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APPROXIMATION BY FREDHOLM OPERATORS IN THE METRIC SPACE OF CLOSED OPERATORS

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1. – *Introduction.* – In this paper we determine those operators on a separable Hilbert Space which can be approximated arbitrarily closely by Fredholm operators of a fixed index. This has already been done using the uniform topology in the space of bounded operators [1]. Here we use the graph topology in the space of closed operators (not necessarily bounded) and obtain the same result. Namely, the closure of the set of Fredholm operators of index k is the complement of the set of semi-Fredholm operators with index different from k. Thus, if A is not semi-Fredholm each neighborhood of A contains Fredholm operators of every finite index, and so the index function has no continuous extension.

2. – *Definitions and Notation.* – Let $\mathcal{C}$ denote the set of operators on a separable Hilbert space H which are closed and have dense domains. If A_1 and A_2 are operators in $\mathcal{C}$, then their graphs G_1 and G_2 are closed subspaces of $H \oplus H$. Let P_1 and P_2 denote the orthogonal projections onto G_1 and G_2. Then one of several equivalent metrics on $\mathcal{C}$ is obtained by taking the operator norm $\| P_1 - P_2 \|$ as the distance between the operators A_1

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and A_2. For the other metrics and a proof that this metric gives the uniform topology on the subset of bounded operators $\mathcal{B}$ see [2].

An operator A is said to be semi-Fredholm if A has closed range $R(A)$ and either the null space of A, $N(A)$, or the orthocomplement of the range $D(A)$ is finite dimensional. If both $N(A)$ and $D(A)$ are finite dimensional then A is called a Fredholm operator. The set of all semi-Fredholm operators is denoted by $\mathcal{S}$ and the set of all Fredholm operators by $\mathcal{F}$. The index of an operator A in $\mathcal{S}$ is defined as

$$k(A) = \dim D(A) - \dim N(A) .$$

It is known that $\mathcal{S}$ is an open set in $\mathcal{C}$ since the components of $\mathcal{S}$ (the sets $\mathcal{F}_k$ consisting of operators of index k) are open in $\mathcal{C}$ [2].

3. – *Preliminary Results.* – Before establishing the main theorem, we need the following results.

LEMMA 1. – *If A is an arbitrary operator in $\mathcal{C}$ then either A or A^* can be written in the form UP where P is a positive self-adjoint operator in $\mathcal{C}$ and U is an isometry.*

Proof. – It is well-known that any operator A in $\mathcal{C}$ can be written in the form $A = VP$ where $(A^* A)^{1/2} = P$ is a positive operator in $\mathcal{C}$ and V is a partial isometry mapping $N(A)^{\perp} = D(P)^{\perp}$ isometrically onto $D(A)^{\perp}$. Since A is closed, $N(A)$ is a (closed) subspace. Thus if $\dim N(A) \leqslant \dim D(A)$ then V can be extended to an isometry U with $A = UP$. Since $D(A^*) = N(A)$, it follows that either A or A^* has the desired form.

It is known that $\mathcal{B}$ is open in $\mathcal{C}$. Using results in [2], the following useful theorem is easily established.

THEOREM 1. – *$\mathcal{B}$ is dense in $\mathcal{C}$.*

Proof. – Since the mapping $A \to A^*$ is continuous on $\mathcal{C}$ [2], it suffices by Lemma 1 to show that every operator of the form UP, where U is an isometry and P is positive self-adjoint, can be approximated arbitrarily closely in $\mathcal{C}$ by bounded operators. Since multiplication by an isometry U is continuous on $\mathcal{C}$ [2], is now suffices to show that P can be approximated arbitrarily closely in $\mathcal{C}$ by bounded operators. In fact, if P has the spectral form

$$P = \int_0^\infty \lambda dE(\lambda) ,$$

then an appropriate approximating sequence can be constructed as follows [2]

$$P_n = \int_0^n \lambda dE(\lambda) + nE(n, \infty) .$$

Here

$$\operatorname*{Lim}_{n\to\infty} P_n = P$$

in the graph topology.

Finally we state a well-known and easily proven result which will be needed in the proof of the main theorem.

LEMMA 2. – *A positive self-adjoint operator P in $\mathcal{C}$ has closed range if and only if* 0 *is not a limit point of the spectrum,* $\sigma(P)$.

4. – *Main Result.* – We can now prove the promised approximation theorem.

THEOREM 2. – *For each integer k,*

$$\overline{\mathcal{F}_k} = \mathcal{F}_k \cup (\mathcal{C} \setminus \mathcal{S}) .$$

Proof. – Since the components of $\mathcal{S}$ are open in $\mathcal{C}$, it is easy to see that

$$\mathcal{F}_k \subset \overline{\mathcal{F}_k} \subset \mathcal{F}_k \cup (\mathcal{C} \setminus \mathcal{S}) .$$

Thus, to establish the theorem we need to show that for each k,

$$\mathcal{C} \setminus \mathcal{S} \subset \overline{\mathcal{F}_k} .$$

In fact, we will now show that any A in $\mathcal{C} \setminus \mathcal{S}$ can be approximated arbitrarily closely by *bounded* Fredholm operators of every finite index k. By continuity of the mapping $A \to A*$ and use of Lemma 1, it suffices to consider A of the form $A = UP$ where U is an isometry and

$$P = \int_0^\infty \lambda dE(\lambda)$$

is positive self-adjoint. Two cases now arise.

Case I. – A has non-closed range. We recall that P is the limit of operators of the form

$$P_n = \int_0^n \lambda dE(\lambda) + nE(n, \infty) .$$

Now if A does not have closed range then P does not have closed range. Suppose P_n has closed range for some n. By standard properties of the spectral projections,

$$E(n, \infty)\int_0^n \lambda dE(\lambda) = 0 .$$

Hence, the range of $E(n, \infty)$ and the range of $\int_0^n \lambda dE(\lambda)$ are orthogonal and easy computation now shows that $\int_0^n \lambda dE(\lambda)$ must also have closed range. Thus, by Lemma 2, 0 is not a limit point of $\sigma\left(\int_0^n \lambda dE(\lambda)\right)$. This implies that for some $\varepsilon > 0$,

$$\int_0^\varepsilon \lambda dE(\lambda) = 0$$

and consequently,

$$P = \int_{\varepsilon}^{\infty} \lambda dE(\lambda)$$

so that again appealing to Lemma 2, we obtain the contradiction that P has closed range. Thus, P_n does not have closed range for any n. Since multiplication by an isometry is continuous we see that

$$\operatorname*{Lim}_{n\to\infty} UP_n = A\ .$$

Now each UP_n is a bounded operator with non-closed range. Hence, by a result in [1], each UP_n can be approximated arbitrarily closely in $\mathcal{B}$ by bounded Fredholm operators of every index. It follows from the equivalence of the graph and uniform topologies on $\mathcal{B}$ that A can be approximated arbitrarily closely in $\mathcal{C}$ by bounded Fredholm operators of every index.

Case II. – A has closed range and

$$\dim N(A) = \dim D(A) = \infty\ .$$

In this case, for each u in the domain of A write

$$A_t u = (A + tV)u$$

where t is a positive real number and V is a partial isometry with $N(V)^{\perp} \subset N(A)$ and $R(V) \subset D(A)$. Using techniques of [2], it is easy to check that the operators A_t are closed with closed range and

$$\operatorname*{Lim}_{t\to\infty} A_t = A\ .$$

Further, $N(A_t) = N(A) \ominus N(V)$ and $D(A_t) = D(A) \ominus R(V)$. Thus, by appropriate choice of V, the operators A_t can be made Fredholm of any desired index. Now noting that each $\mathcal{F}_k$ is open, by use of Theorem 1 we can conclude that A can be approximated arbitrarily closely by operators in $\mathcal{B} \cap \mathcal{F}_k$ for each k. This completes the proof.

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