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Geometric duality

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Duality for geometries (see [4], [3]) may be expressed in terms of complementation of subsets, together with negation of the dependence relation:

$$e \delta^* (X - e) \iff e \bar{\delta} (\subseteq X - e).$$

The dependence relations $\delta$ and $\delta^*$ give rise to closure operators $J$ and $J^*$ with the exchange property. $J$ and $J^*$ may be considered to act on the Boolean algebra $B$ of all subsets of $G$, and on the dual lattice $\widehat{B}$, respectively. Then

$$J^* (\sim x) = J^* (\sim y) \iff J (x) \perp J (y)$$

for all pairs $x, y$ of subsets of $G$, such that $y$ covers $x$ in $B$.

Closure operators with the exchange property also occur as the kernels of strong maps [2] from one geometric lattice to another. This suggests a more general form of duality for geometries. Indeed, we shall prove that if $J$ is a closure satisfying appropriate exchange and finiteness properties on a geometric lattice $P$, and if the dual lattice $\widehat{P}$ is also geometric, then condition (2), above, determines uniquely a closure operator $J^*$ on $\widehat{P}$, satisfying the same exchange and finiteness conditions. The condition on the lattice $P$ is satisfied, for example, if $P$ is a complemented modular lattice of finite height[1].

The relationship of geometric duality holding between the lattices $P/J$ and $\widehat{P}/J^*$ is more general than that obtaining in the theories of Whitney [4]. It coincides with the duality of Whitney if $P$ is a finite Boolean algebra.

Under the same condition on the lattice $P$, namely that $\widehat{P}$ also be geometric, we prove that a closure satisfying finiteness conditions has a dual closure defined by (2) if and only if it has the exchange property.

An element $x$ in a geometric lattice $P$ is cofinite if and only if $x < x \lor p$ for only finitely many atoms $p$ in $P$. A closure $J$ on $P$ is cofinitary if and only if $y \lor x$ and $J(x) \neq J(y)$ imply the existence of a cofinite element $z$ such that $x \leq z$ and $J(z) \neq J(y \lor z)$.

**Proposition 1.** If a lattice $P$ and its lattice dual $\widehat{P}$ are both geometric, and if $J$ is a finitary and cofinitary closure with the exchange property on $P$, then there is a unique closure $J^*$ on $\widehat{P}$ satisfying condition (2), and $\widehat{P}/J^*$ is geometric.

**Proof:** For each element $y \in P$, let $T(y) = \inf \{x : y \lor x$, and $y = x$ or $J(y) \neq J(x)\}$. If $J^*$ is any closure on $\widehat{P}$ satisfying condition (2), then $y \lor x$ implies $\tilde{x} \leq J^*(\tilde{y}) \iff J^*(\tilde{x}) = J^*(\tilde{y}) \iff J(x) \neq J(y)$. Since the lattice $P$ is complemented modular and coatomistic, the interval $[0, y]$ is coatomistic, and $J^*(\tilde{y}) = \widehat{T}(y)$. We prove that $J^*$, thus defined, is a closure operator with the required properties.

$y \geq T(y)$ implies $\tilde{y} \leq J^*(\tilde{y})$. Assume $z \leq y$ and $y \lor x$. Then $T(y) \leq x \iff J(x) < J(y)$. If $J(x) < J(y)$, then $J(x \land z) \leq J(x)$, so $J^*(\tilde{y}) \leq J^*(\tilde{z})$. Assume that for some element $y \in P$, there exists an element $z$ such that $T(y) \lor z$, and such that $J(z) < J(T(y))$. Choose a cofinite element $x$ such that $z \leq x$ and $J(x) \neq J(x \lor T(y))$. Then the interval $[x \land y, y]$ is finite. Let $w$ be a maximal element of $[x \land y, y]$ such that $J(w) \neq J(w \lor T(y))$. If $w \lor T(y) = y$, choose an element $u$ covering $w$ such that $u \lor T(y)$ covers $w \lor T(y)$. Since $w \lor T(y)$
and \( u \lor T(y) \) are in the interval \([T(y), y]\), \( J(u \lor T(y)) < J(u \lor T(y)) \), \( J(v) < J(w) \), and, by the exchange property, \( J(w) < J(u \lor T(y)) \). This contradicts the maximal property of \( w \), so \( w \lor T(y) = y \), and \( T(y) \leq w < y \), by the definition of \( T \). This contradicts the definition of \( w \), so \( T(y) = w \), and \( T(y) \leq w \). Thus \( J^*(\tilde{y}) = J^*(y) \), and \( J^* \) is a closure. \( J^* \) is finitary because \( J \) is cofinitary.

If elements \( x \) and \( y \) cover \( x \land y \) in \( P \), and are thus covered by \( x \lor y \), and if \( J^*(x \lor y) \leq J^*(x) = J \) then \( J(x \land y) \leq J(x) \). If, moreover, \( J^*(x \lor y) < J^*(z) \), then \( J(y) = J(x \lor y) \), \( J(y) \neq J(x \land y) \), and \( J^*(y) = J^*(x \land y) \). Thus \( J^* \) has the exchange property, and \( \tilde{P}/J^* \) is a geometric lattice. 

As an example of duality relative to a complemented modular lattice, consider the seven-point projective plane mapped into a five-point plane in such a way that one line \( j \) is mapped to a point. The empty set, the line \( j \), the four points off \( j \), and the plane are closed relative to this strong map. Only \( \tilde{j} \) and \( \tilde{O} \) are closed relative to the dual closure on the dual plane, and \( \tilde{j} \) is the dual-closure of the empty subset of the dual plane.

A partial converse to proposition 1 is available, which characterizes closures with the exchange property as those closures which have duals.

**Proposition 2.** If a lattice \( P \) and its lattice dual \( \tilde{P} \) are both geometric, if \( J \) is a finitary and cofinitary closure on \( P \), and if \( T \) is a closure on \( \tilde{P} \), where \( T(y) = \inf \{x ; y \mid x, \text{ and } y \neq x \text{ or } J(y) \neq J(x)\} \), then \( J \) has the exchange property.

**Proof:** Assume \( x \) and \( y \) cover \( x \land y \), so \( x \lor y \) covers \( x \) and \( y \). Assume further that \( J(x \land y) < J(x) = J(x \lor y) \) and \( J(x \land y) < J(y) \). If \( J(y) < J(x \lor y) \), then \( T(x \lor y) \leq T(y) \). Since \( J(x \land y) < J(y) \), \( T(y) \leq T(x \land y) \). If \( T \) is a closure, then \( T(x \lor y) = T(x \land y) = T(x) \), contradicting \( J(x) = J(x \lor y) \). Thus \( J(y) = J(x \lor y) \), and \( J \) has the exchange property. 

Added in proof: The following, provided by D. A. Higgs, and printed here with his permission, defines the scope of the preceding duality theory. It is known that every modular geometric lattice is a direct join (cartesian product) of projective geometries. We have considered, above, geometric lattices $L$ whose dual lattices $\sim L$ are continuous. Under this assumption, Higgs proves that the projective geometries involved in the above direct join decomposition must be of finite height. The essential result is as follows.

**Proposition 3.** (D. A. Higgs) A projective geometry $L$ of infinite height cannot be dual continuous.

**Proof.** Let $\{p_i; i = 0, 1, ...\}$ be an independent enumerably infinite set of atoms of $L$, where $L$ is geometric, modular, and every element of rank 2 covers at least 3 atoms. Let $r_n$ be a third atom covered by $p_n \lor p_{n+1}$, $n = 0, 1, ...$. Let $a = \sup r_i$ and $x_i = \sup p_j$. Then $\inf x_i = 0$, because each atom beneath $x_0$ is dependent upon a unique minimal (finite) subset of $\{p_i\}$. Thus $a \lor \inf x_i = a \lor 0 = a$, while $\inf (a \lor x_i) = \inf x_0 = x_0 > a$.

Thus $L$ is not dual-continuous. □

**BIBLIOGRAPHY**


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