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Full rings of continuous real functions

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FULL RINGS OF CONTINUOUS REAL FUNCTIONS

SALVATORE CIAMPA [1])

1. Introduction and results.

1.1. The study of rings of continuous real functions on a space is a main goal of functional topology; one major achievement toward the study of relations between the space topology and such rings has been the Gelfand characterization of the Banach algebras which are isomorphic to the ring of all real continuous functions on a suitable space (which turns out to be compact Hausdorff) [2]). As is wellknown, his methods are concerned with the study of the maximal ideals in the algebra under consideration and the main results (those which connect the space topology with the algebraic structure of the ring of all continuous real functions) rest upon the fact that every maximal ideal in the ring of all continuous real functions on a compact Hausdorff space is fixed (that is, it consists of all functions which vanish at some point) [3]).

1.2. Our concern here is somewhat different; we shall consider a situation in which the set of the points of the space is kept fixed [4]) and we shall try to determine which rings of bounded real

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[1]) Lavoro eseguito nell'ambito del gruppo n. 9 del Comitato Nazionale per la Matematica del C. N. R. (anno 1967-68).

Indirizzo dell'Autore: Scuola Normale Superiore - Pisa.

[2]) See [3] and [6].

[3]) See [4] and [5], chapt. 4,

[4]) That is, we study the classes of all bounded continuous or lower semicontinuous real functions on spaces with a fixed cardinality.

functions on that set are likely to be the rings of all continuous bounded real functions in some suitable topology.

Of course, there is no hope of characterizing the topology by such rings since many different topologies may have the same continuous bounded real functions; it is equally true though that any such ring determines a unique topology among those which are completely regular (see the proof of 3.6.4.).

To be more precise, we prove that :

(i) For every full semiring $\mathcal{F}$ of bounded real functions on a set T (see no. 5.1. (iv)), there exists a unique topology on T such that $\mathcal{F}$ is just the class of all lower semicontinuous bounded real functions in that topology. Moreover, this unique topology is the weakest among those which make every function belonging to $\mathcal{F}$ lower semicontinuous (see no. 5.3. and footnote [13])).

(ii) For every full ring $\mathcal{F}$ of bounded real functions on a set T (see no. 5.1. (v)), there exists a unique completely regular topology on T such that $\mathcal{F}$ is just the class of all continuous bounded real functions in that topology. Moreover, this unique topology is the weakest among those which make every function belonging to $\mathcal{F}$ continuous. (see no. 5.4. and footnote [14])).

We notice that (ii) also solves the problem of giving conditions on a family $\mathcal{F}$ of bounded real functions on a set T to ensure that if we take the class of all bounded continuous real functions in the weakest topology on T which makes every function in $\mathcal{F}$ continuous, we find again $\mathcal{F}$ (analogously, (i) solves the problem in the case in which we consider lower semicontinuity instead of continuity).

2. Notations and definitions.

2.1. With R and R^+ we shall denote respectively the reals and the non-negative reals with the usual algebraic, order and topological properties.

$\{0,1\}$ and N denote the subsets of R consisting respectively of the numbers zero and one and of the positive integers.

Throughout this paper T denotes a fixed non empty set.

For every $a \in R$, $\boldsymbol{a}$ denotes the function from T to R whose constant value is the number a.

For every subset Y of R, $\mathcal{B}(T, Y)$ denotes the set of all bounded functions from T to Y. Algebraic and lattice operations among real functions are meant as pointwise operations; $\sup(f, g)$ and $\inf(f, g)$ will be denoted also by $f \vee g$ and $f \wedge g$, respectively.

When we say that a space is completely regular, we do not include the Hausdorff property.

2.2. $\mathcal{S}$ denotes the class of all sets $\mathcal{H} \subset \mathcal{B}(T, R^+)$ such that

(i) $\boldsymbol{a} \in \mathcal{H}$, for every $a \in R^+$;

(ii) $f + g, fg, f \wedge g$ belong to $\mathcal{H}$ whenever f and g are in $\mathcal{H}$;

(iii) if $\mathcal{L}$ is a non empty subset of $\mathcal{H}$ and $\sup \mathcal{L}$ is a bounded real function on T, then $\sup \mathcal{L} \in \mathcal{H}$;

(iv) for every $a \in R^+$ and $f \in \mathcal{H}$, if $\boldsymbol{a} \leq f$ then $f - \boldsymbol{a} \in \mathcal{H}$.

2.3. $\mathcal{C}$ denotes the class of all sets $\mathcal{K} \subset \mathcal{B}(T, R^+)$ such that

(i) $\boldsymbol{a} \in \mathcal{K}$, for every $a \in R^+$;

(ii) $f + g, fg, f \vee g, f \wedge g$ belong to $\mathcal{K}$ whenever f and g are in $\mathcal{K}$;

(iii) for every $f \in \mathcal{K}$ there exists a positive real number k such that $\boldsymbol{1} - \boldsymbol{k}f \in \mathcal{K}$;

(iv) for every $a \in R^+$ and $f \in \mathcal{K}$, if $\boldsymbol{a} \leq f$, then $f - \boldsymbol{a} \in \mathcal{K}$;

(v) if $\mathcal{L}$ and $\mathcal{M}$ are two non empty subsets of $\mathcal{K}$ such that $\sup \mathcal{L} = \inf \mathcal{M}$, then $\sup \mathcal{L}$ belongs to $\mathcal{K}$.

2.4. $\mathcal{V}$ denotes the class of all non empty sets $\mathcal{G} \subset \mathcal{B}(T, R^+)$ such that, if $\mathcal{L}$ is a non empty subset of $\mathcal{G}$ and $\sup \mathcal{L}$ is a bounded real function on T, then $\sup \mathcal{L}$ belongs to $\mathcal{G}$.

2.5. It is easy to see that $\mathcal{S}, \mathcal{C}, \mathcal{V}$ are set families closed under intersection; each of them, therefore, gives raise to a closure operator in $\mathcal{B}(T, R^+)$. Precisely, we define, for every family of functions $\mathcal{F} \subset \mathcal{B}(T, R^+)$,

$$\mathcal{F}^s = \bigcap \{\mathcal{H} : \mathcal{F} \subset \mathcal{H} \in \mathcal{S}\};$$
$$\mathcal{F}^c = \bigcap \{\mathcal{K} : \mathcal{F} \subset \mathcal{K} \in \mathcal{C}\};$$
$$\mathcal{F}^v = \bigcap \{\mathcal{G} : \mathcal{F} \subset \mathcal{G} \in \mathcal{V}\}.$$

By symbols such as $\mathcal{F}^{cv}$ we denote the $\mathcal{V}$-closure of the $\mathcal{C}$-closure of the family $\mathcal{F}$, and analogously in all other cases.

2.6 For every family $\mathcal{F} \subset \mathcal{B}(T, R^+)$ we define

(i) $\mathcal{F}^0 = \{f \in \mathcal{F} : f(T) \subset \{0, 1\}\}$;

(ii) $\mathcal{F}^* = \{f \in \mathcal{F}$: there exists a positive real number k such that $\mathbf{1} - kf \in \mathcal{F}\}$;

(iii) $\lambda_s(\mathcal{F})$ as the weakest topology on T which makes all functions $f \in \mathcal{F}$ lower semicontinuous;

(iv) $\lambda_c(\mathcal{F})$ as the weakest topology on T which makes every function $f \in \mathcal{F}$ continuous;

(v) $\pi(\mathcal{F}) = \{f^{-1}(1) : f \in (\mathcal{F}^s)^0\}$.

2.7. Let Λ denote the set of all topologies on the set T; then, for every $\lambda \in \Lambda$ we define $\mathcal{S}(\lambda)$ as the class of all real bounded functions on T which, in the topology λ, are lower semicontinuous (if we restrict ourselves to bounded non negative real functions, we write $\mathcal{S}^+(\lambda)$); analogously, $\mathcal{C}(\lambda)$ denotes the class of all functions $f \in \mathcal{B}(T, R)$ which are continuous in the topology λ (as before, $\mathcal{C}^+(\lambda)$ is the class of all functions $f \in \mathcal{B}(T, R^+)$ which are continuous in the topology λ).

3. Preliminary lemmas.

3.1. *Let the family* $\mathcal{F} \subset \mathcal{B}(T, R^+)$ *satisfy conditions* (i), (ii), (iv) *of* no. 2.3. . *Then, for each fixed function* $f \in \mathcal{F}$, *the following propositions are equivalent*

(i) $f \in \mathcal{F}^*$;

(ii) $\sup f - f \in \mathcal{F}$;

(iii) *for every number* $h \in R^+$,

$$h \sup f \leq 1 \Longrightarrow \mathbf{1} - hf \in \mathcal{F}.$$

PROOF. (i) $\Longrightarrow$ (ii) There exists, by definition of $\mathcal{F}^*$, a positive real number k such that $\mathbf{1} - kf \in \mathcal{F}$, which implies that

$$\mathbf{1} \geq kf \quad \text{and} \quad 1 \geq k \sup f;$$

but then, being

$$\mathbf{1} - kf \geq \mathbf{1} - k \sup f,$$

it follows that

$$\boldsymbol{sup\, f} - f = \frac{\boldsymbol{1}}{\boldsymbol{k}} [(\boldsymbol{1} - \boldsymbol{k}f) - (\boldsymbol{1} - \boldsymbol{k}\ \boldsymbol{sup\, f})] \in \mathcal{F}.$$

(ii) $\Longrightarrow$ (iii) It follows from the equality

$$\boldsymbol{1} - \boldsymbol{h}f = (\boldsymbol{1} - \boldsymbol{h}\ \boldsymbol{sup\, f}) + \boldsymbol{h}\,(\boldsymbol{sup\, f} - f).$$

(iii) $\Longrightarrow$ (i) Obvious, because of the definition of $\mathcal{F}^*$.

3.2. *For every family* $\mathcal{F} \in \mathcal{S}$ *we have*

(i) $(\mathcal{F}^0)^s = \mathcal{F}$;

(ii) $\mathcal{F}^* \in \mathcal{C}$;

(iii) $\mathcal{F}^*$ *is the largest part of* $\mathcal{F}$ *which belongs to* $\mathcal{C}$.

PROOF. (i) Since $\mathcal{F}^0 \subset \mathcal{F}$, it is true that $(\mathcal{F}^0)^s \subset \mathcal{F}$; to show the converse inclusion, we note that if for every function $f \in \mathcal{F}$ and every number $a \in R^+$ we set

$$f_a = \sup\{\boldsymbol{n}((f \vee \boldsymbol{a}) - \boldsymbol{a}) \wedge \boldsymbol{1} : n \in N\},$$

the so defined functions f_a are in the family $\mathcal{F}$ and, for every point $x \in T$,

$$f(x) > a \Longrightarrow f_a(x) = 1$$

$$f(x) \leq a \Longrightarrow f_a(x) = 0.$$

This shows also that $f_a \in \mathcal{F}^0$. To complete the proof it is sufficient to notice that, for every number $a \in R^+$, the function $\boldsymbol{a}f_a$ is in $(\mathcal{F}^0)^s$ and that

$$f = \sup\{\boldsymbol{a}f_a : a \in R^+\},$$

thereby showing that $f \in (\mathcal{F}^0)^s$.

(ii) We verify the five properties of no. 2.3.. The constant functions are in $\mathcal{F}^*$, obviously. If f and g are in $\mathcal{F}^*$ (and let h, k be positive real numbers such that $\boldsymbol{1} - \boldsymbol{h}f \in \mathcal{F}$ and $\boldsymbol{1} - \boldsymbol{k}g \in \mathcal{F}$), then

$f + g \in \mathcal{F}^*$ since $f + g \in \mathcal{F}$ and, if $2p = \min(h, k)$,

$$\mathbf{1} - p(f + g) = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{1}[(-\mathbf{2}pf) + (\mathbf{1} - \mathbf{2}pg)] \in \mathcal{F};$$

$fg \in \mathcal{F}^*$ since $fg \in \mathcal{F}$ and

$$\mathbf{1} - hkfg = kg(\mathbf{1} - hf) + (\mathbf{1} - kg) \in \mathcal{F};$$

$f \vee g \in \mathcal{F}^*$ since $f \vee g \in \mathcal{F}$ and, if $p = \min(h, k)$, then

$$\mathbf{1} - p(f \vee g) = (\mathbf{1} - pf) \wedge (\mathbf{1} - pg) \in \mathcal{F};$$

$f \wedge g \in \mathcal{F}^*$ since $f \wedge g \in \mathcal{F}$ and, if $p = \min(h, k)$, then

$$\mathbf{1} - p(f \wedge g) = (\mathbf{1} - pf) \vee (\mathbf{1} - pg) \in \mathcal{F}.$$

Suppose now that $f \in \mathcal{F}^*$, then (by no. 3.1.) $\boldsymbol{sup}\, f - f \in \mathcal{F}$, and, if $p \sup f = 1$, we have

$$\mathbf{1} - p(\boldsymbol{sup}\, f - f) = pf \in \mathcal{F},$$

from which we conclude that $\boldsymbol{sup}\, f - f \in \mathcal{F}^*$ (recall no. 3.1.).

Suppose now that $a \in R^+$, $k \in R^+$, $f \in \mathcal{F}$ are such that $k \neq 0$, $\mathbf{1} - kf \in \mathcal{F}$, $f \geq \boldsymbol{a}$, then $f - \boldsymbol{a} \in \mathcal{F}^*$ since $f - \boldsymbol{a} \in \mathcal{F}$ and

$$\mathbf{1} - k(f - \boldsymbol{a}) = (\mathbf{1} - kf) + k\boldsymbol{a} \in \mathcal{F}.$$

Finally, let $\mathcal{L}$ and $\mathcal{M}$ be non empty subsets of $\mathcal{F}^*$ such that there exists a function $g \in \mathcal{F}$ which equals $\sup \mathcal{L}$ and $\inf \mathcal{M}$. Then, if $g = \boldsymbol{0}$, it is obvious that $g \in \mathcal{F}^*$; if, on the contrary, $g \neq \boldsymbol{0}$, take any function $q \in \mathcal{M}$ and set

$$Q = \{\inf(q, m) : m \in \mathcal{M}\}.$$

We have then that $\inf \mathcal{M} = \inf Q = g$ and, for every $f \in Q$,

$$\sup f \leq \sup q.$$

Now, if a number $k \in R^+$ is chosen in such a way that

$$0 < k \sup q \leq 1,$$

we get

$$\mathbf{1} - kg = \sup \{\mathbf{1} - kf : f \in Q\} \in \mathcal{F},$$

since (by no. 3.1.) the function $\mathbf{1} - kf$ is in $\mathcal{F}$ for every $f \in Q$ (let us notice that $Q \subset \mathcal{F}^*$ since it has been already shown in this proof that from $q, m \in \mathcal{F}^*$ it follows $q \wedge m \in \mathcal{F}^*$).

(iii) It follows from 3.3. (i) and the preceding proposition (ii) noticing that from $\mathcal{A} \subset \mathcal{B}$ it follows $\mathcal{A}^* \subset \mathcal{B}^*$, for any two families $\mathcal{A}, \mathcal{B}$ of functions from $\mathcal{B}(T, R^+)$.

3.3. *For every family of functions* $\mathcal{F} \in \mathcal{C}$ *we have*

(i) $\mathcal{F} = \mathcal{F}^*$;

(ii) $\mathcal{F}^v \in \mathcal{S}$;

(iii) $\mathcal{F}^s = \mathcal{F}^v$;

(iv) $f, g \in \mathcal{F};\ f \geq g \Longrightarrow f - g \in \mathcal{F}$;

(v) $\mathcal{F} = (\mathcal{F}^s)^*$.

PROOF. (i) It follows directly from the definition of $\mathcal{F}^*$.

(ii) To prove the validity of all properties of definition 2.2. it is sufficient to consider that for every function $f \in \mathcal{F}^v$, there exists a set $\mathcal{L} \subset \mathcal{F}$ such that $f = \sup \mathcal{L}$.

(iii) From the preceding proposition (ii) and from the definition of $\mathcal{F}^s$, since $\mathcal{F} \subset \mathcal{F}^v$, we have that $\mathcal{F}^s \subset \mathcal{F}^v$; the converse inclusion follows from $\mathcal{S} \subset \mathcal{V}$.

(iv) Since the functions f and g are in $\mathcal{F}$, also

$$f + (\boldsymbol{sup\ g} - g) \in \mathcal{F}$$

and, being $f \geq g$, we have

$$f + \boldsymbol{sup\ g} - g \geq \boldsymbol{sup\ g},$$

which allows us to conclude that

$$f - g = (f + \boldsymbol{sup\ g} - g) - \boldsymbol{sup\ g} \in \mathcal{F}.$$

(v) From propos. 3.2. (iii) we have the inclusion $\mathcal{F} \subset (\mathcal{F}^s)^*$; to show the converse inclusion, let be $g \in (\mathcal{F}^s)^*$; then (because of the preceding equality (iii)) there exist two sets $\mathcal{L}$, $\mathcal{M}$ in the family $\mathcal{F}$ and a positive real number k in such a way that

$$g = \sup \mathcal{L}, \qquad \mathbf{1} - kg = \sup \mathcal{M};$$

we have then

$$\sup \mathcal{L} = g = \frac{\mathbf{1}}{k}(\mathbf{1} - \sup \mathcal{M}) = \inf\left\{\frac{\mathbf{1}}{k}(\mathbf{1} - m) : m \in \mathcal{M}\right\}$$

and, for the property 2.3. (v), we conclude that $g \in \mathcal{F}$, since the preceding proposition (iv) says that, for every function $m \in \mathcal{M}$, being $\mathbf{1} \geq m$, $\mathbf{1} - m \in \mathcal{F}$.

3.4. *For every topology λ on the set T we have*
(i) $\mathcal{S}^+(\lambda) \in \mathcal{S}$;
(ii) $\mathcal{C}^+(\lambda) \in \mathcal{C}$;
(iii) $\mathcal{C}^+(\lambda) = (\mathcal{S}^+(\lambda))^*$.

PROOF. (i) and (ii) Easy check of the properties in the definitions.

(iii) Since $\mathcal{C}^+(\lambda) \subset \mathcal{S}^+(\lambda)$ and $\mathcal{C}^+(\lambda) \in \mathcal{C}$, from 3.2. (iii) we have that $\mathcal{C}^+(\lambda) \subset (\mathcal{S}^+(\lambda))^*$; let now be $f \in (\mathcal{S}^+(\lambda))^*$, then (because of 3.1.) $\boldsymbol{\sup f} - f \in \mathcal{S}^+(\lambda)$ so that the functions f and $-f$ are both lower semicontinuous in the topology λ and this implies that $f \in \mathcal{C}^+(\lambda)$.

3.5. *For every family of functions $\mathcal{F} \subset \mathcal{B}(T, R^+)$ we have*
(i) $\pi(\mathcal{F})$ *is a topology on T (that is, $\pi(\mathcal{F}) \in \Lambda$)*;
(ii) $\mathcal{F}^s = \mathcal{S}^+(\pi(\mathcal{F}))$.

PROOF. (i) For any non empty set of indices I and if every function f_i is in $(\mathcal{F}^s)^0$, the equalities

$$\bigcup_{i \in I} f_i^{-1}(1) = (\sup_{i \in I} f_i)^{-1}(1)$$

$$\bigcap_{i \in I} f_i^{-1}(1) = (\inf_{i \in I} f_i)^{-1}(1)$$

hold true and these, together with the properties of the family $\mathcal{F}^s$, allow the conclusion that $\pi(\mathcal{F})$ is a topology on the set T.

(ii) We show first that $(\mathcal{F}^s)^0 = (\mathcal{S}^+ (\pi (\mathcal{F})))^0$.

Let f be a function belonging to $(\mathcal{S}^+ (\pi (\mathcal{F})))^0$, then there exists a set $Y \in \pi (\mathcal{F})$ such that $Y = f^{-1}(1)$, that is, there exists a function $g \in (\mathcal{F}^s)^0$ such that $g^{-1}(1) = Y = f^{-1}(1)$ (because of the definition 2.6. (v)); this implies equality between f and g, hence $f \in (\mathcal{F}^s)^0$.

Let now be $f \in (\mathcal{F}^s)^0$, for every number $a \in R^+$ the set $f^{-1}(a, +\infty)$ is either empty or coincides with the set $f^{-1}(1)$: in any case it belongs to the set family $\pi (\mathcal{F})$ and so does, of course, the set $f^{-1}(R^+)$ too: then we may conclude that the function f is lower semicontinuous in the topology $\pi (\mathcal{F})$. So $f \in (\mathcal{S}^+ (\pi (\mathcal{F})))^0$. The equality to be proved, $\mathcal{F}^s = \mathcal{S}^+ (\pi (\mathcal{F}))$, follows now from what has been proved in 3.4. (i) and 3.2. (i).

3.6.1. We recall now some essentially known facts on topologies determined by families of mappings.

We give first a definition: let T and Y be two topological spaces, let $\mathcal{F}$ be a family of continuous mappings from T to Y with the property that for every point $x \in T$ and for every neighborhood H of x there exists a mapping $f \in F$ such that $f(x)$ does not belong to the closure of the set $f(T - H)$. We shall say then that $\mathcal{F}$ is a *separating family of continuous mappings* [5]).

3.6.2. *Let T and Y be two topological spaces; if $\mathcal{F}$ is a separating family of continuous mappings from T to Y, then the topology λ of the space T is the weakest among those which make every mapping $f \in \mathcal{F}$ continuous.*

PROOF. Let us denote by μ the weakest topology in which every mapping $f \in \mathcal{F}$ is continuous; then, obviously, $\mu \subset \lambda$. But, if $A \in \lambda$ and $x \in A$, there exists in $\mathcal{F}$ a mapping f such that, if K_x denotes the closure of the set $f(T - A) \subset Y$, $f(x) \notin K_x$. We have then

$$x \notin f^{-1}(K_x) \qquad \text{and} \qquad x \in T - f^{-1}(K_x) \subset A,$$

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[5]) Separating families of continuous functions into the reals have been considered in [5] no. 3H page 49 and are called *completely regular families.*

and we may write therefore

$$A = \bigcup_{x \in A} (T - f^{-1}(K_x)),$$

thereby showing that $A \in \mu$, since every set K_x is closed in Y.

3.6.3. *Let λ and μ be two topologies on the set T, then*

$$\mathcal{S}^+(\lambda) = \mathcal{S}^+(\mu) \Longleftrightarrow \lambda = \mu.$$

PROOF. If $\mathcal{S}^+(\lambda) = \mathcal{S}^+(\mu)$, the topologies λ and μ have the same lower semicontinuous functions into the closed interval $[0, 1]$. Then propos. 5.2.1. (b) of [2] says that $\lambda = \mu$.

The converse implication is obvious.

3.6.4. *Let λ and μ be two completely regular topologies on the set T, then*

$$\mathcal{C}^+(\lambda) = \mathcal{C}^+(\mu) \Longleftrightarrow \lambda = \mu.$$

PROOF. $\mathcal{C}^+(\lambda)$ and $\mathcal{C}^+(\mu)$ are separating families of real continnous functions for the topologies λ and μ respectively, since the topologies under consideration are completely regular. Then, because of 3.6.2., $\lambda = \mu$ since they both are the weakest topology on T which makes continuous every function $f \in \mathcal{C}^+(\lambda) = \mathcal{C}^+(\mu)$.

The converse inclusion is obvious.

4. Main results.

4.1. *For every family $\mathcal{F} \subset \mathcal{B}(T, R^+)$ we have*

(i) $\mathcal{F}^s = \mathcal{S}^+(\lambda_s(\mathcal{F})) \supset (\mathcal{C}^+(\lambda_s(\mathcal{F})))^s = (\mathcal{C}^+(\lambda_s(\mathcal{F})))^v$;

(ii) $\mathcal{C}^+(\lambda_s(\mathcal{F})) = (\mathcal{F}^s)^*$.

PROOF. (i) The last equality follows from propos. 3.4. (ii) and 3.3. (iii); the middle inclusion is obvious since upper bounds of families of continuous functions are lower semicontinuous. To prove the first equality we observe that, because of propos. 3.2. (i) and 3.4. (i), we only need to prove that $(\mathcal{F}^s)^0 = (\mathcal{S}^+(\lambda_s(\mathcal{F})))^0$.

Since $\mathcal{F} \subset \mathcal{S}^+(\lambda_s(\mathcal{F}))$, from 3.4. (i) we have

$$\mathcal{F}^s \subset \mathcal{S}^+(\lambda_s(\mathcal{F})) \qquad \text{and} \qquad (\mathcal{F}^s)^0 \subset (\mathcal{S}^+(\lambda_s(\mathcal{F})))^0.$$

To show the converse inclusion, since all constant functions belonging to $(\mathcal{S}^+(\lambda_s(\mathcal{F})))^0$ belong also to $(\mathcal{F}^s)^0$, let f be a non constant function in the family $(\mathcal{S}^+(\lambda_s(\mathcal{F})))^0$; there exist then a non empty set J and, for every $j \in J$, a finite non empty set L_j such that

$$f^{-1}(1) = f^{-1}(0, +\infty) = \bigcup_{j \in J} \bigcap_{i \in L_j} f_i^{-1}(a_i, +\infty)$$

with all functions f_i in $\mathcal{F}$ and all numbers a_i in R^+.

If we define now the function

$$u = \sup_{j \in J} \{(\inf_{i \in L_j} \{(f_i \vee a_i) - a_i\}) \wedge 1\}$$

we find that u is a function from the set T to the closed interval $[0, 1]$, moreover, the function u belongs to the family $\mathcal{F}^s$ and it is such that

$$f^{-1}(0) = u^{-1}(0), \qquad f^{-1}(1) = u^{-1}(0, 1].$$

To conclude the proof, it is sufficient now to notice that

$$f = \sup\{h\, u \wedge 1 : h \in N\},$$

this equality implying that $f \in (\mathcal{F}^s)^0$.

(ii) The proof follows from the preceding (i) and from propos. 3.4. (iii).

4.2. *For every family* $\mathcal{F} \subset \mathcal{B}(T, R^+)$ *we have*

(i) $\mathcal{S}^+(\lambda_c(\mathcal{F})) = \mathcal{F}^{cs}$;

(ii) $\mathcal{C}^+(\lambda_c(\mathcal{F})) = \mathcal{F}^c$.

PROOF. (i) Since $\mathcal{F} \subset \mathcal{C}^+(\lambda_c(\mathcal{F}))$, because of 3.4. (ii) and every continuous function being also lower semicontinuous, we have

$$\mathcal{F}^c \subset \mathcal{C}^+(\lambda_c(\mathcal{F})) \subset \mathcal{S}^+(\lambda_c(\mathcal{F})) \qquad \text{and} \qquad \mathcal{F}^{cs} \subset \mathcal{S}^+(\lambda_c(\mathcal{F})).$$

We prove now the converse inclusion. Let $\mathcal{X}$ be the class of all sets $(a, +\infty)$, $[0, b)$ for all positive real numbers a and b. If f is any function in $\mathcal{B}(T, R^+)$ and $H \in \mathcal{X}$, let us define

$$w(f, H) = \begin{cases} (f \vee a) - a & \text{if} \quad H = (a, +\infty) \\ b - (f \wedge b) & \text{if} \quad H = [0, b). \end{cases}$$

We notice that $\mathcal{X}$ is a subbase for the topology of R^+ and, for every $x \in T$,

$$f(x) \in H \Longrightarrow w(f, H)(x) > 0,$$

$$f(x) \notin H \Longrightarrow w(f, H)(x) = 0.$$

Now, let f be a function in $\mathcal{S}^+(\lambda_c(\mathcal{F}))$, then, for every number $k \in R^+$, there exist a non empty set J and, for every $j \in J$, a finite non empty set L_j in such a way that, for some functions $f_i \in \mathcal{F}$ and some sets $H_i \in \mathcal{X}$, we have

$$f^{-1}(k, +\infty) = \bigcup_{j \in J} \bigcap_{i \in L_j} f_i^{-1}(H_i) \text{ [6]}.$$

Let us define now

$$f_k = \sup_{j \in J} \{\inf_{i \in L_j} w(f_i, H_i)\}$$

and observe that since every function $w(f_i, H_i)$ is in $\mathcal{F}^c$, the functions f_k are in $\mathcal{F}^{cs}$, moreover, for every $x \in T$ and every number $k \in R^+$,

$$f(x) > k \Longrightarrow f_k(x) > 0,$$

$$f(x) \leq k \Longrightarrow f_k(x) = 0.$$

If we notice now that

$$f = \sup\{k \beta_k : k \in R^+\}$$

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[6] If $k \geq \sup f$, instead of $J = \emptyset$, as would be natural, we take $J = \{1\}$, $L_1 = \{1, 2\}$, $f_1 = f_2 =$ any function in $\mathcal{F}$, $H_1 = [0, 1)$, $H_2 = (2, +\infty)$. Alternatively, we could have considered only the numbers k in the set $[0, \sup f)$.

where

$$\beta_k = \sup \{(n f_k) \wedge 1 : n \in N\},$$

we may conclude that $f \in \mathcal{F}^{cs}$, being also $\beta_k \in \mathcal{F}^{cs}$, for every $k \in R^+$.

(ii) From the preceding (i) and propos. 3.4. (iii) we have

$$\mathcal{C}^+(\lambda_c(\mathcal{F})) = (\mathcal{F}^{cs})^* ;$$

the equality to be proved follows from $\mathcal{F}^c = (\mathcal{F}^{cs})^*$ (see no. 3.3. (v)).

4.3.1. *For every family* $\mathcal{F} \subset \mathcal{B}(T, R^+)$ *the propositions*

(i) $\mathcal{F}^c = (\mathcal{F}^s)^*$;

(ii) $\mathcal{F} \subset (\mathcal{F}^s)^*$;

(iii) $\mathcal{F}^s = \mathcal{F}^{cs}$;

(iv) $\lambda_s(\mathcal{F}) = \lambda_c(\mathcal{F})$ [7]) ;

(v) *the topology* $\lambda_s(\mathcal{F})$ *is completely regular* ;

(vi) $\mathcal{F}^s = ((\mathcal{F}^s)^*)^s$ [8])

are such that

(i) <==> (ii) <==> (iii) <==> (iv) ==> (v) <==> (vi).

Proof. (i) ==> (ii) Obvious, since $\mathcal{F} \subset \mathcal{F}^c$.

(ii) ==> (iii) That $\mathcal{F}^s \subset \mathcal{F}^{cs}$ is obvious, since $\mathcal{F} \subset \mathcal{F}^c$; for the converse inclusion, being $\mathcal{F} \subset (\mathcal{F}^s)^*$, we have $\mathcal{F}^c \subset (\mathcal{F}^s)^*$ (see no. 3.2. (ii)) hence $\mathcal{F}^{cs} \subset ((\mathcal{F}^s)^*)^s \subset \mathcal{F}^s$ because $(\mathcal{F}^s)^* \subset \mathcal{F}^s$.

(iii) ==> (iv) It follows from propos. 4.1. (i), 4.2. (i), 3.6.3..

(iv) ==> (v) Obvious, since the topology $\lambda_c(\mathcal{F})$ is completely regular (see [5] no. 3.7.).

(v) ==> (vi) We have only to prove that $\mathcal{F}^s \subset ((\mathcal{F}^s)^*)^s$, since the converse inclusion is obvious being $(\mathcal{F}^s)^* \subset \mathcal{F}^s$. Let be $f \in \mathcal{F}^s$, then (because of 4.1. (i)) f is lower semicontinuous in the topology $\lambda_s(\mathcal{F})$, hence (see [1], § 1 prop. 5) there exists a family of functions $\mathcal{L} \subset \mathcal{C}(\lambda_s(\mathcal{F}))$ such that $f = \sup \mathcal{L}$; it easy to see now that the

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[7]) In general, we have only $\lambda_s(\mathcal{F}) \subset \lambda_c(\mathcal{F})$.

[8]) This condition is equivalent to the requirement that the inclusion in 4.1. (i) is actually an equality.

function f is in $((\mathcal{F}^s)^*)^s$ since

$$f = \sup\{g \vee \boldsymbol{O} : g \in \mathcal{L}\}$$

and every function $g \vee \boldsymbol{O}$ is in $\mathcal{C}^+(\lambda_s(\mathcal{F})) = (\mathcal{F}^s)^*$ because of 4.1. (ii).

(vi) $\Longrightarrow$ (v) The hypothesis implies that $\mathcal{S}^+(\lambda_s(\mathcal{F})) = (\mathcal{C}^+(\lambda_s(\mathcal{F})))^v$ (recall propos. 3.3. (iii), 4.1. (i) and (ii)): because of prop. 5, § 1 of [1] the topology $\lambda_s(\mathcal{F})$ is completely regular [9]).

(iv) $\Longrightarrow$ (i) Obvious, because of propos. 4.1. (ii) and 4.2. (ii).

4.3.2. The implication (iv) $\Longrightarrow$ (v) in the preceding proposition cannot be reversed, as the following counterexample shows.

Let T be the real line and let μ be the euclidean topology on T. Take $\mathcal{G} = \mathcal{C}^+(\mu)$ and $\mathcal{F} = \mathcal{S}^+(\mu)$, then $\mathcal{F} = \mathcal{G}^s$ and, because of 3.3. (v), $\mathcal{G} = (\mathcal{G}^s)^* = \mathcal{F}^* = (\mathcal{F}^s)^*$, hence $\mathcal{F}^s = \mathcal{F} = ((\mathcal{F}^s)^*)^s$ and proposition 4.3.1. (vi) holds true. But, being $\mathcal{G}$ properly contained in $\mathcal{F}$, proposition 4.3.1. (ii) is false, that is $\mathcal{F} \not\subset (\mathcal{F}^s)^*$.

4.3.3. *If the family of functions $\mathcal{F}$ belongs to the class $\mathcal{C}$, all six propositions of* 4.3.1. *hold true.*

PROOF. Since, obviously, $\mathcal{F} \subset \mathcal{F}^s$, being $\mathcal{F} \in \mathcal{C}$, propos. 3.2. (iii) says that $\mathcal{F} \subset (\mathcal{F}^s)^*$.

4.4.1. We recall that in 2.6. (v) we have defined, for every family $\mathcal{F} \subset \mathcal{B}(T, R^+)$, $\pi(\mathcal{F})$ as a class of subsets of T and in 3.5. (i) we have proved that $\pi(\mathcal{F})$ actually is a topology on the set T. We may therefore speak of the mapping π from $\mathcal{P}(\mathcal{B}(T, R^+))$, family of all subsets of $\mathcal{B}(T, R^+)$, into the class Λ of all topologies on T. We prove now some properties of this mapping.

4.4.2. *For every family $\mathcal{F} \subset \mathcal{B}(T, R^+)$ we have $\pi(\mathcal{F}) = \lambda_s(\mathcal{F})$.*

PROOF. Propositions 3.5. (ii) and 4.1. (i) affirm that the two topologies $\pi(\mathcal{F})$ and $\lambda_s(\mathcal{F})$ admit the same non negative bounded

[9]) Actually, the Bourbaki's proposition deals with functions from a space to the extended reals; however, the proof is valid also in our case, as is easily seen.

lower semi-continuous real functions. They coincide, then, because of prop. 3.6.3..

4.4.3. (i) *If* λ *is a topology on* T, *then*

$$\lambda = \pi(\mathcal{S}^{+}(\lambda));$$

(ii) *the restriction of the mapping* π *to the class* $\mathcal{S}$ *is a bijection onto the class* Λ *of all topologies on* T.

PROOF. (i) From 4.4.2. we draw the equality

$$\pi(\mathcal{S}^{+}(\lambda)) = \lambda_s(\mathcal{S}^{+}(\lambda))$$

and from 4.1. (i), 3.4. (i) we deduce that the two topologies λ and $\pi(\mathcal{S}^{+}(\lambda))$ have the same non negative bounded lower semicontinuous real functions. Proposition 3.6.3. concludes the proof.

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are families in the class $\mathcal{S}$ and $\pi(\mathcal{F}) = \pi(\mathcal{G})$, then, being also $\lambda_s(\mathcal{F}) = \lambda_s(\mathcal{G})$ because of 4.4.2., from proposition 4.1. (i) we have $\mathcal{F}^s = \mathcal{G}^s$, that is $\mathcal{F} = \mathcal{G}$: hence the mapping π restricted to the class $\mathcal{S}$ is injective. Its surjectivity follows directly from the preceding proposition (i).

4.4.4. (i) *If* λ *is a completely regular topology on* T, *then*

$$\lambda = \pi(\mathcal{C}^{+}(\lambda));$$

(ii) *The restriction of the mapping* π *to the class* $\mathcal{C}$ *is a bijection onto the class* $\Sigma \subset \Lambda$ *of all completely regular topologies on the set* T.

PROOF. (i) Let us notice first that, λ being completely regular, $\mathcal{C}^{+}(\lambda)$ is a separating family of continuous real functions, so (because of propos. 3.6.2.) $\lambda = \lambda_c(\mathcal{C}^{+}(\lambda))$. On the other hand, by proposition 4.4.2., we have $\pi(\mathcal{C}^{+}(\lambda)) = \lambda_s(\mathcal{C}^{+}(\lambda))$ and, being $\mathcal{C}^{+}(\lambda) \in \mathcal{C}$, from propos. 4.3.3. we have $\lambda_s(\mathcal{C}^{+}(\lambda)) = \lambda_c(\mathcal{C}^{+}(\lambda))$ and we conclude that $\lambda = \pi(\mathcal{C}^{+}(\lambda))$.

(ii) If $\mathcal{F} \in \mathcal{C}$, $\pi(\mathcal{F})$, being equal to $\lambda_s(\mathcal{F})$ because of 4.4.2., is also equal to $\lambda_c(\mathcal{F})$ because of 4.3.3.: this means that $\pi(\mathcal{F})$ is a completely regular topology on the set T (see [5] no. 3.7.). The surjectivity of the mapping $\pi: \mathcal{C} \rightarrow \Sigma$ has been shown in the preceding proposition (i). To prove the injectivity, let $\mathcal{F}$, $\mathcal{G}$ be two families in $\mathcal{C}$ such that $\pi(\mathcal{F}) = \pi(\mathcal{G})$; for what has been said above, this implies that $\lambda_c(\mathcal{F}) = \lambda_c(\mathcal{G})$, hence $\mathcal{C}^+(\lambda_c(\mathcal{F})) = \mathcal{C}^+(\lambda_c(\mathcal{G}))$, and recalling propos. 4.2. (ii), $\mathcal{F} = \mathcal{G}$.

5. Full rings of continuous functions.

5.1. We start with some definitions.

Let $\mathcal{F}$ be a subset of $\mathcal{B}(T, R)$, then:

(i) $\mathcal{F}$ is *full* if, and only if, for every $a \in R$ and every $f \in \mathcal{F}$, $a + f \in \mathcal{F}$;

(ii) the *full closure* of $\mathcal{F}$ is the family of functions

$$\mathcal{F}^{\sim} = \{a + f: \ a \in R, f \in \mathcal{F}\};$$

(iii) the *positive part* of $\mathcal{F}$ is the family of functions

$$\mathcal{F}^+ = \{f: f \in \mathcal{F} \ \text{ and } \ \inf f \geq 0\}$$ [10];

(iv) $\mathcal{F}$ is a *full semiring* (of bounded real functions on T) if, and only if, it is full and its positive part $\mathcal{F}^+$ belongs to the class $\mathcal{S}$[11];

(v) $\mathcal{F}$ is a *full ring* (of bounded real functions on T) if, and only if, it is full and its positive part $\mathcal{F}^+$ belongs to the class $\mathcal{C}$[12].

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[10]) The notation $\mathcal{F}^+$ for the positive part of a family $\mathcal{F} \subset \mathcal{B}(T, R)$ is in accordance with the definitions of $\mathcal{S}^+(\lambda)$ and $\mathcal{C}^+(\lambda)$ in 2.7..

[11]) If we call, as it is usual, *semiring* a set with two (binary) associative operations, say addition and multiplication, such that the addition is commutative, has an identity and is cancellative, the multiplication is distributive over the addition, then a full semiring is *not* a semiring; only its positive part is such.

[12]) A full ring is a lattice ordered commutative ring of real functions, as is easily seen if we bear in mind propos. 5.4..

5.2.1. *For every family* $\mathcal{F} \subset \mathcal{B}(T, R^+)$ *we have*

(i) $\mathcal{S}(\lambda_s(\mathcal{F})) = (\mathcal{F}^s)^\sim$;

(ii) $\mathcal{C}(\lambda_c(\mathcal{F})) = (\mathcal{F}^c)^\sim$.

PROOF. (i) From 4.1. (i) we have $\mathcal{S}^+(\lambda_s(\mathcal{F})) = \mathcal{F}^s$ so that, obviously, $(\mathcal{F}^s)^\sim \subset \mathcal{S}(\lambda_s(\mathcal{F}))$; on the other hand, if $f \in \mathcal{S}(\lambda_s(\mathcal{F}))$, then $f + |\inf f| \in \mathcal{S}^+(\lambda_s(\mathcal{F})) = \mathcal{F}^s$, hence $f \in (\mathcal{F}^s)^\sim$.

(ii) Same argument as in case (i).

5.2.2. *If* $\mathcal{F}$ *is a full family contained in* $\mathcal{B}(T, R)$, *then*

$$\mathcal{F} = (\mathcal{F}^+)^\sim .$$

PROOF. Since $\mathcal{F}^+ \subset \mathcal{F}$ and being $\mathcal{F}$ full, we have $(\mathcal{F}^+)^\sim \subset \mathcal{F}$. Conversely, let f be a function in the family $\mathcal{F}$, then

$$f + |\inf f| \in \mathcal{F}^+$$

and

$$f = (f + |\inf f|) - |\inf f| \in (\mathcal{F}^+)^\sim .$$

5.3. THEOREM. *If* $\mathcal{F}$ *is a full semiring of bounded real functions on the set* T, *then* $\lambda = \pi(\mathcal{F}^+)$ *is the unique topology on* T *such that*

$$\mathcal{F} = \mathcal{S}(\lambda);$$

hence λ *may be characterized as the weakest topology on* T *which makes every function* $f \in \mathcal{F}$ *lower semicontinuous* [13]).

PROOF. From definition 5.1. (iv) and from propos. 4.4.2., 5.2.1. (i) and 5.2.2. we have $\mathcal{S}(\lambda) = (\mathcal{F}^+)^\sim = \mathcal{F}$. To prove uniqueness, let μ be any topology on T such that $\mathcal{S}(\mu) = \mathcal{F}$; then the topologies λ

([13]) Conversely, as is easily seen from 3.4. (i), for every topology on T, the class of all lower semicontinuous bounded real functions is a full semiring.

and μ have the same lower semicontinuous functions into the closed interval $[0, 1]$, they coincide because of propos. 5.2.1. (*b*) of [2]. The last statement, that is $\lambda = \lambda_s(\mathcal{F})$, follows easily from the uniqueness property of λ, since any topology on T whose class of all lower semicontinuous bounded real function contains $\mathcal{F}$ is larger than λ.

5.4. THEOREM. *If $\mathcal{F}$ is a full ring of bounded real functions on the set T, then $\lambda = \pi(\mathcal{F}^+)$ is the unique completely regular topology on T such that*

$$\mathcal{F} = \mathcal{C}(\lambda);$$

moreover, λ is the weakest topology on T which makes every function $f \in \mathcal{F}$ continuous [14]).

PROOF. Because of propos. 4.4.2. and 4.3.3. we have $\lambda = \lambda_s(\mathcal{F}^+) = \lambda_c(\mathcal{F}^+)$. Then, from definition 5.1. (v) and from propos. 5.2.1. (ii), 5.2.2. we draw the equalities $\mathcal{C}(\lambda) = (\mathcal{F}^+)^{\sim} = \mathcal{F}$. The topology λ is completely regular because of propos. 4.3.3.. To prove uniqueness, let μ be a completely regular topology on T such that $\mathcal{C}(\mu) = \mathcal{F}$; then, since $\mathcal{F}^+ = \mathcal{C}^+(\mu) = \mathcal{C}^+(\lambda)$, the two topologies λ, μ coincide because of propos. 3.6.4.. To prove the equality $\lambda = \lambda_c(\mathcal{F})$ it is sufficient to observe that the family $\mathcal{F}$ of all bounded real continuous functions in the topology λ is separating (because λ is completely regular); then, because of propos. 3.6.2., λ is the weakest topology which makes every function $f \in \mathcal{F}$ continuous.

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([14]) Conversely, as is easily seen from 3.4. (ii), for every topology on T, the class of all continuous bounded real functions is a full ring.

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