0$, so (e) holds. Since $m = 1$, dC/dv does not have 1 as a root. Thus by [1; § 12 (b)], 1 is not a critical monomials of Λ, so N is not critical of Γ by [1; § 11 (a)], proving the lemma.

7. Example: The example we consider here illustrates the fact that in the theorem of § 3, only the terms of degree p (and in a negative way, those of degree r) play an essential role, since in this example the terms of degree $< r$ are essentially arbitrary.

Consider the example,

$$\Omega(y) = (y')^{n+2} - y^n (y'')^2 + x^{-3} (\log x)^t y^{n+1} y' + \\ + \psi(x)\, y^{r-(m+q)} (y')^m (y'')^q + \Lambda(y),$$

where n is any integer > 2, t is any strictly positive real number, ψ is any finite sum of logarithmic monomials, r is any non-negative integer $\leq n + 1$, m and q are any non-negative integers such that $m + q \leq r$ and where Λ is any second order differential polynomial with coefficients in an LDF over $F(-\pi, \pi)$, each of whose terms has total degree $\leq r - 1$ in y, y', y''. In this case, $G(z)$ and $H(z)$ of § 3 are as follows: $G(z) = z^{n+2} - (z')^2 - 2z^2 z' - z^4 + x^{-3} (\log x)^t z$, and $H(z) = \psi(x)\, z^m (z' + z^2)^q$. Using [1; § 26], we find that $G(z)$ has the $(n + 1)$ simple, non-parametric (see § 6) critical monomials consisting of $M_1 = x^{-1} (\log x)^{t/3}$, $M_2 = e^{2\pi i/3} M_1$, $M_3 = e^{4\pi i/3} M_1$ and the $(n - 2)\,nd$ roots of unity. Using [1; §§ 21, 26], we find that H has the one critical monomial x^{-1} if $q > 0$, and no critical monomials if $q = 0$. Since $\partial G/\partial z' = -2z' - 2z^2$, we find that $\partial G/\partial z'$ has the one critical monomial x^{-1}.

Now consider M_1, M_2 and M_3. Clearly $IF(M_1) \equiv 1$ while $IF(M_2) = IF(M_3) \equiv -1/2$. Thus M_1 is the only one of these three which satisfies (ii) of § 3. For M_1, we now compute the function T of § 3. Since $\partial G/\partial z = (n+2)z^{n+1} - 4zz' - 4z^3 + x^{-3}(\log x)^t$, an easy calculation shows that $T \sim (3/2)x^{-1}(\log x)^{t/3}$. Thus $IF(T) \equiv 1$. However, we see that $T \approx M_1$ so we must verifiy (vii) of § 3. But here $\sigma = 3/2$, so clearly $\sigma \notin \{p-r, 2(p-r)\}$ (since $p-r$ is a positive integer), and $IF((\sigma+(r-p))M_1)$ is either $+1$ or -1 depending on whether $p-r < 3/2$ (i. e. $p = r+1$) or $p-r > 3/2$. Thus by § 3, we can conclude that e. f. d. in $F(-\pi, \pi)$ the equation $\Omega(y) = 0$ possesses a one parameter family of solutions $y_c^* = \exp \int W_c$ where $W_c \sim M_1$.

Now let N be any $(n-2)$ *nd* root of unity, say $N = e^{i\alpha}$ where $-\pi < \alpha \leq \pi$. For N, we now compute the function T of § 3. Since $N^{n-2} = 1$, an easy calculation shows that $T \sim (1-(n/2))N$, so since N is a constant, clearly $IF(T) \not\equiv 0$. However, since $T \approx N$ we must again verify (vii) of § 3. But here $\sigma = 1-(n/2)$, so $\sigma < 0$ since $n > 2$. Thus clearly $\sigma \notin \{p-r, 2(p-r)\}$, and since $(\sigma+(r-p))N$ is a costant monomial, clearly $IF((\sigma+(r-p))N) \not\equiv 0$. Now $IF(N)(\varphi) = \operatorname{Cos}(\varphi+\alpha)$. Thus by determining the largest open subintervals of $(-\pi, \pi)$ on which $IF(N)$ is positive, we obtain the following conclusions by § 3; If $-\pi/2 \leq \alpha \leq \pi/2$, then e. f. d. in $F(-\alpha-\pi/2, -\alpha+\pi/2)$, the equation $\Omega(y) = 0$ possesses a one-parameter family of solutions $\exp \int V_c$ where $V_c \sim N$. If $\pi/2 \leq \alpha \leq \pi$ (respectively, $-\pi < \alpha \leq -\pi/2$), then e. f. d. in each of $F(-\pi, -\alpha+\pi/2)$ and $F(-\alpha+3\pi/2, \pi)$ separately (respectively, $F(-\pi, -\alpha-3\pi/2)$ and $F(-\alpha-\pi/2, \pi)$ separately), the equation $\Omega(y) = 0$ possesses a one parameter family of solutions $\exp \int V_c$, where $V_c \sim N$.

All solutions found in this example are $>$ all logarithmic monomials by § 3 (A).

BIBLIOGRAPHY

[1] S. BANK, *On the instability theory of differential polynomials*, Ann. Mat. Pura Appl., 74 (1966), 83-112.

[2] S. BANK, *On the asymptotic behavior of solutions near an irregular singularity*, Proc. Amer. Math. Soc., 18 (1967), 15-21.

[3] E. W. CHAMBERLAIN, *Families of principal solutions of ordinary differential equations*, Trans. Amer. Math. Soc., 107 (1963), 261-272.

[4] W. STRODT, *Contributions to the asymptotic theory of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc., No. 13, (1954), 81 pp.

[5] W. STRODT, *Principal solutions of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc., No. 26, (1957), 107 pp.

[6] W. STRODT, *On the algebraic closure of certain partially ordered fields*, Trans. Amer. Math. Soc., 105 (1962), 229-250.

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