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ANOTHER SAASCHÜTZIAN THEOREM FOR DOUBLE SERIES

*Nota *) di L. CARLITZ (a Durham)*

Put

$$S(m, n) = \sum_{r=0}^m \sum_{s=0}^n \frac{(-m)_r (-n)_s (\alpha)_{r+s} (\beta)_r (\beta')_s}{r! s! (\gamma)_{r+s} (\delta)_r (\delta')_s}.$$

The writer [1] has proved that if

$$(1) \quad \begin{cases} \gamma = \beta + \beta', \\ \gamma + \delta = \alpha + \beta - m + 1, \\ \gamma + \delta' = \alpha + \beta' - n + 1, \end{cases}$$

then

$$(2) \quad S = \frac{(\beta + \beta' - \alpha)_{m+n} (\beta')_m (\beta)_n}{(\beta + \beta')_{m+n} (\beta' - \alpha)_m (\beta - \alpha)_n}.$$

Moreover (2) is equivalent to the transformation formula

$$(3) \quad \begin{aligned} F_1(\gamma - \alpha; \beta', \beta; \gamma; x, y) &= (1-x)^{\beta'-\alpha} (1-y)^{\beta-\alpha} \\ F_1(\alpha; \beta, \beta'; \gamma; x, y), \end{aligned}$$

where $\gamma = \beta + \beta'$ and

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m! n! (\gamma)_{m+n}} x^m y^n.$$

*) Pervenuto in Redazione il 31 ottobre 1963.

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It may be of interest to note another result somewhat similar to (2). Put

$$S(n) = \sum_{r+s \geq n} \frac{(-n)_{r+s}(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{r!s!(\gamma)_{r+s}(\delta)_r(\delta')_s}.$$

Then

$$S(n) = \sum_{r=0}^n \frac{(-n)(\alpha)(\beta)_r}{r!(\gamma)_r(\delta)_r} \sum_{s=0}^{n-r} \frac{(-n+r)(\alpha')(\beta')_s}{s!(\gamma+r)_s(\delta')_s}.$$

If we assume that

$$(4) \quad \gamma + \delta' = \alpha' + \beta' - n + 1,$$

then inner sum is Saaschutzian and therefore equal to

$$\begin{aligned} & \frac{(\gamma - \alpha' + r)_{n-r}(\gamma - \beta' + r)_{n-r}}{(\gamma + r)_{n-r}(\gamma - \alpha' - \beta' + r)_{n-r}} = \\ & = \frac{(\gamma - \alpha')_n(\gamma - \beta')_n}{(\gamma)_n(\gamma - \alpha' - \beta')_n} \frac{(\gamma)_r(\gamma - \alpha' - \beta')_r}{(\gamma - \alpha')_r(\gamma - \beta')_r}. \end{aligned}$$

It follows that

$$S_n = \frac{(\gamma - \alpha')_n(\gamma - \beta')_n}{(\gamma)_n(\gamma - \alpha' - \beta')_n} \sum_{r=0}^n \frac{(-n)_r(\alpha)_r(\beta)_r(\gamma - \alpha' - \beta')_r}{r!(\gamma - \alpha')_r(\gamma - \beta')_r(\delta)_r}.$$

If we assume that

$$(5) \quad \gamma = \beta + \beta',$$

the last sum becomes

$$\sum_{r=0}^n \frac{(-n)_r(\alpha)_r(\beta - \alpha')_r}{r!(\gamma - \alpha')_r(\delta)_r}.$$

If in addition we take

$$(6) \quad \gamma + \delta = \alpha + \beta - n + 1,$$

the sum reduces to

$$\frac{(\gamma - \alpha - \alpha')_n(\gamma - \beta)_n}{(\gamma - \alpha')_n(\gamma - \alpha - \beta)_n},$$

so that finally

$$(7) \quad S(n)_r = \frac{(\beta)_n(\beta')_n(\gamma - \alpha - \alpha')_n}{(\gamma)_n(\beta - \alpha')_n(\beta' - \alpha)_n},$$

provided

$$(8) \quad \begin{cases} \gamma = \beta + \beta' \\ \gamma + \delta = \alpha + \beta - n + 1, \\ \gamma + \delta' = \alpha' + \beta' - n + 1. \end{cases}$$

If $\gamma = \beta + \beta'$ and λ is arbitrary, it follows from (7) that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\beta)_n(\beta')_n(\gamma - \alpha - \alpha')_n}{n! (\gamma)_n(\lambda)_n} x^n &= \sum_{n=0}^{\infty} \frac{(\beta - \alpha')_n(\beta' - \alpha)_n}{n! (\lambda)_n} x^n, \\ &\quad \sum_{r+s > n} \frac{(-n)_{r+s}(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{r! s! (\gamma)_{r+s}(\alpha - \beta' - n + 1)_r(\alpha' - \beta - n + 1)_s}. \end{aligned}$$

Since

$$(\alpha - \beta' - n + 1)_r = (-1)^r \frac{(\beta' - \alpha)_n}{(\beta' - \alpha)_{n-r}},$$

the right member reduces to

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{x^n} \sum_{r+s \leq n} (-1)^{r+s} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s}{(n-r-s)! r! s! (\gamma)_{r+s}} \cdot \\ \cdot (\beta' - \alpha)_{n-r}(\beta - \alpha')_{n-s} &= \\ &= \sum_{r,s=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s(\beta - \alpha')_r(\beta' - \alpha)_s}{r! s! (\gamma)_{r+s}(\lambda)_{r+s}} x^{r+s} \cdot \\ \cdot \sum_{n=0}^{\infty} \frac{(\beta - \alpha' + r)_n(\beta' - \alpha + s)_n}{n! (\lambda + r + s)_n} x^n. \end{aligned}$$

We have therefore

$$(9) \quad \begin{aligned} {}_3F_2 \left[\begin{matrix} \beta, \beta', \gamma - \alpha - \alpha'; x \\ \gamma, \lambda \end{matrix} \right] &= \\ &= \sum_{r,s=0}^{\infty} \frac{(\alpha)_r(\alpha')_s(\beta)_r(\beta')_s(\beta - \alpha')_r(\beta' - \alpha)_s}{r! s! (\gamma)_{r+s}(\lambda)_{r+s}} x^{r+s} \cdot \\ &\quad \cdot F[\beta - \alpha' + r, \beta' - \alpha + s; \lambda + r + s; x], \end{aligned}$$

where $\gamma = \beta + \beta'$ and λ is arbitrary. Alternatively this can be written in the form

$$\begin{aligned}
 (10) \quad {}_3F_2 \left[\frac{\beta, \beta', \gamma - \alpha - \alpha'; x}{\gamma, \lambda} \right] &= \\
 &= \sum_{n=0}^{\infty} \frac{(\beta - \alpha)_n (\beta' - \alpha)_n}{n! (\lambda)_n} \cdot \\
 &\cdot \sum_{r, s=0}^{\infty} \frac{(\alpha)_r (\alpha')_s (\beta)_r (\beta')_s (\beta - \alpha' + n)_r (\beta' - \alpha + n)_s}{r! s! (\gamma)_{r+s} (\lambda + n)_{r+s}} x^{r+s}.
 \end{aligned}$$

REFERENCE

- [1] CARLITZ L.: *A Saalschützian theorem for double series*. Journal of the London Mathematical Society, Vol. 38 (1963), 415-418.