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N. Abdul-Halim
W. A. Al-Salam

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A CHARACTERIZATION OF THE LAGUERRE POLYNOMIALS

Nota *) *di* N. ABDUL-HALIM *e di* W. A. AL-SALAM (*a Lubbock*)

1. – Recently Carlitz [1] showed that if $\Psi(z)$ is analytic in a neighborhood *of* $z = 0$ then

$$e^t\Psi(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{n!} \tag{1.1}$$

and

$$f_n(xy) = \sum_{k=0}^{n} \binom{n}{k} y^k(1-y)^{n-k} f_k(x) , \tag{1.2}$$

where $\{f_k(x)\}$ is a simple set of polynomials, are equivalent.

Since a sequence of polynomials $\{g_n(x)\}$ is Appell ($g_n' = ng_{n-1}$) if and only if they possess a generating function of the form

$$A(t)e^{xt} = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!}$$

where $A(t)$ is analytic near $t = 0$, we can restate Carlitz' result in the following form:

THEOREM: Given a sequence of polynomials $\{f_n(x)\}$ a necessary and sufficient condition for $\{f_n(x)\}$ to be Appel is that

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Indirizzo degli AA.: Department of mathematics and astronomy, Texas Technological College, Lubbock, Texas (U.S.A.).

$\{g_n(x)\}$, where $g_n(x) = x^n f_n(1/x)$, should possess a multiplication formula of the form (1.2).

This result can be employed to obtain the following characterization of the Laguerre polynomials:

THEOREM 2: The only orthogonal polynomials of the form

$$ {}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} x \right] $$

where n is a non-negative integer and the α's and β's are independent of x and n, are the Laguerre polynomials ($p = 0$, $q = 1$).

Proof. We have after Rainville [2, p. 267]

$$ e^t {}_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} -xt \right] = \sum_{n=0}^{\infty} {}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} x \right] \frac{t^n}{n!} $$

Hence the polynomials ${}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} x \right]$ satisfy a multiplication formula of the form (1.2). But Feldheim [3] proved that the only orthogonal polynomials which satisfy such a multiplication formula are those of Laguerre. Hence the theorem follows.

This result can be also stated in the following way:

THEOREM 3: The function $e^t \ F[\alpha_1, \ldots, \alpha_p; \beta_1 \ldots, \beta_q; -xt]$ generates a set of orthogonal polynomials if and only if $p = 0$, $q = 1$.

2. – We now give an independent and direct proof of our result. Let

$$ (2.1) \qquad \varnothing_n(x) = {}_{p+1}F_q[-n, \alpha_1, \ldots, \alpha_p; \ \beta_1, \ldots, \beta_q; \ x] . $$

It is well known that in order for $\{\varnothing_n(x)\}$ to be orthogonal there exist constants A_n, B_p, C_p so that

$$ (2.2) \qquad \varnothing_{n+1}(x) = (A_n x + B_n) \varnothing_n(x) + C_n \varnothing_{n-1}(x) $$

where $A_n C_n \neq 0$.

Substituting (2.1) in (2.2) and equating coefficients of powers of x we get

$$n+1=(n-k+1)B_n+$$
$$+\frac{(n-k)(n-k+1)}{n}C_n-k\frac{\prod_{j=1}^{q}(\beta_j+k-1)}{\prod_{i=1}^{p}(\alpha_i+k-1)}A_n$$
$$(k=0,1,2,\dots).$$

Putting $k=0,\ n+1,\ n,\ n-1$ we get, respectively,

(2.3) $$B_n+C_n=1\,,$$

(2.4) $$A_n=-\frac{(\alpha_1+n)(\alpha_2+n)\dots(\alpha_p+n)}{(\beta_1+n)(\beta_2+n)\dots(\beta_q+n)}$$

(2.5) $$n+1=B_n-n\cdot$$
$$\cdot\frac{(\beta_1+n-1)(\beta_2+n-1)\dots(\beta_q+n-1)}{(\alpha_1+n-1)(\alpha_2+n-1)\dots(\alpha_p+n-1)}A_n$$

(2.6) $$n+1=2B_n+\frac{2}{n}C_n-(n+1)\cdot$$
$$\cdot\frac{(\beta_1+n-2)(\beta_2+n-2)\dots(\beta_q+n-2)}{(\alpha_1+n-2)(\alpha_2+n-2)\dots(\alpha_p+n-2)}A_n\,.$$

Formulas (2.3), (2.4), and (2.5) give

$$C_n=\frac{n}{2}\left\{\frac{\prod_{j=1}^{q}(\beta_j+n-2)\prod_{i=1}^{p}(\alpha_i+n)}{\prod_{k=1}^{q}(\beta_k+n-2)\prod_{i=1}^{p}(\alpha_i+n)}-1\right\}$$

and

$$B_n=1+\frac{n}{2}-\frac{n}{2}\frac{\prod_{j=1}^{q}(\beta_j+n-2)\prod_{i=1}^{p}(\alpha_i+n)}{\prod_{k=1}^{q}(\beta_k+n)\prod_{j=1}^{p}(\alpha_j+n-2)}\,.$$

Thus (2.6) becomes

$$(2.7) \qquad n+1 = 1 + \frac{n}{2} - \frac{n}{2} \cdot$$

$$\cdot \frac{\prod_{i=1}^{q} (\beta_j + n - 2) \prod_{j=1}^{p} (\alpha_i + n)}{\prod_{i=1}^{p} (\alpha_i + n - 2) \prod_{j=1}^{q} (\beta_j + n)} + n \frac{\prod_{i=1}^{q} (\beta_j + n - 1) \prod_{j=1}^{p} (\alpha_i + n)}{\prod_{i=1}^{p} (\alpha_i + n - 1) \prod_{j=1}^{q} (\beta_j + n)} .$$

If we now let $K_n = 1/A_n$, (2.7) becomes

$$K_n = 2K_{n-1} - K_{n-2}, \qquad K_0 = \frac{\beta_1\beta_2 \dots \beta_q}{\alpha_1\alpha_2 \dots \alpha_p}.$$

Thus

$$(2.8) \qquad K_n = nD + E = \frac{(\beta_1 + n)(\beta_2 + n) \dots (\beta_q + n)}{(\alpha_1 + n)(\alpha_2 + n) \dots (\alpha_p + n)}$$

where D and E are arbitrary constants.

The case $D = 0$ leads to $B_n = 1$ and hence $C_n = 0$ which contradicts the restriction mentioned in (2.2).

If $D \neq 0$ then the only way for (2.8) to hold is that $p + 1 = q$ and $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2, \dots, \beta_{q-1} = \alpha_p$ and $\beta_q = \beta$, $D = 1$.

Thus

$$\varnothing_n(x) = {}_1F_1[-n; \beta; x]$$

which is essentially the Laguerre polynomials. This evidently completes the proof of our theorem 3.

REFERENCES

[1] L. Carlitz: *Some multiplication formulas*, Rendiconti del Seminario Matematico della Università di Padova, vol. 32 (1962), pp. 239-242.

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[3] E. Feldheim: *Une propriete caracteristique des polynomes de Laguerre*, Commentarii Matematici Helvetici, vol. 13 (1940), pp. 6-10.

Duke University, and *Texas Technological College*.