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ON THE APPROXIMATE SOLUTIONS OF LINEAR EQUATIONS IN THE l_2-SPACE

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In the present paper we give a method of solving approximately linear equations in l_2-space. This method is an extension of Altman's method of approximate solution of linear algebraic equations (Altman, 1957). It is indicated, in this paper, that this method is immediately applicable to any linear equation in a separable Hilbert Space H and consequently to linear functional equations in L_2.

1\. – Let us consider the l_2-space. Let K be a bounded linear transformation of a subspace X of l_2 into a subspace Y of l_2. We suppose that K is one-to-one.

Consider the linear equation,

$$Kx = y(x \in X, \ y \in Y) . \tag{1}$$

Let $\{e_i\}$ be a complete orthonormal system. Since, K is one-to-one, $\{a_i\}$ where $a_i = Ke_i$ is linearly independent.

For if a_i's are dependant then there exist constants α_i's not

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all zeros such that,

$$\Sigma\alpha_i a_i = 0 \quad \text{or}$$

$$\text{or} \quad \Sigma\alpha_i K e_i = 0 \quad \text{or,} \quad K\Sigma\alpha_i e_i = 0 \quad \text{since,} \quad K \text{ is bounded}$$

$$\therefore \Sigma\alpha_i e_i = 0$$

i.e., e_i's are not independent, which is a contradiction.

The sequence $\{a_i\}$ is weakly convergent. For every $\varphi \in l_2$,

$$|(a_m, \varphi) - (a_n, \varphi)| = |(Ke_m, \varphi) - (Ke_n, \varphi)|.$$

$$= |(e_m, K^*\varphi) - (e_n, K^*\varphi)|.$$

$$\longrightarrow 0$$

$$\text{as } m, n \text{ both} \longrightarrow \infty.$$

because $\{e_i\}$ is weakly convergent to θ.

Again, for every $\varphi \in l_2$,

$$(a_i, \varphi) = (Ke_i, \varphi) = (e_i, K^*\varphi) \longrightarrow \theta$$

$$\text{as } i \longrightarrow \infty.$$

Therefore, for every $\varphi \in l_2$,

$$(a_i, \varphi) \longrightarrow (\theta, \varphi) \qquad \text{as } i \longrightarrow \infty.$$

and hence $a_i \longrightarrow \theta$ weakly as $i \longrightarrow \infty$.

Define the infinite set of vectors as follows:

(2) $$y_1 = y - P_1 y, \quad y_2 = y_1 - P_2 y_1, \dots$$

$$y_n = y_{n-1} - P_n y_{n-1}, \quad y_{2n} = y_{2n-1} - P_{2n} - P_{2n-1}$$

where P_i denotes the orthogonal projection on

a_i $(i = 1, 2, 3, \dots n, \dots)$

(3) Then, $$\|y_{k+1}\|^2 = \|y_k\|^2 - \|P_{k+1} y_k\|^2.$$

$$\text{where } P_k y = \frac{(y, Ke_k)}{\|Ke_k\|^2} Ke_k,$$

It follows from (3) that $\{\|y_k\|^2\}$ is monotonously decreasing and bounded below and is therefore also convergent.

We shall next show that $\{y_k\}$ $(K = 1, 2, ..., n, ...)$ is weakly convergent.

From (3), we have $\{I_k\}$ is bounded, and hence contains a weakly convergent subsequence $\{y_{k_p}\}$.

Let, $$y_{k_p} \xrightarrow{\text{weakly}} v \text{ as } p \longrightarrow \infty .$$

Now, $$y_{k_p+1} = y_{k_p} - \alpha_{k_p+1}\, a_{k_p+1} .$$

where $$\alpha_{k_p+1} = \frac{(y_{k_p}, a_{k_p+1})}{\|a_{k_p+1}\|^2}$$

Let, $$\alpha_{i_p} = (y_{k_p}, a_i) .$$

Now, as $i \longrightarrow \infty$ $\alpha_{i_p} \longrightarrow 0$, for all I_{k_p}'s. $(p = 1, 2, ...)$.

$$\text{because } a_i \xrightarrow{\text{weakly}} \theta .$$

Also,

$$\alpha_{i_p} \longrightarrow (v, a_i) \quad \text{for all} \quad a_i\text{'s} .$$

as $p \longrightarrow \infty$

$$\text{because } y_{k_p} \xrightarrow{\text{weakly}} v \quad \text{as } p \longrightarrow \infty .$$

So, $$\alpha_{i_p} \longrightarrow 0 \quad \text{as both } i \,\&\, p \longrightarrow \infty .$$

$$\therefore\ \alpha_{k_p+1} \longrightarrow 0 \quad \text{as} \quad p \longrightarrow \infty .$$

Hence, $$y_{k_p+1} = y_{k_p} - \alpha_{k_p+1}\, a_{k_p+1}$$

$$\xrightarrow{\text{weakly}} v \quad \text{as} \quad p \longrightarrow \infty .$$

$$\text{because, } a_{k_p+1} \xrightarrow{\text{weakly}} \theta \quad \text{as} \quad p \longrightarrow \infty .$$

Similarly, we can prove that,

$$y_{k_p+i} \xrightarrow{\text{weakly}} v \quad \text{as} \quad p \longrightarrow \infty .$$

for all i (finite).

$$\text{If,} \quad i \longrightarrow \infty, \quad p \longrightarrow \infty \quad \text{also.}$$

$$y_{k_p+1} \xrightarrow{\text{weakly}} v \quad \text{as} \quad i \longrightarrow \infty .$$

$$\therefore \ \{y_i\} \ \text{is weakly convergent}$$

and converges to

$$v = y - \sum_{i=1}^{\infty} \alpha_i a_i = y^{(1)} . \qquad \text{(say)}$$

Now, again we define the infinite set of vectors as follows:

$$y^{(1)}_{(1)} = y^{(1)} - \frac{(y^{(1)}, a_1)}{\|a_1\|^2} a_1, \quad y^{(1)}_{(2)} = y^{(1)}_{(1)} - \frac{(y^{(1)}_{(1)}, a_2)}{\|a_2\|^2} a_2,$$

$$\text{(4)} \qquad y^{(1)}_{(i)} = y^{(1)}_{(i-1)} - \frac{(y^{(1)}_{(i-1)}, a_i)}{\|a_i\|^2} a_i, \ldots \ (i = 1, 2, \ldots n, \ldots).$$

We can prove similarly that $\{y^{(1)}_{(i)}\} \xrightarrow{\text{weakly}} y^{(2)}$.
We thus obtain the following infinite sequences:

$$y^{(i)}_{(j)} = y^{(i)}_{(j-1)} - \frac{(y^{(i)}_{(j-1)}; a_j)}{\|a_j\|^2} a_j, \ldots \quad \begin{array}{l} i = 0, 1, 2, \ldots, n, \ldots \\ j = 0, 1, 2, \ldots, n, \ldots \end{array}$$

with $\quad y^{(0)}_{(0)} = y, \ y^{(0)}_{(j)} = y_{(j)} = y_j$ and $y^{(i)}_{(0)} = y^{(i)}$.

The above sequences can be written in the following array:

$$\begin{array}{lllll} y, & y_{(1)}, & y_{(2)}, \ldots & y_{(i)}, \ldots \\ y^{(1)}, & y^{(1)}_{(1)}, & y^{(1)}_{(2)}, \ldots & y^{(1)}_{(i)}, \ldots \\ \vdots & & & \\ y^{(n)}, & y^{(n)}_{(1)}, & y^{(n)}_{(2)}, \ldots & y^{(n)}_{(i)}, \ldots \\ \vdots & & & \end{array}$$

Now, from (3) and (4),

$$\text{(5)} \qquad (y_{(1)}, a_1) = 0, \quad (y^{(1)}_{(1)}, a_1) = 0, \ \ldots \ (y^{(i)}_{(1)}, a_1) = 0, \ \ldots$$

Also

$$(y_{(2)}, a_2) = 0\,, \quad (y^{(1)}_{(2)}, a_2) = 0\,, \dots \quad (y^{(i)}_{(2)}, a_2) = 0\,, \dots$$

$$\dots\dots\dots\dots\dots \quad \dots\dots\dots\dots\dots \quad \dots\dots\dots \quad \dots\dots$$

$$(y_{(k)}, a_k) = 0\,, \quad (y^{(1)}_{(k)}, a_k) = 0\,, \dots \quad (y^{(i)}_{(k)}, a_k) = 0\,, \dots$$

Now, $\{y^{(n)}_{(1)}\}$ is bounded and contains a weakly convergent subsequence $\{y^{(n_p)}_{(1)}\}$.

Now, let $\{y^{(n_p)}_{(1)}\} \xrightarrow{\text{weakly}} v_1$ as $p \longrightarrow \infty$

$$\therefore\ (y^{(n_p)}_{(1)}, a_1) = 0 \ \&\ \text{since,}\quad y^{(n_p)}_{(1)} \xrightarrow{\text{weakly}} v_1\,,$$

$$\because\ (v_1, a_1) = 0\,.$$

$$y^{(n_p)}_{(2)} = y^{(n_p)}_{(1)} - \frac{(y^{(n_p)}_{(1)}, a_2)}{\|a_2\|^2}\, a_2\,,$$

$$y^{(n_p)}_{(1)} - y^{(n_p)}_{(2)} = \frac{(y^{(n_p)}_{(1)}, a_2)}{\|a_2\|^2}\, a_2 \xrightarrow{\text{weakly}} \frac{(v_1, a_2)}{\|a_2\|^2}\, a_2\,.$$

$$\therefore\ y^{(n_p)}_{(2)} \xrightarrow{\text{weakly}} v_2 = v_1 - \frac{(v_1, a_2)}{\|a_2\|^2}\, a_2\,.$$

(6) $$\|y^{(n_p)}_{(2)}\|^2 = \|y^{(n_p)}_{(1)}\|^2 - \|P_2 y^{(n_p)}_{(1)}\|^2\,.$$

Also,

$$\|y\| \geqslant \|y^{(1)}\| \geqslant \|y^{(2)}\| \geqslant \|y^{(3)}\| \geqslant \dots$$

Also

(7) $$\|y^{(n_p)}\| \geqslant \|y^{(n_p)}_{(1)}\| \geqslant \|y^{(n_p)}_{(2)}\| \geqslant \dots \geqslant \|y^{(n_p+1)}\| \geqslant$$
$$\geqslant \|y^{(n_p+1)}_{(1)}\| \geqslant \|y^{(n_p+1)}_{(2)}\| \geqslant \dots$$

$\therefore$ $\{\|y^{(n_p)}_{(1)}\|\}$ is a monotonic decreasing sequence and bouded below and converges to V_1 (say).

Also, $\{\|y^{(n_p)}_{(2)}\|\}$ is a monotonic decreasing sequence and bounded below and converges to V_2 (say).

From (7),

$$V_1 \geqslant V_2 \geqslant V_1\,.$$

$$\therefore\ V_1 = V_2\,.$$

From (6), we have $\{\|P_2 y_{(1)}^{(n_p)}\|^2\}$ is convergent and converges to zero.

$$\therefore \{P_2 y_{(1)}^{(n_p)}\} \text{ convergers strongly to } \theta\,.$$

$$\operatorname*{Lt}_{p\to\infty} y_{(2)}^{(n_p)} = \operatorname*{Lt}_{p\to\infty} y_{(1)}^{(n_p)} - \operatorname*{Lt}_{p\to\infty} P_2 y_{(1)}^{(n_p)} \qquad \text{(weak limit)}\,.$$

$$\therefore v_2 = v_1\,.$$

Similarly, we can prove, $v_{i+1} = v_i$.

$$\therefore v_1 = v_2 = v_3 = \dots = v_i = \dots = v \qquad \text{(say)}$$

Now, $\quad (v_1, a_1) = 0\,,\quad (v_2; a_2) = 0\,,\quad \dots,\quad (v_i, a_i) = 0$ for all i.

Or, $\quad (v, a_i) = 0$ for al i.

$$(K^*v, e_i) = 0 \qquad \text{for all } i.$$

$$\therefore K^*v = \theta\,.$$

$\because K^*$ is bounded and has an inverse $[a\bar{l}_2 = l_2]$.

$$\therefore v = \theta\,.$$

Define,

(8)
$$\omega_1 = \frac{(y, Ke_1)}{\|Ke_1\|^2}\, e_1\,,\quad \omega_2 = \frac{(y_{(1)}, Ke_2)}{\|Ke_2\|^2}\, e_2; \dots$$
$$\omega_k = \frac{(y_{(k=1)}, a_k)}{\|a_k\|^2}\, e_k\,, \dots$$

and

$$\omega_1^{(i)} = \frac{(y^{(i)}, Ke_1)}{\|Ke_1\|^2}\, e_1\,,\quad \omega_2^{(i)} = \frac{(y_{(1)}^{(i)}, Ke_2)}{\|Ke_2\|^2}\, e_2\,, \dots$$
$$\omega_k^{(i)} = \frac{(y_{(k-1)}^{(i)}, Ke_k)}{\|Ke_k\|^2}\, e_k\,.$$

Define, $\quad x_k^{(0)} = \omega_1 + \omega_2 + \dots + \omega_k\,.$

(9) $\quad x_{(k)}^{(i)} = \omega_1^{(i)} + \omega_2^{(i)} + \dots + \omega_k^{(i)}\,.$

Now, $\quad Kx_{(k)}^{(0)} = K\omega_1 + K\omega_2 + \dots + K\omega_k$
$$= y - y_k\,.$$

$$\therefore\ Kx_k^{(0)} \xrightarrow{\text{weakly}} y - y^{(1)} \quad \text{as} \quad k \longrightarrow \infty .$$

$$Kx_k^{(1)} = K\omega_{(1)}^{(1)} + K\omega_{(2)}^{(1)} + \dots + K\omega_k^{(1)} =$$

$$= y^{(1)} - y_{(k)}^{(1)} \xrightarrow{\text{weakly}} y^{(1)} - y^{(2)}$$

$$\text{as} \quad k \longrightarrow \infty .$$

$$\therefore\ K(x_k^{(0)} + x_k^{(1)} + x_k^{(2)} + \dots + x_k^{(i)}) =$$

$$= (y - y_{(k)}) + (y^{(1)} - y_{(k)}^{(1)}) + \dots + (y^{(i-1)} - y_{(k)}^{(i)}) .$$

$$\text{as} \quad k \longrightarrow \infty \quad \text{R.H.S.} \xrightarrow{\text{weakly}} y - y^{(i)} .$$

$$\text{and as} \quad i \longrightarrow \infty \quad y^{(i)} \xrightarrow{\text{weakly}} \theta .$$

$$(10) \qquad KX_{ik} = K(x_k^{(0)} + x_k^{(1)} + \dots + x_k^{(i)}) , \qquad \text{where } X_{ik} = \sum_0^i x_k^{(l)} .$$

R.H.S. $\xrightarrow{\text{weakly}} y$ as $k \longrightarrow \infty$ and then $i \longrightarrow \infty$.

Hence, $\lim\limits_{i\to\infty} (\lim\limits_{k\to\infty} X_{ik})$ is the solution of the given equation $Kx = y$.

Now, L_2-space is isomorphic and isometric to l_2-space. Hence, any linear equation in L_2-space can be reduced to an infinite dimensional matrix equation in l_2 and hence can be solved by the above method.

Remark 1:

For the error estimation we have the following recurrent formula obtained from (3).

$$\|y_{(k+1)}\|^2 = \|y_{(k)}\|^2 - \frac{|(y_{(k)}, a_{k+1})|^2}{\|a_{k+1}\|^2}$$

or,

$$(11) \qquad \|y_{(k)}\|^2 - \|y_{(k+1)}\|^2 = \frac{|(y_{(k)}, a_{k+1})|^2}{\|a_{k+1}\|^2} .$$

We put an end to computations when the R.H.S. of (11) is sufficiently small and then we estimate the valye of $y^{(1)}$.

Proceeding in this manner, we estimate the value of $y^{(i)}$

$$\text{(12)} \qquad \text{Now } y^{(i+1)} = y^{(i)} - \sum_{k=1}^{\infty} \alpha_k^{(i)} a_k,$$

where

$$\alpha_k^{(i)} = \frac{y_{k-1}^{(i)},\ a_k}{\|a_k\|^2}$$

Also,

$$\|y^{(i+1)}\| \leqslant \|y^{(i)}\| .$$

So, we put an end to computations when $\|y^{(i)}\|$ is sufficiently small.

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