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THE PROJECTIVE INVARIANTS OF THE CONFIGURATION OF TWO LINES AND A THIRD ORDER DIFFERENTIAL ELEMENT

Nota *) *di* Rodney Angotti (*a Buffalo*)

1. – Bompiani [1] has shown that the configuration of two lines and a third order differential element in the projective space $P_3(K)$, K complex, has two projective invariants. The purpose of this paper is to discuss the invariant properties of this configuration in the event that the two lines are incident; in particular, we will show that the configuration of two incident lines and a third order differential element has a third invariant and, in addition, that each of the invariants which Bompiani discusses is not geometric when the two lines are incident, i.e., each has a constant value for every pair of incident lines.

2. – Points in $P_3(K)$ will be represented by the projective homogeneous coordinates x^i, $(i = 0, 1, 2, 3)$, or alternatively by the ordered tetrad (t, x, y, z). Non-homogeneous coordinates can be introduced by the embedding map

$$X = x/t\,, \qquad Y = y/t\,, \qquad Z = z/t\,.$$

Consider a curve C, given parametrically by the equations, $x^i = x^i(s)$. The totality of curves having a contact of order three

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with C at a point corresponding to a value $s = s_0$ defines, by abstraction, a differential curvilinear element of order three E_3 with center $(x^i(s_0))$. Let us further assume that the curves we are considering are of at least class C^2 in a neighborhood of s_0. We can, consequently, represent the element E_3 by the developments

$$x^i(s) = a_0^i + a_1^i s + a_2^i s^2 + a_3^i s^3 + o(s^4) \tag{2.1}$$

where the coefficients a_j^i, $(j = 0, 1, 2, 3)$, are determined by the usual formulas and the symbol $o(s^4)$ indicates that the coefficients of the terms of degree $\geqslant 4$ are arbitrary.

By a proper choice of the system of reference, we can always reduce the representation (2.1) to

$$\begin{aligned} y &= ax^2 + o(x^4) \\ z &= bx^3 + o(x^4) \end{aligned}, \qquad t = 1, \ a \cdot b \neq 0. \tag{2.2}$$

In this representation the center of the element E_3 is 0 (1, 0, 0, 0), the tangent to E_3 at 0, i.e., the common tangent of all the curves making up the element, is $y = z = 0$ and the osculating plane of E_3 at 0, defined similarly, is $z = 0$.

A (non-singular) collineation of $P_3(K)$ which leaves the form (2.2) invariant, i.e., leaves the element E_3 invariant, is of the form

$$\begin{aligned} \lambda t' &= p_1 t + p_2 x + p_3 y + p_4 z \\ \lambda x' &= q_1 x + q_2 y + q_3 z \\ \lambda y' &= p_1^{-1} q_1^2 y + q_4 z \\ \lambda z' &= p_1^{-2} q_1^3 z \end{aligned} \tag{2.3}$$

where

$$q_4 = ab^{-1} p_1^{-2} q_1 (2ap_1 q_2 - p_2 q_1).$$

Of necessity, we will reproduce here some of the results contained in [1]. Since each collineation of the form (2.3) maps a generic line r, whose equation we can always write in the form

$$(2.4)\qquad \begin{aligned} t - s_2 y - s_3 z &= 0 \\ x + h_2 y + h_3 z &= 0, \end{aligned}$$

onto the line $t' = x' = 0$, we can, without any loss of generality, express a collineation having the form (2.3) in the more convenient form

$$(2.5)\qquad \begin{aligned} \lambda t' &= t - s_1 x - (s_2 + h_2 s_1) y - (s_3 + h_3 s_1) z \\ \lambda x' &= n(x + h_2 y + h_3 z) \\ \lambda y' &= n^2 (y + pz) \\ \lambda z' &= n^3 z \end{aligned}$$

where

$$(2.6)\qquad p = ab^{-1}(s_1 + 2ah_2).$$

For a fixed line r, the relation (2.6) represents a projectivity between the pencil of planes through the tangent (depending on p) and the pencil of planes through r (depending on s_1). The locus of line intersections of corresponding planes in this projectivity are generators of the quadric Q_r

$$(2.7)\qquad az(t - s_2 y - s_3 z) + (by + 2a^2 h_2 z)(x + h_2 y + h_3 z) = 0.$$

By varying r, we obtain a linear system of quadrics of freedom three. For a geometrical characterization of these quadrics, see [1].

3. - Consider the configuration of the element E_3 and two generic lines. We can always choose these lines as the line r, given by equations (2.4), and the line $q : [t = x = 0]$, i.e., $s_2 = s_3 = h_2 = h_3 = 0$.

In [1], it is shown that this configuration has two projective invariants. In order to formulate these, consider among the collineations (2.5), which leave the element E_3 invariant, those which map q onto itself. These are given by

$$
\begin{aligned}
&\lambda t' = t - s_1 x, \qquad \lambda x' = nx, \\
&\lambda y' = n^2(y + ab^{-1}s_1 z), \qquad \lambda z' = n^3 z.
\end{aligned}
\tag{3.1}
$$

The line r, under transformations of this type, describes a congruence. One of the projective invariants of the element E_3 and the two lines can be expressed as the cross ratio of the focal points of the line r describing the congruence with respect to the points of intersection of this line with the quadric Q_a, i.e., the quadric (2.7) with $s_2 = s_3 = h_2 = h_3 = 0$; the other invariant, as the cross ratio of the focal planes of the line r with respect to the tangent planes to the quadric Q_a through r.

Let us represent the points of the line r parametrically by the equations

$$
t = s_2 m + s_3, \qquad x = - h_2 m - h_3, \qquad y = m, \qquad z = 1. \tag{3.2}
$$

The focal points of the line r correspond to the values of m satisfying the equation

$$
\begin{aligned}
&h_2^2 m^2 + (ab^{-1}s_2 + 3h_3)h_2 m + 2h_3^2 + \\
&3ab^{-1}h_2 s_3 - 2ab^{-1}s_2 h_3 = 0^1,
\end{aligned}
\tag{3.3}
$$

and the points $r \cap Q_a$ to the values of m determined by

$$
bh_2 m^2 + (bh_3 - as_2)m - as_3 = 0. \tag{3.4}
$$

In the pencil of planes (with axis r)

$$
(t - s_2 y - s_3 z) + w(x + h_2 y + h_3 z) = 0,
$$

the focal planes of r are given by the roots of

$$
\begin{aligned}
&ah_2^2 w^2 - h_2(3as_2 + bh_3)w + 2as_2^2 + \\
&3bh_2 s_3 - 2bs_2 h_3 = 0
\end{aligned}
\tag{3.5}
$$

[1. Given erroneously in [1].]

and the tangent planes to Q_q by the values of w satisfying

$$ah_2w^2 + (bh_3 - as_2)w - bs_3 = 0. \tag{3.6}$$

It is immediately seen that the cross ratios of the above values of m and of w are left invariant by a collineation having the form (2.5). Using these cross ratios, it is not difficult to obtain analytical expressions for these invariants.

If the two lines r and q are incident, it is only a matter of calculation to verify that both of the above described cross ratios have the value zero (one, or infinity) and, consequently, are not geometric.

If we express the condition that r and q are incident in the form

$$s_2h_3 - s_3h_2 = 0, \tag{3.7}$$

it is easy to see that the equations (3.3) and (3.4) share the root

$$m = - h_3h_2^{-1};$$

and, consequently, their four roots (in some order) yield a cross ratio of zero.

Similarly, for values of s_2, s_3, h_2, and h_3 satisfying (3.7), the equations (3.5) and (3.6) have the common root

$$w = s_2h_2^{-1}$$

and we are lead to the same conclusion.

4. – The configuration of the element E_3 and two incident lines, however, has an addition projective invariant. In order to construct this, consider the element E_3' obtained by projecting the element E_3 onto its osculating plane from the point of intersection of the two incident lines. This element obviously has the same center and tangent as the element E_3. The conics in the osculating plane containing E_3' form a hyper-osculating pencil, i.e., a pencil of conics with only one base point, each conic of the pencil being tangent to $y = z = o$ at $0(1, 0, 0, 0)$. Therefore, the polar line of a point on the tangent line with respect to any conic of this pencil passes through the center $0(1, 0, 0, 0)$ and, moreover, is the same for each conic. The element E_3', consequently, determines a projectivity between the points of the tangent line and the pencil of lines in the osculating plane of the element E_3 through $0(1, 0, 0, 0)$. In particular, in this projectivity the point $0(1, 0, 0, 0)$ corresponds to the tangent line $y = z = o$. Furthermore, the two incident lines determine three additional lines in this pencil; namely, the line corresponding in the projectivity to the point of the tangent line belonging to the plane of the incident lines, and the two lines determined by the center of the element E_3 and the points in which the two given lines pierce the osculating plane. The configuration of the element E_3 and the two incident lines, therefore, completely determines four concurrent lines in the osculating plane of the element E_3. The cross ratio of these four lines is obviously left invariant by collineations having the form (2.5).

This invariant can also be obtained by considering among those quadrics which contain the element E_3, i.e., the ∞^5 system

$$xz + c_1yz + c_2y^2 + c_3z^2 + c_4(bxy - azt) + c_4(yt - ax^2) = 0,$$

the quadric cones, each having its vertex at the point of intersection of the two incident lines. These cones form a pencil which has only one degenerate cone. Furthermore the two lines determine three addition cones in the pencil; namely, the two cones having a given line as a generator and the cone tangent to the plane of the two (incident) lines.

It is not difficult to obtain an analytical expression for this invariant.

REFERENCE

[1] BOMPIANI E.: *Configuration associées à un élément curviligne du troisième ordre dans l'espace projectif*, Journal des Math. pures et appl. 41 (1962).